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THE

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EDITED BY

WILLIAM RUTHERFORD,

AND

STEPHEN FENWICK.

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PREFACE.

In concluding their third and last volume, the Editors of the "Mathematician" beg to return their heartfelt thanks to their contributors and supporters, who have enabled them to continue the work to its termination. They cannot look at the number, value, and variety of the papers which appear in the work, without a feeling of lively satisfaction that they have been the means of bringing before the public so many valuable additions to pure science. Nor was it without deep regret that they found themselves obliged to abandon the further prosecution of their original design—especially at the time when the value of the work seemed to be appreciated by the mathematical public in general. Though this step was inevitable, they for a while entertained a hope that it would be taken up under other auspices, and under such management, that they could confidently recommend it to the continued support of all those friends who had so generously contributed to its support hitherto. That hope has not been realized; and they now issue the supplementary number (as promised on the cover of Vol. III., No. 6) without being able to hold out the least hope of the blank, which is thus created in our mathematical literature, being filled up.

The design of the work was so amply explained in the original prospectus, (reprinted in the first number) that the Editors feel it unnecessary to recapitulate any part of it in this place. They rely with much confidence on the hope that they will be considered to have fully redeemed the pledges there given; and they entertain

the belief that they have been able to afford that encouragement to the younger aspirants to mathematical honours, which will produce permanent effects upon their future career.

That the Editors have spared no trouble must be evident from an inspection of the work. In order to economise the funds so as to be able to continue the work thus far, they have been obliged (in addition to the ordinary duties of Editors) to take upon themselves the labour of the entire reading of the proof sheets from the beginning. This labour, indeed, became so continuous, that it occupied almost the whole of their disposable time.

This explanation will, the Editors trust, be satisfactory, both as a reason for the discontinuance of the work, and for any oversights that may be detected in any part of it: but they cannot adequately express their pleasure in the many agreeable associations which their task has produced—associations which will not be severed with the existence of the work in which they originated.

[Since the appearance of the sixth number of this volume, the Editors have received a copy of a "*Treatise on the more difficult Geometrical Problems*," from the Author, Dr. August Wiegand, of Halle; and they have much gratification in presenting to the readers of the Mathematician the subjoined extract of a letter from that gentleman:—

"*To Messrs. Rutherford and Fenwick.*"

"Esteemed Friends,

"I have taken the liberty of dedicating to you the enclosed "Treatise on the more difficult Geometrical Problems," trusting you will look upon my doing so as a mark of my high esteem. You will find in it the most important Theorems contained in your Mathematician. Present my most kind regards to those gentlemen whose contributions I have reproduced, and whose names I have given. I heartily regret that the Mathematician, which formed so pleasing a bond of union between us, has been discontinued, but I trust, however, that our mutual communications will not be interrupted."

The following are the names of those gentlemen whose contributions to the Mathematician, have been reproduced in Germany, by Dr. Wiegand :—

Mr. James Anderson, Montrose.
,, W. S. Ashton, Blackburn.
,, Samuel Bills, Hawton.
Dr. W. Burns, Rochester.
Lieutenant A. R. Clarke, R. E.
Mr. W. Collins, Limerick.
,, James Dalmahoy, Edinburgh.
,, T. S. Davies, Little Heath, Charlton.
,, T. Dobson, Totteridge, Herts.
,, W. F., Durham.
,, J. W. Elliott, Greatham.
,, P. Elliott, Lamesley.
,, S. Fenwick, Woolwich.
,, R. Harley, Blackburn.
,, G. W. Hearn, R. M. College, Sandhurst.
,, H. Hewett, Shooter's Hill.
,, G. F. Lacey, Woolwich Common.
,, J. Laws, Newcastle.
,, W. H. Levy, Shalbourne.
,, W. Mortimer, Seedley.
,, J. Paul, Shooter's Hill.
,, John Riddle, Greenwich Hospital.
,, Andrew Roy, Dundee Academy.
Dr. William Rutherford, Woolwich.
Mr. N. Smyth, Galway.
,, Mr. John Walker, Trin. Coll., Dublin.
,, Thomas Weddle, Wimbledon, Surrey.
,, R. H. Wright, London.]

Royal Military Academy,
September, 1850,

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THE MATHEMATICIAN.

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DEMONSTRATION OF A USEFUL THEOREM, WITH SOME IMPORTANT APPLICATIONS.

[*Mr. Andrew Roy, Dundee Academy.*]

Before proceeding to the demonstration of the theorem, the following investigation of some well known properties may be premised, as introductory to the main object of this paper.

In Barlow's Theory of Numbers, p. 152, the author observes in the solution to prop. xx., "It is undoubtedly true that the n^{th} differences of consecutive n^{th} powers are constant and equal to $1.2.3\ldots n$; but it is difficult to demonstrate this on pure elementary principles. The demonstration appears to rest on the following theorem, viz. the product $1.2.3.4\ldots n = n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - etc.$;" and he refers to the differential calculus for the proof of this equation.

The following is an elementary and very simple proof of the subsequent propositions :

1°. That the n^{th} differences of consecutive n^{th} powers are constant.

2°. That said differences $= n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - etc.$

3°. That $n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - etc. = n(n-1)(n-2)..2.1.$

Taking the differences of 1^n, 2^n, 3^n, 4^n, 5^n, *etc.* we have

1st order of diff., $2^n - 1$; $3^n - 2^n$; $4^n - 3^n$; $5^n - 4^n$; *etc.*

2nd ——— $3^n - 2.2^n + 1$; $4^n - 2.3^n + 2^n$; $5^n - 2.4^n + 3^n$; *etc.*

3rd ——— $4^n - 3.3^n + 3.2^n - 1$; $5^n - 3.4^n + 3.3^n - 2^n$; *etc.*

4th ——— $5^n - 4.4^n + 6.3^n - 4.2^n + 1$; $6^n - 4.5^n + 6.4^n - 4.3^n + 2^n$; *etc.*

$\vdots \qquad \vdots \qquad \vdots$

In these orders of differences let 2^n, 3^n, 4^n, 5^n, *etc.* be changed into $(1+1)^n$, $(2+1)^n$, $(3+1)^n$, *etc.* and we have the above different orders transformed into the following,

(1st.) $n+n.\frac{n-1}{2}+n.\frac{n-1}{2}.\frac{n-2}{3}+etc.$;

$$n.2^{n-1}+n.\frac{n-1}{2}.2^{n-2}+n.\frac{n-1}{2}.\frac{n-2}{3}.2^{n-3}+etc.;\ etc.$$

(2nd.) $n(2^{n-1}-1)+n.\frac{n-1}{2}(2^{n-2}-1)+n.\frac{n-1}{2}.\frac{n-2}{3}(2^{n-3}-1)+etc.;\ etc.$

(3rd.) $n(3^{n-1}-2.2^{n-1}+1)+n.\frac{n-1}{2}(3^{n-2}-2.2^{n-2}+1)+etc.;\ etc.$

$\vdots \qquad \vdots \qquad \vdots$

From this transformation it is manifest that the n^{th} order of differences of 1^n, 2^n, 3^n, *etc.* is made up of the $(n-1)^{\text{th}}$ order of differences of the $(n-1)^{\text{th}}$ and lower powers of 1, 2, 3, 4, *etc.* Now as the first order of differences of 1, 2, 3, 4, *etc.* is constant, it follows from this that the second order of differences of 1^2, 2^2, 3^2, 4^2, *etc.* is constant:—that the third order of differences of 1^3, 2^3, 3^3, *etc.* is constant, and so on to the n^{th} order of differences of 1^n, 2^n, 3^n, *etc.*, which is therefore constant.

2°. In the transformed values of the above orders of differences it is manifest that the different terms follow, in their formation, the law of a Binomial raised to a power an unit less than the order of the differences; hence the n^{th} order of differences

$$=n\left\{n^{n-1}-(n-1)(n-1)^{n-1}+(n-1)\frac{n-2}{2}(n-2)^{n-1}-etc.\right\}$$
$$+n.\frac{n-1}{2}\left\{n^{n-2}-(n-1)(n-1)^{n-2}+etc.\right\}.$$

But all the parts of this expression after the first vanish; hence n^{th} order of differences

$$=n^n-n(n-1)^n+n.\frac{n-1}{2}(n-2)^n-n.\frac{n-1}{2}.\frac{n-2}{3}(n-3)^n+etc.$$

3°. In the preceding transformed values of the different orders of differences, it is evident that in taking the successive orders of differences all the parts in each order, except the first part, may be neglected. Beginning with the second therefore, and taking only

$$n(2^{n-1}-1);\ n(3^{n-1}-2^{n-1});\ n(4^{n-1}-3^{n-1});\ etc.$$

and taking the differences of those parts, putting $(1+1)^{n-1}$, $(2+1)^{n-1}$, $(3+1)^{n-1}$, *etc.* for 2^{n-1}, 3^{n-1}, 4^{n-1}, *etc.* we shall have

3rd order of diff. $=n(n-1)(2^{n-2}-1)+etc.$; $n(n-1)(3^{n-2}-2^{n-2})+etc.$; and continuing in this way to put $1+1$, $2+1$, $3+1$, *etc.* for 2, 3, 4, *etc.*, we get finally the n^{th} order of differences of

$$1^n, 2^n, 3^n, etc.=n(n-1)(n-2)(n-3)........2.1.$$

Hence

$$n(n-1)(n-2)....2.1=n^n-n(n-1)^n+n.\frac{n-1}{2}.(n-2)^n-etc.$$

From the foregoing investigation it is evident that when n is less than m, then

$$m^n-m(m-1)^n+m.\frac{m-1}{2}.(m-2)^n-m.\frac{m-1}{2}.\frac{m-2}{3}(m-3)^n+..=0.$$

Hence also it is manifest that if the n^{th} order of differences of 1^n, 2^n, 3^n, *etc.* are constant, then the n^{th} order of differences of a^n, $(a+d)^n$, $(a+2d)^n$, *etc.* must be constant.

We now proceed to demonstrate the following

THEOREM.

Let $$\frac{Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots S}{A'x^n + B'x^{n-1} + C'x^{n-2} + \ldots S'} = V \ldots\ldots\ldots\ldots (1),$$

where V is constant with respect to x, and where A, B, C, *etc.* and A′, B′, C′, *etc.*, are also constant and not indefinitely great, then if x be arbitrary through any finite extent p, how small soever p may be, we shall have

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}, \text{ etc. to } \frac{S}{S'} = V.$$

DEMONSTRATION.

Multiplying and transposing (1) we have

$$(A-A'V)x^n + (B-B'V)x^{n-1} + (C-C'V)x^{n-2} + etc. \ldots S - S'V = 0 \ldots (2).$$

Now since x, in this equation, is supposed to be arbitrary from x to $x+p$, we may divide p into $n+1$ equal parts, each equal to h, and for x put $x+h$, or $x+2h$, or $x+3h$, *etc.*, up to $x+(n+1)h$ in equation (2), and we shall have the following,

$$(A-A'V)(x+h)^n + (B-B'V)(x+h)^{n-1} + etc. \ldots S - S'V = 0 \ldots (3),$$
$$(A-A'V)(x+2h)^n + (B-B'V)(x+2h)^{n-1} + etc. \ldots S - S'V = 0 \ldots (4),$$
$$(A-A'V)(x+3h)^n + (B-B'V)(x+3h)^{n-1} + etc. \ldots S - S'V = 0 \ldots (5),$$

. . . .

. . . .

$$(A-A'V)\{x+(n+1)h\}^n + (B-B'V)\{x+(n+1)h\}^{n-1} + etc. \ldots S - S'V = 0.$$

If now the equation (3) be taken from (4), then (4) from (5), and so on to the last, and the first of the resulting differences from the second, the second from the third, and so on to the last; and if this process be continued in the ordinary way of taking differences, as far as possible, it is manifest that the terms S—S′V and the powers of x wanting h will disappear in the first differences, then the simple powers of h in the second differences, then the squares of h in the third differences, and so on, till we come at last to the following equation,

$$(A-A'V).\, n(n-1)(n-2)\ldots 2.1.h^n = 0;$$

or, $$(A-A'V).\, n(n-1)(n-2)\ldots 2.1.\, \frac{p^n}{(n+1)^n} = 0.$$

Hence, $$A - A'V = 0, \text{ and } \frac{A}{A'} = V.$$

The same reasoning applied to what remains of the equation (2) after the term $(A-A'V)x^n$ is thus removed, shews that $\frac{B}{B'} = V$, and so on of the remaining terms to $\frac{S}{S'} = V$.

Cor. 1. This proves the truth of the theory of indeterminate or conditional coefficients without supposing x to become zero at any step of the

proof, how far soever the number n, in the order of the equation, may be carried.

Cor. 2. In the equation (2) the quantity x cannot be zero, because in that case S—S′V would be zero, which is contrary to hypothesis. Neither can x be indefinitely great which would make $\frac{1}{x}$ indefinitely small, and consequently $A-A'V=0$, which is also contrary to hypothesis. Nor can x be arbitrary through any finite extent p from $-\infty$ to $+\infty$, how small soever p may be taken, else A—A′V, B—B′V and to S—S′V would all be zero; x must therefore have one determinate value. That value may be only symbolical, because n and p in the preceding demonstration may be mere symbols; but the value of x, whether real or symbolical, must be fixed and determinate. Our theorem therefore seems to furnish a simple and elementary proof of the important proposition, that in every equation of the ordinary form the unknown quantity has at least one value.—Even if the coefficients be imaginary the demonstration is equally valid.

Cor. 3. The foregoing proof may be extended to the following.

Let

$$\frac{Ax^n + Bx^{n-1} + etc.\ A'y^n + B'y^{n-1} + etc.\ A''z^n + B''z^{n-1} + etc.\ S}{ax^n + bx^{n-1} + etc.\ a'y^n + b'y^{n-1} + etc.\ a''z^n + b''z^{n-1} + etc.\ S'} = V;$$

where V is constant with respect to x, y, and z,—where the coefficients are also constant and not indefinitely great,—and where x, y, and z are independent and arbitrary through any finite extent, however small: then $\frac{A}{a}$, *etc.*; $\frac{A'}{a'}$, *etc.*; $\frac{A''}{a''}$ *etc.* $= V$.

This case of the theorem contains the whole doctrine of relative maxima and minima. It may be readily applied to prove the fundamental proposition in the doctrine of rotatory motion as follows.

Let m be a point in the revolving body referred to the rectangular axes Ox, Oy, and Oz, the coordinates of the point m being x, y, and z respectively. Let v_1 be the angular velocity of the point m round Ox, v_2 the angular velocity round Oy, and v_3 that round Oz; and if these three motions are equivalent to a single motion round any other axis AB, then mB, being perpendicular to AB, the condition of these compounded motions is this, viz.

$$\text{angular velocity round AB} = \frac{\text{absolute velocity of } m}{Bm}.$$

Let m, n, p be the coordinates of the point A, and a, b, c the direction cosines of AB to Ox, Oy, and Oz respectively: then as the coordinates of the point B are $AB.a+m$; $AB.b+n$; and $AB.c+p$; and those of m are x, y, and z, we have

$$(Bm)^2=(x-m-AB.a)^2+(y-n-AB.b)^2+(z-p-AB.c)^2$$
$$=(x-m)^2+(y-n)^2+(z-p)^2-AB^2,$$

from which $AB = a(x-m) + b(y-n) + c(z-p)$,

and hence $(Bm)^2=(1-a^2)(x-m)^2+(1-b^2)(y-n)^2+(1-c^2)(z-p)^2$
$-2ab(x-m)(y-n)-2ac(x-m)(z-p)-2bc(y-n)(z-p).$

Again, by resolving the tangential velocities of m round Ox, Oy and Oz in the ordinary way, we have the whole velocity of m

$$\begin{aligned} &\text{parallel to } Ox = zv_2 - yv_3 \\ &\ldots\ldots\ldots\ Oy = xv_3 - zv_1 \\ &\ldots\ldots\ldots\ Oz = yv_1 - xv_2. \end{aligned}$$

and hence the absolute velocity of m will be

$$\{(zv_2 - yv_3)^2 + (xv_3 - zv_1)^2 + (yv_1 - xv_2)^2\}^{\frac{1}{2}}$$
$$=(v_2^2+v_3^2)x^2+(v_1^2+v_3^2)y^2+(v_1^2+v_2^2)z^2-2xyv_1v_2-2xzv_1v_3-2yzv_2v_3.$$

Now if v be the angular velocity of rotation of m round the axis AB, the above expressions give

$$\frac{(v_2^2+v_3^2)x^2+(v_1^2+v_2^2)y^2+(v_1^2+v_2^2)z^2-2xyv_1v_2-2xzv_1v_3-2yzv_2v_3}{(1-a^2)x^2+(1-b^2)y^2+(1-c^2)z^2+ \textit{etc., etc.}} = v^2;$$

from which, by applying our theorem, we have

$$\frac{v_2^2+v_3^2}{1-a^2} = \frac{v_1^2+v_3^2}{1-b^2} = \frac{v_1^2+v_2^2}{1-c^2} = v^2, \text{ and hence } v_1^2+v_2^2+v_3^2 = v^2.$$

The same equations give $a = \dfrac{v_1}{v}$; $b = \dfrac{v_2}{v}$; and $c = \dfrac{v_3}{v}$.

The coefficients of x, y and z in the numerator and denominator of the value of v^2 being made equal to one another and each equal to v^2, give

$$\frac{2yv_1v_2+2zv_1v_3}{2m(1-a^2)+2ab(y-n)+2ac(z-p)} = \frac{2xv_1v_2+2zv_2v_3}{2n(1-b^2)+2ab(x-m)+2bc(z-p)}$$
$$= \frac{2xv_1v_3+2yv_2v_3}{2p(1-c^2)+2ac(x-m)+2bc(y-n)} = v^2;$$

and if in these we substitute for v_1, v_2 and v_3 the foregoing values, viz. av, bv and cv, we get $\dfrac{a}{b} = \dfrac{m}{n}$ and $\dfrac{b}{c} = \dfrac{n}{p}$.

These same values might have been found from that part of the denominator of v^2 which wants x, y and z, viz. $(b^2+c^2)m^2+(a^2+c^2)n^2+(a^2+b^2)p^2 -2abmn-2acmp-2bcnp=(an-bm)^2+(np-cm)^2+(bp-cn)^2$, which part must be equated to zero. Hence $\dfrac{a}{b} = \dfrac{m}{n}$; $\dfrac{a}{c} = \dfrac{m}{p}$; and $\dfrac{b}{c} = \dfrac{n}{p}$ as before.

These equations shew that the axis round which the resultant rotation takes place passes through the origin of coordinates.

PROPOSITIONS CONCERNING THE MOTION OF A SYSTEM OF PARTICLES,

SHEWING THAT D'ALEMBERT'S PRINCIPLE IS NOT NECESSARY TO THE THEORY OF DYNAMICS.

[*The Rev. Alexander Q. G. Craufurd, M.A., of Jesus College, Cambridge.*]

Let m, m', m'', *etc.* be a collection of material particles, any how connected with one another; let P, P′, P″, *etc.* denote the forces applied to these particles, and $a, b, c,\ a', b', c',\ a'', b'', c''$, *etc.* the angles which the directions of P, P′, P″, *etc.* make with the rectangular axes of x, y, and z respectively: let (m, m'), (m, m''), (m', m''), *etc.* denote the distances of m from m', m from m'', m' from m'', *etc.*: let $[m, m']$, $[m, m'']$, $[m' m'']$,

etc. denote the forces exerted by m on m', by m on m'', by m' on m'', *etc.*: and lastly, let $x, y, z, x', y', z', x'', y'', z''$, *etc.* be the coordinates of m, m', m'', *etc.*

By the third law of motion we are entitled to assume, that m' will exert on m a force equal and contrary to $[m, m']$, that m'' will exert on m a force equal and contrary to $[m', m'']$, that m'' will exert on m' a force equal and contrary to $[m', m'']$, *etc.*; and that the forces which each particle exerts on any other is in the direction of the line joining those particles.

By the principles of analytical geometry, we have for the cosines of the angles which the line (m, m') makes with the axes of x, y, and z, respectively, the expressions

$$\frac{x - x'}{(m, m')}, \quad \frac{y - y'}{(m, m')}, \quad \frac{z - z_{,}}{(m, m')};$$

and for the other lines which join particles we have similar expressions.

From this notation and these principles it is easy to form the equations of of motion of each of the particles m, m', m'', *etc.*

The equations of motion of m will be

$$m\frac{d^2x}{dt^2} = \mathrm{P}\cos a - [m, m']\frac{x - x'}{(m, m')} - [m, m'']\frac{x - x''}{(m, m'')}, \text{ etc.}$$

$$m\frac{d^2y}{dt^2} = \mathrm{P}\cos b - [m, m']\frac{y - y'}{(m, m')} - [m, m'']\frac{y - y''}{(m, m'')}, \text{ etc.}$$

$$m\frac{d^2z}{dt^2} = \mathrm{P}\cos c - [m, m']\frac{z - z'}{(m, m')} - [m, m'']\frac{z - z''}{(m, m'')}, \text{ etc.}$$

The equations of motion of m' will be

$$m'\frac{d^2x'}{dt^2} = \mathrm{P}'\cos a' + [m, m']\frac{x - x'}{(m, m')} - [m', m'']\frac{x' - x''}{(m', m'')}, \text{ etc.}$$

$$m'\frac{d^2y'}{dt^2} = \mathrm{P}'\cos b' + [m, m']\frac{y - y'}{(m, m')} - [m', m'']\frac{y' - y''}{(m', m'')}, \text{ etc.}$$

$$m'\frac{d^2z'}{dt^2} = \mathrm{P}'\cos c' + [m, m']\frac{z - z'}{(m, m')} - [m', m'')\frac{z' - z''}{(m', m'')}, \text{ etc.}$$

The equations of motion of m'' will be

$$m''\frac{d^2x''}{dt^2} = \mathrm{P}''\cos a'' + [m, m'']\frac{x - x''}{(m, m'')} + [m', m'']\frac{x' - x''}{(m', m'')}, \text{ etc.}$$

$$m''\frac{d^2y''}{dt^2} = \mathrm{P}''\cos b'' + [m, m'']\frac{y - y'''}{(m, m'')} + [m', m'']\frac{y' - y''}{(m', m'')}, \text{ etc.}$$

$$m''\frac{d^2z''}{dt^2} = \mathrm{P}''\cos c'' + [m, m'']\frac{z - z''}{(m, m'')} + [m', m'']\frac{z' - z''}{(m', m'')}, \text{ etc.}$$

If we now add together the first equations of each set, all the terms depending on the mutual action disappear, and the resulting equation is simply

$$\Sigma\, m\frac{d^2x}{dt^2} = \Sigma\, \mathrm{P}\cos a.$$

In like manner by adding the second and third equations of each set, we obtain the results

$$\Sigma\, m \frac{d^2y}{dt^2} = \Sigma\, \mathrm{P} \cos b.$$

$$\Sigma\, m \frac{d^2z}{dt^2} = \Sigma\, \mathrm{P} \cos c.$$

In order to form three more equations which are independent of the terms resulting from the mutual action of the particles, multiply the first of the first set by y and the second by x, and subtract the former from the latter; do the same with each set, and add all the results; it will easily be seen that the equation thus formed is simply

$$\Sigma\, m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right) = \Sigma\, \mathrm{P}(x \cos b - y \cos a.)$$

And by combining, in like manner, the first and third, and the second and third equations of each set, and adding the results, we form the two similar equations

$$\Sigma\, m \left(x \frac{d^2z}{dt^2} - z \frac{d^2x}{dt^2} \right) = \Sigma\, \mathrm{P}(x \cos c - z \cos a),$$

$$\Sigma\, m \left(y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2} \right) = \Sigma\, \mathrm{P}(y \cos c - z \cos b).$$

The three equations of fluid motion which are ordinarily derived from D'Alembert's principle, may be proved without it in the following manner.

In every mass to the particles of which forces are applied, each particle will be subject to a force in consequence of its connection with the other particles, besides the force which is applied to it immediately. Hence arises pressure, or tension, to which every element of mass is subject. Let af be a rectangular element of a mass, the side $acgh$ being perpendicular to the axis of x. Let p be the pressure referred to unity of surface, perpendicular to this side; then $pdydz$ will represent the pressure on the side, and $\left(p + \frac{dp}{dx}\, dx\right) dy\, dz$ will be the pressure upon the side $bdef$ opposite to it. Let X represent the resolved part (parallel to the axis of x) of all the accelerative forces applied to the element by forces external to the mass; and let ρ denote the density of the element. Then $\mathrm{X}\rho dxdydz$ will represent the corresponding moving force. Therefore $\mathrm{X}\rho dxdydz + pdxdydz - \left(p + \frac{dp}{dx}\, dx\right) dydz$, or $\mathrm{X}\rho dxdydz - \frac{dp}{dx}\, dxdydz$, represents the resolved part, parallel to the axis of x, of all the moving forces applied to the element. Hence, by the general equations of movement, we have

$$\rho dxdydz \left(\frac{du}{dt} \right) = \mathrm{X}\rho dxdydz - \frac{dp}{dx}\, dxdydz,$$

or more generally $$\left(\frac{du}{dt} \right) dm = \mathrm{X}\, dm - d.\mathrm{R} \cos a.$$

Where u represents the velocity of the element parallel to the axis of x, and $\left(\frac{du}{dt} \right)$ the total differential coefficient of u with regard to t. Therefore $\left(\frac{du}{dt} \right)$ is equivalent to

$$\frac{du}{dt} + \frac{du}{dx} \frac{dx}{dt} + \frac{du}{dy} \frac{dy}{dt} + \frac{du}{dz} \frac{dz}{dt}.$$

Let v and w denote the velocities of the element parallel to the axes of y and z, then $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$, and our equation becomes, by substituting, dividing by $\rho dxdydz$, and transposing,

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}.$$

This equation applies to any mass whatever, p denotes the pressure perpendicular to the plane of yz referred to unity of surface at the point of which the coordinates are x, y, z. If p' and p'' have similar significations with reference to the planes xz and yx, we shall have the equations

$$\frac{1}{\rho}\frac{dp'}{dy} = Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz},$$

and

$$\frac{1}{\rho}\frac{dp''}{dz} = Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}.$$

In a fluid p, p', and p'' are equal.

In any mass, in a state of equilibrium, take an element of any shape whatever. Let R be the resultant of all the forces applied to this element in consequence of its connection with the other particles of which the mass is composed; a the angle which the direction of this resultant makes with a line drawn through the element in any *given* direction; X the resolved part, parallel to this line, of all the forces applied to the element; dm the mass of the element.

Then $$d.\mathrm{R}\cos a = \mathrm{X}dm.$$

If the element be a rectangular parallelopiped of which the faces are parallel to three rectangular axes; if p denote the pressure referred to unity of surface in the direction of the axis of x; and if a be referred to this axis, then

$$\mathrm{R}\cos a = pdydz, \text{ and } d.\mathrm{R}\cos a = \frac{dp}{dx}dxdydz.$$

Also, $dm = \rho dxdydz$; therefore, by substitution, $\frac{1}{\rho}\frac{dp}{dx} = \mathrm{X}$.

And if p' and p'' be the corresponding pressures for the axes of y and z,

$$\frac{1}{\rho}\frac{dp'}{dy} = \mathrm{Y}, \quad \frac{1}{\rho}\frac{dp''}{dz} = \mathrm{Z};$$

or, $$d.\mathrm{R}\cos a = \mathrm{X}\rho dxdydz, \quad d.\mathrm{R}\cos\beta = \mathrm{Y}\rho dxdydz,$$

and $$d.\mathrm{R}\cos\gamma = \mathrm{Z}\rho dxdydz;$$

in which a, β, γ are connected by the equation

$$\cos^2 a + \cos^2\beta + \cos^2\gamma = 1.$$

In the case of a flexible string in which the tension is in the direction of the length, let w denote the area of a section of the string made by a plane perpendicular to its length, ds the length of the element, T the tension; then $\rho w ds$ is the mass of the element on dm, and if our equation be applied to each of the three axes of coordinates in succession, it gives

$$d.\mathrm{T}\frac{dx}{ds} = \mathrm{X}\rho w ds, \quad d.\mathrm{T}\frac{dy}{ds} = \mathrm{Y}\rho w ds, \quad d.\mathrm{T}\frac{dz}{ds} = \mathrm{Z}\rho w ds.$$

From which it is easy to deduce the well known equations of the catenary.

ON THE TRANSFORMATION OF LINEAR DIFFERENTIAL EQUATIONS.

[*The Rev. Brice Bronwin.*]

The following general theorem, which may be proved by performing the differentiations indicated in the second member, will be useful.

$$XD^n y = D^n Xy - nD^{n-1}X^{(1)}y + \frac{n(n-1)}{2}D^{n-2}X^{(2)}y - \ldots\ldots(a),$$

where D stands for $\frac{d}{dx}$, and $X^{(1)}$, $X^{(2)}$, *etc.*, are the first, second, *etc.* differential coefficients of X relative to x. We may change the sign of n, then

$$XD^{-n}y = D^{-n}Xy + nD^{-n-1}X^{(1)}y + \frac{n(n+1)}{2}D^{-n-2}X^{(2)}y + \ldots(b),$$

which may be verified by continued differentiation.

Let $U = Xy + X_1 Dy + X_2 D^2 y + \ldots + X_m D^m y$; and let this be transformed into

$$U = Vy + DV_1 y + D^2 V_2 y + \ldots + D^m V_m y \ldots\ldots(1).$$

By (a) we find

$$\begin{aligned} V &= X - X_1^{(1)} + X_2^{(2)} - \ldots \\ V_1 &= X_1 - 2X_2^{(1)} + 3X_3^{(2)} - \ldots \\ V_2 &= X_2 - 3X_3^{(1)} + 6X_4^{(2)} - \ldots \\ V_3 &= X_3 - 4X_4^{(1)} + 10X_5^{(2)} - \ldots \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

Now make $y = vz + Dv_1 z + D^2 v_2 z + \ldots + D^p v_p z$; and let (1) by this be transformed into

$$U = Wz + DW_1 z + D^2 W_2 z + \ldots + D^{m+p} W_{m+p} z \ldots\ldots\ldots(2).$$

The reductions being made by (a), we have

$$W = vV - v_1 V^{(1)} + v_2 V^{(2)} - \ldots; \quad W_1 = \begin{cases} v_1 V - 2v_2 V^{(1)} + 3v_3 V^{(2)} - \ldots \\ v V_1 - v_1 V_1^{(1)} + v_2 V_1^{(2)} - \ldots \end{cases}$$

$$W_2 = \left\{ \begin{array}{l} v_2 V - 3v_3 V^{(1)} + 6v_4 V^{(2)} - \ldots \\ v_1 V_1 - 2v_2 V_1^{(1)} + 3v_3 V_1^{(2)} - \ldots \\ v V_2 - v_1 V_2^{(1)} + v_2 V_2^{(2)} - \ldots \end{array} \right\}; \quad W_3 = \begin{cases} v_3 V - 4v_4 V^{(1)} + 10v_5 V^{(2)} - \ldots \\ v_2 V_1 - 3v_3 V_1^{(1)} + 6v_4 V_1^{(2)} - \ldots \\ v_1 V_2 - 2v_2 V_2^{(1)} + 3v_3 V_2^{(2)} - \ldots \\ v V_3 - v_1 V_3^{(1)} + v_2 V_3^{(2)} - \ldots \end{cases}$$

etc. *etc.*

We now resolve the second member of (2) into

$$(P + DP_1 + D^2 P_2 + \ldots + D^p P_p)(Qz + DQ_1 z + D^2 Q_2 z + \ldots + D^m Q_m z).$$

The new values of W, W_1, *etc.* will obviously be obtained by putting P, P_1, *etc.* for V, V_1, *etc.*, and Q, Q_1, *etc.* for v, v_1, *etc.* in the former values. Therefore

$$W = PQ - P^{(1)}Q_1 + P^{(2)}Q_2 - \ldots;$$

$$W_1 = \left\{ \begin{array}{l} PQ_1 - 2P^{(1)}Q_2 + 3P^{(2)}Q_3 - \ldots \\ P_1 Q - P_1^{(1)}Q_1 + P^{(2)}Q_2 - \ldots \end{array} \right\}; \quad W_2 = \begin{cases} PQ_2 - 3P^{(1)}Q_3 + 6P^{(2)}Q_4 - \ldots \\ P_1 Q_1 - 2P_1^{(1)}Q_2 + 3P_1^{(2)}Q_3 - \ldots \\ P_2 Q - P_2^{(1)}Q_1 + P_2^{(2)}Q_2 - \ldots \end{cases}$$

etc. *etc.*

These values of W, W_1, *etc.* being made equal to the former give equations for determining Q, Q_1, *etc.* in terms of V, V_1, *etc.*, and the contrary. And if we make $(P+DP_1+D^2P_2+..+D^pP_p)^{-1}U=U_1$, we have

$$U_1=Qz+DQ_1z+D^2Q_2z+..+D^mQ_mz \ldots\ldots\ldots (3),$$

and we may either transform (1) into (3), or (3) into (1).

But as we cannot operate with the factor $P+DP_1+..+D^pP_p$, when it contains more terms than two, except for particular values of P, P_1, *etc.*; and as the binomial form is the most convenient, we will make $y=vz+Dv_1z=(Dv_1+v)z$, and therefore $z=(Dv_1+v)^{-1}y$, and also $(DP_1+P)^{-1}U=U_1$, $U=(DP_1+P)U_1$. Then we shall have

$$vV-v_1V^{(1)}=\ PQ-\ P^{(1)}Q_1+\ \ P^{(2)}Q_2-..$$

$$v_1V+vV_1-v_1V_1^{(1)}=\begin{cases}PQ_1-2P^{(1)}Q_2+\ 3P^{(2)}Q_3-..\\ P_1Q-\ P_1^{(1)}Q_1+\ \ P_1^{(2)}Q_2-..\end{cases}$$

$$v_1V_1+vV_2-v_1V_2^{(1)}=\begin{cases}PQ_2-3P^{(1)}Q_3+\ 6P^{(2)}Q_4-..\\ P_1Q_1-2P_1^{(1)}Q_2+\ 3P_1^{(2)}Q_3-..\end{cases}$$

$$v_1V_2+vV_3-v_1V_3^{(1)}=\begin{cases}PQ_3-4P^{(1)}Q_4+10P^{(2)}Q_5-..\\ P_1Q_2-3P_1^{(1)}Q_3+\ 6P_1^{(1)}Q_4-..\end{cases}$$

etc., where the law of continuation is obvious ....(4).

We shall have one more equation than we have quantities to determine, and therefore an equation of relation between the given coefficients and the four arbitrary quantities v, v_1, P, P_1. These quantities must therefore be so determined as to satisfy that equation. Or if this cannot be done, we shall have an equation of relation between the given coefficients in order that the tranformation may be possible. The solutions of these equations in the general case is very prolix; but by way of example we will take

$$U=Vy+DV_1y+D^2y,\quad U_1=Qz+DQ_1z+D^2z;$$

where $V_2=Q_2=1$, and we will suppose $v_1=P_1=1$; then we find

$$Q_1=V_1+v-P,\quad Q=V+vV_1-V_1^{(1)}-PV_1-Pv+P^2+2P^{(1)};$$

and the equation of condition may be put under the form

$$v=P+\frac{V^{(1)}-PV_1^{(1)}-P^{(1)}V_1+2PP^{(1)}+P^{(2)}}{V-PV_1+P^2+P^{(1)}}=P+D\log(V+P^{(1)}-PV_1+P^2),$$

a result very remarkable.

If we make $v=P$, we must have $V+P^{(1)}-PV_1+P^2=$ a constant. We cannot integrate this so as to determine P; but by giving a particular value to P, we have a relation between V and V_1, which will render the given equation susceptible of this transformation.

We have also $Q=V-V_1^{(1)}+2P^{(1)}+(V_1-P)D\log(V+P^{(1)}-PV_1+P^2)$.

Suppose $Q=0$, the transformed equation is then integrable. If $P=V_1$, $V-V_1^{(1)}+2P^{(1)}=0$, or $V+V_1^{(1)}=0$, then $Q=0$. Or if $V-V_1^{(1)}+2P^{(1)}=0$, and $V+P^{(1)}-PV_1+P^2=$ a constant, then $Q=0$; and by eliminating P between these, we shall have two equations of relation between V and V_1.

Again we find

$$V_1=Q_1+P-v,\ V=Q+PQ_1+Q_1^{(1)}-vQ_1-vP+v^2-P^{(1)}-v^{(1)}.$$

This value of V substituted in the first of (4) will give an equation of relation between Q and Q_1; but it will be very complicated. However, by giving particular values to v and P, it would give a relation between Q and Q_1 which would render the equation in z susceptible of the transformation. And moreover by determining Q so as to make $V=0$, the transformed would be integrable. And it may be observed of the general equations (1), (2), (3), that if $V=0$, or $Q=0$, or $W=0$, $W_1=0$, one integration may be effected.

But it is when V, V_1, *etc.* Q, Q_1, *etc.* are integer functions of x, that this method is chiefly applicable; when it may happen that the given equation is susceptible of a series of similar transformations, till it becomes ultimately integrable. And here v, v_1, P, P_1 must also be integer functions of x. Suppose then $v=P=n$, $v_1=P_1=x$. The equations (4) become

$$nV-xV^{(1)}=nQ,$$
$$xV+nV_1-xV_1^{(1)}=nQ_1+xQ-Q_1,$$
$$xV_1+nV_2-xV_2^{(1)}=nQ_2+xQ_1-2Q_2,$$
$$xV_2+nV_3-xV_3^{(1)}=nQ_3+xQ_2-3Q_3, \quad etc.$$

Eliminating all the Q's, we shall have the equation of condition

$$f(x, v, V_1 \ldots .)=0.$$

And supposing $V=a+bx+cx^2+\ldots .$, $V_1=a_1+b_1x+c_1x^2+\ldots . etc.$; substituting these values, and making the coefficients of the several powers of x zero; we shall find the required relation between the constants that the transformation may be possible. Then finding the values of Q, Q_1, *etc.*, we shall easily give to the constants such values that a second transformation will be possible, and so many that the equation will ultimately become integrable once.

If we wish to transform the other way, we must eliminate the V's, and find $\phi(x, Q, Q_1 \ldots \ldots)=0$; with which we must proceed in the same way as before.

It may happen that we may find more than one set of constants that will answer. We shall certainly find all the equations that can be transformed by making $y=(Dx+n)z$, and $z=(Dx+n)^{-1}y$. And we may make other assumptions for v, v_1, P, P_1; and may find all the transformable equations corresponding in the same manner.

Put (1) under the form $U=Vy+D\phi V_1 y+D^2\phi^2 V^2 y+ \ldots \ldots \ldots \ldots (5)$, where ϕ is a function of x; make $\rho=\phi D$; then reducing when necessary by (a), we have

$$D\phi V_1 y=\phi D V_1 y+\phi^{(1)} V_1 y=(\rho+\phi^{(1)}) V_1 y.$$

Change $V_1 y$ into $(\rho+2\phi^{(1)})V_2 y$; then as

$$D\phi(\rho+2\phi^{(1)})V_2 y=D\phi^2 D V_2 y+2D\phi\phi^{(1)}V_2 y=D^2\phi^2 V_2 y, \text{ we have}$$
$$D^2\phi^2 V_2 y=(\rho+\phi^{(1)})(\rho+2\phi^{(1)}) V_2 y.$$

Change now $V_2 y$ into $(\rho+3\phi^{(1)})V_3 y$, and we find in the same manner

$$D^3\phi^3 V_3 y=(\rho+\phi^{(1)})(\rho+2\phi^{(1)})(\rho+3\phi^{(1)}) V_3 y, \; etc.$$

Therefore (5) is changed into

$$U = Vy + (\rho+\phi^{(1)})V_1 y + (\rho+\phi^{(1)})(\rho+2\phi^{(1)})V_2 y + (\rho+\phi^{(1)})(\rho+2\phi^{(1)})(\rho+3\phi^{(1)})V_3 y + \dots\dots \quad (6),$$

which may be put under the form

$$U = Ty + \rho T_1 y + \rho^2 T_2 y + \dots\dots \quad (7),$$

by making $T = V + \phi^{(1)}V_1 + 2\phi^{(1)2}V_2 + 6\phi^{(1)3}V_3 + \dots;$

$T_1 = V_1 + 3\phi^{(1)}V_2 + 11\phi^{(1)2}V_3 + \dots$; $T_2 = V_2 + 6\phi^{(1)}V_3 + \dots$; $T_3 = V_3 + \dots$ *etc.*

If we can give to $\phi^{(1)}$ such a value that $T=0$, we shall have

$$U = \rho T_1 y + \rho^2 T_2 y + \dots\dots \text{ and } \rho^{-1}U = T_1 y + \rho T_2 y + \dots\dots,$$

and the equation will be once integrated.

Again make $\pi = \phi D + \lambda = \rho + \lambda$, $\pi_1 = \phi D + \lambda_1 = \rho + \lambda_1$, $\pi_2 = \rho + \lambda_2$, *etc.*; $\rho = \pi - \lambda = \pi_1 - \lambda_1 = \pi_2 - \lambda_2$ *etc.*; and we can transform (7) into

$$U = Sy + \pi S_1 y + \pi\pi_1 S_2 y + \pi\pi_1\pi_2 S_3 y + \dots\dots \quad (8),$$

where S, S_1, *etc.* are functions of T, T_1, *etc.*, and λ, λ_1, *etc.*; and there is no difficulty in eliminating ρ and finding their values, but it is rather a long and troublesome operation. Now if $S=0$ in (8), or if we can give to λ, λ_1, *etc.* and ϕ such values as make $S=0$, we have $U = \pi S_1 y + \pi\pi_1 S_2 y + \dots$; and operating with π^{-1} on both members, making $\pi^{-1}U = U_1$, we have

$$U_1 = S_1 y + \pi_1 S_2 y + \dots\dots,$$

and the equation is once integrated.

Other transformations might be employed, and each increases the chances of integration. Also each transformation will enable us to determine a relation between the coefficients, which will give an integrable case. Instead of the first transformation, we might employ the following.

Let $U = Xy + X_1 Dy + X_2 D^2 y$, and make $y = (\lambda D + \phi)z$.

By substitution the given equation becomes

$$U = (\phi X + \phi^{(1)}X_1 + \phi^{(2)}X_2)z + \big(\lambda X + (\phi+\lambda^{(1)})X_1 + (2\phi^{(1)}+\lambda^{(2)})X_2\big)Dz + \big(\lambda X_1 + (\phi+2\lambda^{(1)})X_2\big)D^2 z + \lambda X_2 D^3 z.$$

Make $U = (\beta D + \theta)(Az + A_1 Dz + A_2 D^2 z) = (\theta A + \beta A^{(1)})z$
$+ (\theta A_1 + \beta A + \beta A_1^{(1)})Dz + (\theta A_2 + \beta A_1 + \beta A_2^{(1)})D^2 z + \beta A_2 D^3 z.$

Comparing like terms we have

$$\theta A + \beta A^{(1)} = \phi X + \phi^{(1)}X_1 + \phi^{(2)}X_2,$$
$$\theta A_1 + \beta A + \beta A_1^{(1)} = \lambda X + (\phi+\lambda^{(1)})X_1 + (2\phi^{(1)}+\lambda^{(2)})X_2,$$
$$\theta A_2 + \beta A_1 + \beta A_2^{(1)} = \lambda X_1 + (\phi + 2\lambda^{(1)})X_2,$$
$$\beta A_2 = \lambda X_2.$$

These results are perhaps not quite so simple as those obtained by the first method. That has also some other advantages. But we might employ methods which would give the new coefficients Q, Q_1, *etc.* at once, without any elimination.

Gunthwaite Hall, August 6th, 1847.

NOTE ON CERTAIN IMAGINARY EXPONENTIAL EXPRESSIONS.

[*Mr. Andrew Roy, Dundee Academy.*]

In the report of the third meeting of the British Association, note page 347, Professor PEACOCK, quoting from page 287 of CRELLE's Journal, says, " Since $e^{2n\pi\sqrt{-1}}=1$, when n is a whole number, we get $e^{1+2n\pi\sqrt{-1}}=e$, and therefore $e^{(1+2n\pi\sqrt{-1})^2}=e^{1+2n\pi\sqrt{-1}}=e$ $=e^{1+4n\pi\sqrt{-1}-4n^2\pi^2}$, and consequently, $e^{-4n^2\pi^2}=1$; a conclusion which M. CLAUSEN, the writer in CRELLE's Journal, characterizes as absurd; but which, Dr. PEACOCK says, involves no other difficulty than that which is included in the equation $e^{2n\pi\sqrt{-1}}=1$. The explanation, he adds, must be sought for in the circumstances which accompany the transition from a function to its equivalent series when a strict arithmetical equality does not exist between them. But it must be confessed, that these difficulties are of a very serious nature, and are in every way deserving of a more careful examination and analysis than they have hitherto received."

It appears from the preceding statement, either that Professor PEACOCK does not perceive the source of the difficulty in question, or that the one to which he alludes is something different from the difficulty which presents itself to me. It is certainly, therefore, not without some misgiving that the following explanation is offered where such authority is at fault.

Taking the equation $e^{2n\pi\sqrt{-1}}=1$, we have by multiplying both sides by e, $e^{1+2n\pi\sqrt{-1}}=e\times 1$; and therefore $e^{(1+2n\pi\sqrt{-1})^2}=e^{1+2n\pi\sqrt{-1}}$ $\times 1^{1+2n\pi\sqrt{-1}}$. Now this is the expression which M. CLAUSEN, and after him Professor PEACOCK, puts equal to $e^{1+2n\pi\sqrt{-1}}$ alone, without the multiplier $1^{1+2n\pi\sqrt{-1}}$, which they thus confound with or make equal to unity, going upon the assumption that $1^{1+2n\pi\sqrt{-1}}=1$. If this last equation were true, we should also have $1^{2n\pi\sqrt{-1}}=1$, because $1^{1+2n\pi\sqrt{-1}}=1^1\times 1^{2n\pi\sqrt{-1}}$. But since $e^{2n\pi\sqrt{-1}}=1$, we shall also have $e^{(2n\pi\sqrt{-1})(2n\pi\sqrt{-1})}=1^{2n\pi\sqrt{-1}}$, or $e^{-4n^2\pi^2}=1^{2n\pi\sqrt{-1}}=1$; the same result as that arrived at by M. CLAUSEN.

It would hence appear that the difficulty involved in this conclusion is contained in the equation $1^{1+2n\pi\sqrt{-1}}=1$ or $1^{2n\pi\sqrt{-1}}=1$, and not in the equation $e^{2n\pi\sqrt{-1}}=1$; or that the conclusion is erroneous, and that the error arises from a want of attention to the distinction between e and $\pm 1\times e$, or to the difference between unity and the powers of unity when the indices of these powers are not whole numbers; and no example could shew more

clearly that in symbolical equations which have to be operated upon by multiplication, division, *etc.*, we should never neglect or lay aside any quantity or symbol unless we are certain that the suppression does not assume some unwarranted equality, or take something for granted which we are not authorized to decide upon. This is a very likely source of error in the symbolical representation of zero as well as of infinity, as $\frac{1}{n}$, $\frac{1}{n^2}$, *etc.*, where n is indefinitely great; or $\frac{1}{a}$, $\frac{1}{a^2}$, where a is indefinitely small. The importance of the distinction between e and $\pm 1 \times e$ receives a beautiful illustration in the difference between common logarithms and the more general logarithms pointed out by EULER, and the still more general logarithms subsequently investigated by Mr. GRAVES. Thus in the equation $a^x = \text{N}$, x is the common log. of N to the base a; but if $(\pm 1 \times a)^y = \pm 1 \times \text{N}$, then y is the general log. of N, which may be exhibited as follows. For $\pm 1 \times a$ put $e^{n\pi\sqrt{-1}} \times a$, and for $\pm 1 \times \text{N}$ put $e^{n'\pi\sqrt{-1}} \times \text{N}$ and taking the logs. in both sides to the base e, we have

$$y\{\log a \times n\pi\sqrt{-1}\} = \log \text{N} \times n'\pi\sqrt{-1},$$

$$\text{or} \qquad y = \frac{\log \text{N} \times n'\pi\sqrt{-1}}{\log a \times n\pi\sqrt{-1}}.$$

This is Mr. GRAVE's general expression for the log. of N. If $a = e$, and if the ± 1 before e be neglected, then Mr. GRAVE's logarithms become the same as EULER's.

Dundee, August 2nd, 1847.

PROPERTIES OF THE HYPERBOLA AND PARABOLA.

[*Mr. Thomas Weddle, Newcastle-upon-Tyne.*]

Division of an hyperbolic sector into equal parts.—Maximum polygon inscribed in an hyperbolic or a parabolic segment.—Minimum polygon circumscribed about the same.

In a previous communication * I discussed with respect to the ellipse the topics which are now to be considered with respect to the other curves of the second order. In that paper, by putting the equation to the ellipse under the form, $x = a\cos\phi$, $y = b\sin\phi$, I was enabled to reduce the problems with respect to the ellipse to the same problems with respect to the circle. By assuming an analogous form for the equation of the hyperbola, namely, $x = a.\cos\phi\sqrt{-1}$, $y = b\sqrt{-1}.\sin\phi\sqrt{-1}$, that is, $x = \frac{a}{2}\left\{e^{\phi} + e^{-\phi}\right\}$, $y = \frac{b}{2}\left\{e^{\phi} - e^{-\phi}\right\}$, the problems for the hyperbola might be reduced to those for the *equilateral* hyperbola, but there would be no advantage in doing so; for it would remain to discuss the problems with respect to the latter curve, and thus we should find ourselves not a single step nearer our object than at first. (This disadvantage does not occur in the case of the ellipse, seeing that the properties of the circle are so well known.) It hence appears that no method analogous to that pursued in the paper referred to

* Mathematician, No. 4., Vol. II.

can be applied to the hyperbola, and the discussion must therefore be based upon entirely different considerations. The same is obviously true of the parabola; also since this curve is without a centre, there is no such figure as a parabolic sector, and hence one of the problems considered with respect to the ellipse and hyperbola cannot be proposed with respect to the parabola. Before going further I would observe however, that the discussion in reference to the hyperbola and parabola is not quite so interesting as that in reference to the ellipse, for the latter being *closed*, we can consider the whole curve, whereas in the others we are confined to a sector or segment merely. Nevertheless I trust this paper will not be found entirely unworthy of perusal; for a parabolic segment especially the constructions are of the simplest character, and have the advantage of being always *geometrical.*

I proceed, after these brief remarks, with the discussion in reference to the hyperbola.

Let OC and OD be the asymptotes to the hyperbola, which are inclined to each other at an angle θ. Taking OC and OD for axes, the equation to the hyperbola will be

$$xy = \gamma^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\gamma = \frac{1}{4}$ distance between the foci.

Throughout this paper the point P_m in the hyperbola will be denoted by (x_m, y_m); also if $P_m Q_m (= y_m)$ be drawn from P_m parallel to OD it will meet OC in Q_m;

$$\therefore \Delta P_1OP_2 = \tfrac{1}{2}(x_2y_1 - x_1y_2)\sin\theta \ldots\ldots\ldots\ldots (2),$$

and trapezoid

$$P_1Q_1Q_2P_2 = \tfrac{1}{2}(y_1+y_2)(x_2-x_1)\sin\theta = \tfrac{1}{2}(x_2y_1 - x_1y_2 + x_2y_2 - x_1y_1)\sin\theta;$$

$$\therefore, (1), \text{ trapezoid } P_1Q_1Q_2P_2 = \tfrac{1}{2}(x_2y_1 - x_1y_2)\sin\theta \ldots\ldots\ldots (3).$$

Hence also (2, 3) $\quad \Delta P_1OP_2 = \text{trapezoid } P_1Q_1Q_2P_2 \ldots\ldots\ldots (4).$

Deduct the area of the segment on P_1P_2 from each side of (4),

$$\therefore \quad \text{sector } P_1OP_2 = \text{mixtilineal quad. } P_1Q_1Q_2P_2 \ldots\ldots (5).$$

Now it is easily shown (see Hymers's Conic Sections) that

$$\text{mixt. quad. } P_1Q_1Q_2P_2 = \gamma^2 \sin\theta . \log_e \frac{x_2}{x_1} \ldots\ldots\ldots (6);$$

$$\therefore \quad \text{sector } P_1OP_2 = \gamma^2 \sin\theta . \log_e \frac{x_2}{x_1} \ldots\ldots\ldots (7).$$

Let the sector P_1OP_n be divided into $n-1$ equal sectors by the straight lines OP_2, $OP_3 \ldots . OP_{n-1}$, so that sector $P_1OP_2 =$ sector $P_2OP_3 = \ldots .$ $=$ sector $P_{n-1}OP_n$; hence (7),

$$\frac{x_2}{x_1} = \frac{x_3}{x_2} = \frac{x_4}{x_3} = \ldots . = \frac{x_n}{x_{n-1}} = v \text{ (suppose)} \ldots\ldots (8).$$

From (8) we see that

$$\frac{x_n}{x_1} = v^{n-1}, \text{ or } v = \left(\frac{x_n}{x_1}\right)^{\frac{1}{n-1}} \ldots\ldots\ldots\ldots (9),$$

which determines v, since x_1 and x_n are both known. Hence (8),

$$x_2 = vx_1,\ x_3 = v^2x_1,\ x_4 = v^3x_1, \ldots\ldots x_{n-1} = v^{n-2}x_1 \ldots\ldots (10)$$

are all known. Hence (8), we have the following construction,

(A). From P_1 and P_n the extremities of the *base* P_1P_n of the hyperbolic sector P_1OP_n, draw straight lines P_1Q_1 and $P_n\ Q_n$ parallel to OD one of the asymptotes, meeting OC the other asymptote, in Q_1 and Q_n; between OQ_1 and OQ_n insert $n-2$ mean proportionals OQ_2, $OQ_3 \ldots OQ_{n-1}$, which set off along OC; through Q_2, $Q_3 \ldots Q_{n-1}$, draw straight lines P_2Q_2, $P_3Q_3 \ldots P_{n-1}\ Q_{n-1}$ parallel to OD meeting the hyperbolic arc in P_2, $P_3 \ldots P_{n-1}$ and join OP_2, $OP_3 \ldots OP_{n-1}$. The sector P_1OP_n is divided into the $n-1$ equal sectors P_1OP_2, $P_2OP_3 \ldots P_{n-1}OP_n$.

It is almost unnecessary to remind the reader that these mean proportionals cannot be found *geometrically* except in a very few cases. They can easily be *calculated*, however, by aid of (9) and (10), to any degree of accuracy.

It is evident from (5) that the mixtilineal quadrilateral $P_1Q_1\ Q_n\ P_n$ is divided into $n-1$ equal figures by the lines P_2Q_2, $P_3Q_3, \ldots P_{n-1}\,Q_{n-1}$.

The next problem to be discussed is—To inscribe a maximum polygon of n sides in a given hyperbolic segment P_1P_n; and it will be most convenient first to consider the case in which $n=3$, the polygon then becoming a triangle.

If $P_1P_2P_3$ be the maximum triangle inscribed in the segment P_1P_3, it is clear that the tangent at P_2 must be parallel to the chord P_1P_3. Now the equation to the tangent at P_2 is

$$y-y_2=-\frac{\gamma^2}{x_2^{\,2}}(x-x_2)\ldots\ldots\ldots\ldots(11),$$

and the equation to the chord P_1P_3 is

$$y-y_1=\frac{y_3-y_1}{x_3-x_1}(x-x_1)=(1)\gamma^2\left(\frac{1}{x_3}-\frac{1}{x_1}\right)\cdot\frac{x-x_1}{x_3-x_1}=-\frac{\gamma^2}{x_1x_3}(x-x_1)\ldots(12).$$

Since (11) and (12) are parallel we must have, $-\dfrac{\gamma^2}{x_2^{\,2}}=-\dfrac{\gamma^2}{x_1x_3}$,

$$\therefore\quad x_2^{\,2}=x_1x_3 \text{ or } \frac{x_2}{x_1}=\frac{x_3}{x_2}\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Now, let $P_1P_2 \ldots P_n$ be the maximum polygon of n sides inscribed in the segment P_1P_n. As before, it is evident that the tangent at any angle P_m must be parallel to the chord $P_{m-1}P_{m+1}$; hence (13), $\dfrac{x_2}{x_1}=\dfrac{x_3}{x_2}$, $\dfrac{x_3}{x_2}=\dfrac{x_4}{x_3}$, *etc.*;

$$\therefore\quad \frac{x_2}{x_1}=\frac{x_3}{x_2}=\frac{x_4}{x_3}=\ldots\ldots=\frac{x_n}{x_{n-1}}\ldots\ldots\ldots\ldots(14),$$

and this coinciding with (8), we infer the following construction.

(B). Find the points P_2, $P_3, \ldots P_{n-1}$ as in (A), and join P_1P_2, $P_2P_3 \ldots P_{n-1}P_n$; then will $P_1P_2 \ldots P_n$ be the maximum polygon of n sides inscribed in the segment P_1P_n

Let the tangents at P_m and P_{m+1} intersect in R_m, and denote R_m by (X_m, Y_m). The equations to the tangents at P_1 and P_2 give

$$y_1X_1+x_1Y_1=2\gamma^2\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

and, $$y_2X_1+x_2Y_1=2\gamma^2\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

also, $$x_1y_1=\gamma^2=x_2y_2;\text{ hence}$$

$$X_1=\frac{2x_1x_2}{x_1+x_2},\quad Y_1=\frac{2\gamma^2}{x_1+x_2}\ldots\ldots\ldots\ldots\ldots(17).$$

Suppose now that the points P_1 and P_3 remain constant while P_2 may vary; let us endeavour to find the position of the latter, so that the quadrilateral $P_1R_1R_2P_3$ may be a minimum, or which is the same thing, that the rectilinear figure $Q_1P_1R_1R_2P_3Q_3$ may be a maximum. Now twice the area of the latter figure divided by $\sin\theta$ is evidently

$$(y_1+Y_1)(X_1-x_1)+(Y_1+Y_2)(X_2-X_1)+(y_3+Y_2)(x_3-X_2),$$

or, since $x_1y_1=\gamma^2=x_2y_2$, this becomes

$$y_1X_1-x_1Y_1+x_3Y_2-y_3X_2+X_2Y_1-X_1Y_2 \ldots\ldots (18).$$

Now (1), $y_1=\dfrac{\gamma^2}{x_1}$, $y_3=\dfrac{\gamma^2}{x_3}$, also (17), $X_1=\dfrac{2x_1x_2}{x_1+x_2}$, $X_2=\dfrac{2x_2x_3}{x_2+x_3}$, $Y_1=\dfrac{2\gamma^2}{x_1+x_2}$, and $Y_2=\dfrac{2\gamma^2}{x_2+x_3}$, hence (18) after a little reduction becomes

$$\frac{8\gamma^2(x_3-x_1)x_2}{(x_1+x_2).(x_2+x_3)};$$

and this must be a maximum, x_2 being variable. Hence omitting constant factors, and inverting

$$\frac{(x_1+x_2)(x_2+x_3)}{x_2}=(x_1^{\frac{1}{2}}+x_3^{\frac{1}{2}})^2+\left(x_2^{\frac{1}{2}}-\frac{x_1^{\frac{1}{2}}x_3^{\frac{1}{2}}}{x_2^{\frac{1}{2}}}\right)$$

must be a minimum, which it will evidently be, when $x_2^{\frac{1}{2}}=\dfrac{x_1^{\frac{1}{2}}x_3^{\frac{1}{2}}}{x_2^{\frac{1}{2}}}$,

$$\therefore\ x_2^2=x_1x_3, \text{ or } \frac{x_2}{x_1}=\frac{x_3}{x_2} \ldots\ldots (19),$$

and this equation determines the value of x_2, and hence the position of P_2.

At the points $P_1, P_2, P_3, .. P_n$ draw tangents to the hyperbola forming by their intersections the polygon $P_1R_1R_2..R_{n-1}P_n$ of $n+1$ sides circumscribed about the segment P_1P_n. If this be the minimum polygon of $n+1$ sides circumscribed about the segment, it is clear by (19), that we must have

$$\frac{x_2}{x_1}=\frac{x_3}{x_2}=\frac{x_4}{x_3}=\ldots\ldots=\frac{x_n}{x_{n-1}} \ldots\ldots (20).$$

Now this coincides with (8), and hence the following solution of the problem:—To circumscribe the minimum polygon of $n+1$ sides about a given hyperbolic segment.

(C). Find the points $P_2, P_3 .. P_{n-1}$ as in (A); tangents at $P_1, P_2, P_3 .. P_n$ will, by their intersections, form the minimum polygon $P_1R_1R_2..R_{n-1}P_n$ of $n+1$ sides circumscribed about the segment P_1P_n.

From (17) we have $X_m Y_m=\dfrac{4\gamma^2x_m x_{m+1}}{(x_m+x_{m+1})^2}$, but (10) $x_{m+1}=v.x_m$,

$$\therefore\ X_m Y_m=\frac{4v\gamma^2}{(1+v)^2} \ldots\ldots (21).$$

Hence,

(D). The angular points of a minimum polygon circumscribed about a segment of an hyperbola are situated in a concentric similar hyperbola.

Again, (10,17), $X_m = \dfrac{2x_m\, x_{m+1}}{x_m + x_{m+1}} = \dfrac{2v^m\, x_1}{1+v}$, hence $\dfrac{X_{m+1}}{X_m} = v$,

$$\therefore \frac{X_2}{X_1} = \frac{X_3}{X_2} = \frac{X_4}{X_3} = \ldots = \frac{X_{n-1}}{X_{n-2}} \ldots\ldots\ldots\ldots(22).$$

Comparing this with (8), we see that,

(E). If $P_1R_1R_2\ldots\ldots R_{n-1}\,R_n$ be the minimum polygon of $n+1$ sides circumscribed about the hyperbolic segment P_1P_n, and R_1R_{n-1} be joined, the polygon $R_1R_2\ldots\ldots R_{n-1}$ will be the maximum polygon of $n-1$ sides inscribed in the segment R_1R_{n-1} of the hyperbola in which (D) the points R_1, R_2,.. ..R_{n-1} are situated.

From (D) and (E) the following is easily inferred.

(F). The sides of a maximum polygon inscribed in a segment of an hyperbola touch a concentric similar hyperbola.

Having now considered all the three problems with respect to the hyperbola, I shall next discuss, for the parabola, those two that can be proposed in the case of the latter curve; and, in order to point out the analogy of the following propositions to (B)..(F), I shall denote them by (b)..(f), omitting (a), as there is here nothing analogous to (A).

Let $P_1P_2\ldots\ldots P_n$ be the maximum polygon of n sides inscribed in the segment P_1P_n of a parabola, and from P_2, P_3,..P_{n-1} draw diameters (which will be parallel to each other), meeting the chord P_1P_n in the points Q_2, Q_3..Q_{n-1} respectively. Since the polygon is a maximum, the tangent at P_2 must be parallel to the chord P_1P_3; hence P_1P_3 is bisected by the diameter P_2Q_2, wherefore also, P_3Q_3 is bisected in Q_2, and therefore $P_1Q_2 = Q_nQ_3$; similarly it may be shown that $Q_2Q_3 = Q_3Q_n$, *etc.*

$$\therefore P_1Q_2 = Q_2Q_3 = Q_3Q_4 = \ldots\ldots\ldots = Q_{n-2}\,Q_{n-1} = Q_{n-1}\,P_n \ldots.(23).$$

Hence,

(b). Divide the chord or base P_1P_n of the parabolic segment into $n-1$ equal parts in the points Q_2,Q_3,....Q_{n-1}, and through these points draw straight lines parallel to a diameter of the parabola, meeting the arc in P_2, P_3, ..P_n: also join P_1P_2, P_2P_3,..$P_{n-1}P_n$. The figure P_1P_2..P_n is the maximum polygon of n sides inscribed in the parabolic segment P_2P_n.

Let the conjugate axes to which the parabola is referred be so taken that the axis of x may coincide with the diameter that bisects the chord P_1P_n, and let P' be the origin and Q' the middle point of P_1P_n. If $P'Q' = a$, $P_1Q' = P_nQ' = b$, and the point P_m be denoted by (x_m, y_m), we shall evidently have from (23),

$$y_1 = -b,\ y_2 = \left(\frac{2}{n-1} - 1\right)b, \ldots\ y_m = \left(\frac{2(m-1)}{n-1} - 1\right)b \ldots\ldots (24).$$

At P_1, P_2, P_3, three points in a parabola, apply tangents forming by their intersections the quadrilateral $P_1R_1R_2R_3$ circumscribed about the segment P_1P_3. Suppose the points P_1,P_3 to remain fixed, and let us endeavour to find the position of P_2 so that the quadrilateral may be a minimum. Taking the conjugate axes in such a way that the axis of x may bisect P_1P_3, the equation to the parabola will be

$$y^2 = px \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Also the points P_1, P_2, P_3, R_1, R_3 may be denoted by $(x_1, -y_1)$,

(x_2, y_2), (x_1, y_1),) (X_1, Y_1), (X_2, Y_2) respectively. The equations to the tangents at the points P_1, P_2 give

$$-2Y_1y_1 = pX_1 + y_1^2, \text{ and } 2Y_1y_2 = pX_1 + y_2^2,$$

$$\therefore \quad X_1 = \frac{-y_1y_2}{p}, \; Y_1 = \tfrac{1}{2}(y_2 - y_1) \ldots\ldots\ldots\ldots\ldots (26).$$

Similarly, $$X_2 = \frac{y_1y_2}{p}, \; Y_2 = \tfrac{1}{2}(y_2 + y_1) \ldots\ldots\ldots\ldots\ldots (27).$$

Also if the tangents at P_1 and P_3 intersect in R, and R be denoted by (X,Y), we shall easily get

$$X = -x_1 = (25) - \frac{y_1^2}{p}, \; Y = 0 \ldots\ldots\ldots\ldots\ldots (28).$$

Since the quadrilateral $P_1R_1R_2P_3$ is to be a minimum, the triangle RR_1R_2 must be a maximum. Now twice the area of this triangle divided by the sine of the inclination of the axes is

$$X_1Y_2 - X_2Y_1 + X_2Y - XY_2 + XY_1 - X_1Y,$$

which by (26, 27, 28) becomes

$$\left(y_1^2 - y_2^2\right)\frac{y_1}{p};$$

and this must be a maximum, y_2 being variable; now this is evidently the case when $y_2 = 0$, and hence x_2 also $= 0$. The point of contact P_2 must therefore coincide with the origin, and hence the tangent at P_2, that is, the side R_1R_2 of the quadrilateral $P_1R_1R_2P_3$ must be parallel to the chord or base P_1P_3.

Let P_1, P_2, .. P_n be the points of contact of the sides of the minimum polygon $P_1R_1R_2 \ldots R_{n-1}P_n$ of $n+1$ sides circumscribed about the parabolic segment P_1P_n. From what we have just seen it appears that the side $R_{m-1}R_m$ which is the tangent at P_m must be parallel to the chord $P_{m-1}P_{m+1}$, and this is the very property from which the construction (b) was deduced; hence,

(c). Having found the points $P_2, P_3, \ldots P_{n-1}$ as in (b), tangents at P_1, $P_2 \ldots\ldots P_n$ will by their intersections form the minimum polygon $P_1R_1R_2 \ldots\ldots R_{n-1}P_n$ of $n+1$ sides circumscribed about the parabolic segment P_1P_n.

Taking the axes as in (24), and denoting the point R_m by (X_m, Y_m), it may easily be shown as in (26) or (27), that

$$X_m = \frac{y_m y_{m+1}}{p}, \; Y_m = \tfrac{1}{2}(y_m + y_{m+1}) =, (24), \left(\frac{2m-1}{n-1} - 1\right)b \,..(29);$$

$$\therefore y_m =, (24), \left(\frac{2(m-1)}{n-1} - 1\right)b = Y_m - \frac{b}{n-1}, \text{ and}$$

$$y_{m+1} = \left(\frac{2m}{n-1} - 1\right)b = Y_m + \frac{b}{n-1},$$

hence, $pX_m = y_m y_{m+1} = Y_m^2 - \dfrac{b^2}{(n-1)^2} = Y_m^2 - \dfrac{pa}{(n-1)^2}$, for $b^2 = pa$,

$$\therefore y_m^2 = p.\left(X_m + \frac{a}{(n-1)^2}\right) \ldots\ldots\ldots\ldots\ldots (30).$$

Hence,

(d). The angular points of a minimum polygon circumscribed about a segment of a parabola, are situated in an equal parabola, and the two parabolas have their principal axes in the same straight line.

Since, (29), $Y_2-Y_1=Y_3-Y_2=\ldots\ldots=Y_{n-1}-Y_{n-2}$, it is evident that if diameters to the parabola mentioned in (d), be drawn through $R_2, R_3 \ldots R_{n-2}$, they will divide the chord R_1R_{n-1} into $n-2$ equal parts, and hence, (b, d),

(e). If $P_1R_1R_2\ldots R_{n-1}P_n$ be the minimum polygon of $n+1$ sides circumscribed about the parabolic segment P_1P_n, and R_1R_{n-1} be joined; the polygon $R_1R_2\ldots\ldots R_{n-1}$ will be the maximum polygon of $n-1$ sides inscribed in the segment R_1R_{n-1} of the parabola in which, (d), the points $R_1, R_2\ldots R_{n-1}$ are situated.

Finally, from (d) and (e) the following readily follows.

(f). The sides of a maximum polygon inscribed in a segment of a parabola touch an equal parabola, and the two parabolas have their principal axes in the same straight line.

Cottenham Street, Newcastle on Tyne,
May 1*st*, 1847.

APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

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SECTION III.

OF POLAR RECIPROCAL CURVES AND SURFACES.

The principal object of this paper is to develope, in an easy manner, the most simple and elementary properties of polar reciprocal surfaces. For the convenience of the reader we shall begin with reciprocal curves in a plane, and then proceed to the more general properties of space. The theory of poles and polars having been previously developed by an able hand in the pages of the Mathematician, we might proceed at once to the theory of polar reciprocal curves and surfaces; but it may be expedient to recapitulate, in a slightly different form, a few propositions of the former theory, in order to place the latter in as clear a light as possible.

OF POLAR RECIPROCAL CURVES.

I.

If a straight line be drawn from a given point P, meeting a given curve of the second degree in Q and Q′; and if P and P′ be conjugate harmonic points with respect to Q and Q′, it is required to find the locus of P′.

Let x_1y_1 be the coordinates of P, $x'y'$ those of P′, and xy those of Q; and put $PQ=r$, $PP'=\rho$, and $\theta=$ the angle which PQ makes with the axis of x. The coordinates being rectangular, we shall have

$$\frac{x-x_1}{r}=\cos\theta=\frac{x'-x_1}{\rho},\quad \frac{y-y_1}{r}=\sin\theta=\frac{y'-y_1}{\rho};$$

$$\therefore\quad x=x_1+\frac{r}{\rho}(x'-x_1),\quad y=y_1+\frac{r}{\rho}(y'-y_1).$$

Let the equation of the given curve be

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0\ldots\ldots\ldots\ldots\ldots(1),$$

then by substituting the above values of x and y, and making slight reductions, we obtain

$$\begin{aligned}&Ax_1^2+2Bx_1y_1+Cy_1^2+2Dx_1+2Ey_1+F\\&+2\Big\{Ax_1(x'-x_1)+Bx_1(y'-y_1)+By_1(x'-x_1)+Cy_1(y'-y_1)\\&\qquad+D(x'-x_1)+E(y'-y_1)\Big\}\frac{r}{\rho}\\&+\Big\{Ax_1(x'-x_1)^2+2B(x'-x_1)(y'-y_1)+C(y'-y_1)^2\Big\}\frac{r^2}{\rho^2}=0.\end{aligned}$$

Now if r_1, r_2 denote the roots of this equation, we have by hypothesis

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{\rho} \text{ or } \frac{\rho}{r_1}+\frac{\rho}{r_2}=2;$$

hence by quadratics

$$\frac{2\{Ax_1(x'-x_1)+Bx_1(y'-y_1)+By_1(x'-x_1)+Cy_1(y'-y_1)+D(x'-x_1)+E(y'-y_1)\}}{Ax_1^2+2Bx_1y_1+Cy_1^2+2Dx_1+2Ey_1+F}+2=0,$$

or, by slight reductions,

$$Ax_1x'+B(x_1y'+y_1x')+Cy_1y'+D(x_1+x')+E(y_1+y')+F=0;$$

hence it is evident that the equation of the locus of P′ is

$$(Ay_1+By_1+D)x+(Bx_1+Cy_1+E)y+Dx_1+Ey_1+F=0 \ldots\ldots (2).$$

The straight line represented by this equation is called the *polar* of the point $P(x_1, y_1)$ with regard to the curve (1), and the point P is called the *pole* of the straight line (2).

II.

If x_2y_2 be the coordinates of any point R, the condition in order that the straight line (2) may pass through R will be

$$Ax_1x_2+B(x_1y_2+y_1x_2)+Cy_1y_2+D(x_1+x_2)+E(y_1+y_2)+F=0\ldots.(3);$$

now since the equation of the polar of R is

$$(Ay_2+By_2+D)x+(Bx_2+Cy_2+E)y+Dx_2+Ey_2+F=0,$$

equation (3) also expresses the condition in order that the latter line may pass through P. Hence we have the following theorem:

If the polar (2) *of a given point* P *pass through another point* R, *then reciprocally the polar* (3) *of the latter point* R *will pass through the former point* P.

Hence if any two points be taken in a straight line, the intersection of their polars will be the pole of that line.

III.

If any two curves PQR and P′Q′R′, in the plane of a conic section ABCD, be so related that the points P, Q, *etc.* of the one are the poles of tangents applied to the other at P′, Q′, *etc.*; then reciprocally the points P′, Q′, *etc.* of the latter are the poles of tangents applied to the former at P, Q, *etc.*

For, since P is the pole of P′T′, and Q the pole of Q′T′, T′ is the pole of PQ (II). Let P and P′ remain fixed, while Q′ moves up to P′; then when Q′ arrives at P′ the three points P′, Q′, T′ ultimately coincide, and PQ becomes a tangent at P; hence P′ is the pole of the tangent at P; and in like manner it may be demonstrated that Q′ is the pole of the tangent at Q, *etc.*

Hence it is evident that each of the curves may be considered either as the envelope of the polars of the points of the other, or as the locus of the poles of the tangents of the other. On account of this remarkable relation the curves PQR and P′Q′R′ are called *polar reciprocals* with respect to the conic section ABCD, which is called the *director*.

IV.

Let x_1y_1 be the coordinates of any point P on one of the curves, $x'y'$ those of the corresponding point P′ on the reciprocal curve, and

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

the equation of the director. If a diameter of the director be drawn through the point x_1y_1, the equation of its ordinates is

$$(Ax_1 + By_1 + D)y + (Bx_1 + Cy_1 + E)x + G = 0,$$

where G is indeterminate; again, since the tangent at $x'y'$ is the polar of x_1y_1 (III), its equation is

$$(Ax_1 + By_1 + D)y + (Bx_1 + Cy_1 + E)x + Dx_1 + Ey_1 + F = 0;$$

and since this line is parallel to the former for any value of G, we have the following theorem.

A tangent applied to one of the curves at any point, is parallel to the ordinates of the diameter of the director which passes through the corresponding point of its reciprocal curve.

Since any diameter of a circle is perpendicular to its ordinates, it follows that *when the director is a circle, a straight line drawn from its centre to any point in one of the curves, is perpendicular to the tangent applied at the corresponding point to the reciprocal curve.*

V.

To find the curve reciprocal to any curve of the second degree with respect to a circle.

The circle being referred to any rectangular axes passing through its centre, if the radius be taken as the linear unit, its equation is

$$x^2 + y^2 = 1,$$

and the equation of the given curve is

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \ldots\ldots\ldots\ldots \quad (4).$$

Let $x'y'$ be any point on the given curve, and let x_1y_1 be the corresponding point of the required curve. The tangent applied to the given curve at the former point being

$$(Ax'+By'+D)x+(Bx'+Cy'+E)y+Dx'+Ey'+F=0,$$

and the polar of the latter

$$x_1x + y_1y = 1;$$

since these two lines are identical (III), we have the conditions

$$x_1 = -\frac{Ax' + By' + D}{Dx' + Ey' + F}, \quad y_1 = -\frac{Bx' + Cy' + E}{Dx' + Ey' + F} \ldots\ldots\ldots (a).$$

Clearing these of fractions, we get

$$(A + Dx_1)x' + (B + Ex_1)y' + D + Fx_1 = 0,$$
$$(B + Dy_1)x' + (C + Ey_1)y' + E + Fy_1 = 0;$$

from which we obtain by elimination

$$x' = \frac{CD-BE+(FC-E^2)x_1+(DE-FB)y_1}{B^2-AC+(BE-CD)x_1+(BD-AE)y_1},$$
$$y' = \frac{AE-BD+(DE-FB)x_1+(FA-D^2)y_1}{B^2-AC+(BE-CD)x_1+(BD-AE)y_1};$$

now since the polar of x_1y_1 passes through $x'y'$ we have

$$x_1x' + y_1y' = 1,$$

hence, by substituting the preceding values of x', y', and clearing the result of fractions, we obtain

$$(FC-E^2)x_1^2+2(DE-FB)x_1y_1+(FA-D^2)y_1^2$$
$$+2(CD-BE)x_1+2(AE-BD)y_1-(B^2-AC)=0;$$

and therefore the equation of the required curve is

$$(FC-E^2)x^2+2(DE-FB)xy+(FA-D^2)y^2$$
$$+2(CD-BE)x_1+2(AE-BD)y_1-(B^2-AC)=0 \ldots\ldots(5).$$

The second members of equations (a) are the reciprocals of the segments which the tangent applied to the curve (4) at the point $x'y'$ intercepts on the axes of x and y respectively. Hence we see that when the director is a circle, if any two rectangular axes be drawn through the centre of that circle, and a tangent applied to one of the curves at any point, the reciprocals of the segments which the tangent intercepts on these axes are the coordinates of the corresponding point of the reciprocal curve referred to the same axes.

VI.

A comparison of the equations (4) and (5) of the polar reciprocal curves leads to a great variety of properties, the following of which are the most obvious and interesting.

1st. When $B^2-AC=0$ the curve (5) evidently passes through the origin. Hence *when one of the curves* (4) *is a parabola, the reciprocal curve* (5) *passes through the centre of the director.* Conversely when the curve (4) passes through the centre of the director, the other curve (5) is a parabola. For when $F=0$, the first three terms of equation (5) constitute a complete square.

2nd. When $D^2-AF=0$, the curve (4) touches the axis of x, and (5) is an hyperbola having an asymptote parallel to the axis of y. Hence *the asymptotes of one of the curves are perpendicular to the tangents drawn to the other from the centre of the director*, and the converse.

It is well known that two tangents drawn to a parabola from any point in its directrix are perpendicular to one another; hence *when the centre of the director is on the directrix of a parabola, the reciprocal curve is an equilateral hyperbola, whose asymptotes are tangents to the parabola.*

3rd. When $B=0$ and $D=0$, or $B=0$ and $E=0$, we have $DE-FB=0$; hence *when the centre of the director is on an axis of one of the reciprocal curves* (4), *the axes of those curves are parallel.*

4th. When $D=0$ and $F=0$, the axes of x and y are evidently a tangent and normal to the curve (4); in this case the equation of the reciprocal curve is

$$\left(x+\frac{B}{E}\right)^2=\frac{A}{E}\left(2y+\frac{C}{E}\right),$$

which denotes a parabola having its axis parallel to the axis of y. Hence *when one of the reciprocal curves* (4) *passes through the centre of the director, the other curve* (5) *is a parabola whose axis is perpendicular to the tangent applied to the former at the centre of the director.*

When $B=0$, $C=0$, $D=0$, and $F=0$, the equations of the reciprocal curves are

$$Ax^2+2Ey=0, \text{ and } Ex^2-2Ay=0;$$

hence *when the centre of the director is at the principal vertex of a parabola, the reciprocal curve is a parabola, having the same principal vertex, the axes of the parabolas being in the same straight line, and their parameters reciprocally proportional.*

5th. When $B=0$, $D=0$, and $E=0$, the equations of the reciprocal curves become

$$Ax^2+Cy^2+F=0, \quad \frac{x^2}{A}+\frac{y^2}{C}+\frac{1}{F}=0;$$

hence *when one of the curves is concentric with the director, the axes of the reciprocal curves coincide in direction, and the lengths of the coincident axes are reciprocals.*

6th. The polar of the origin taken with respect to the curve (5) being (I.)

$$(CD-BE)x+(AE-BD)y-(B^2-AC)=0,$$

may also be written in the form

$$\alpha x+\beta y-1=0,$$

$\alpha\beta$ being the centre of the curve (4); but the last equation also represents the polar of $\alpha\beta$ with respect to the director: hence we see that *the polar of the centre of the director taken with respect to one of the reciprocal curves coincides with the polar of the centre of the other curve taken with respect to the director.*

VII.

Equation (4) may evidently be written in the form

$$(FA-D^2)x^2+2(FB-DE)xy+(FC-E^2)y^2+(Dx+Ey+F)^2=0 \ldots\ldots (4');$$

now when $FB-DE=0$, and $FA-D^2=FC-E^2$, this equation becomes

$$(FA-D^2)(x^2+y^2)+(Dx+Ey+F)^2=0,$$

and equation (5) is reduced to

$$(FA-D^2)(x^2+y^2)+(B^2-AC)(\alpha x+\beta y-1)=0;$$

hence it is evident that *when the centre of the director coincides with a focus of one of the curves* (4), *the reciprocal curve* (5) *is a circle; and the centres of the three curves lie in the same straight line.*

VIII.

By reducing equation (5) to a form similar to (4′) we obtain

$$Cx^2-2Bxy+Ay^2=\frac{B^2-AC}{M}(\alpha x+\beta y-1)^2 \ldots\ldots\ldots\ldots (5');$$

where $M=CD^2+AE^2+FB^2-ACF-2BDE$: hence it is evident that if $M'=C'D'^2+A'E'^2+F'B'^2-A'C'F'-2B'D'E'$, the curves represented by the equations

$$A'x^2+2B'xy+C'y^2+2D'x+2E'y+F=0 \ldots\ldots \ldots\ldots (6),$$

and $$C'x^2-2B'xy+A'y^2=\frac{B'^2-A'C'}{M'}(\alpha'x+\beta'y-1)^2 \ldots\ldots\ldots (7)$$

are also polar reciprocals. Now when the curves (4) and (6) are similar we have

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}=k,$$

hence multiplying (7) by k and subtracting (5′), we obtain

$$\frac{(B'^2-A'C')k}{M'}(\alpha'x+\beta'y-1)^2-\frac{B^2-AC}{M}(\alpha x+\beta y-1)^2=0,$$

which evidently represents two straight lines passing through the intersection of the straight lines

$$\alpha x+\beta y-1=0, \text{ and } \alpha'x+\beta'y-1=0,$$

and forming with them a harmonic pencil. Hence we have the following theorem:

When two conic sections are similar, their polar reciprocals intersect on two straight lines which pass through the pole of the straight line joining the centres of the two given curves ; and those conjugate common secants together with the polars of these centres form a harmonic pencil.

Hence if a system of similar conic sections have their centres in the same straight line, the pole of that line is a point of intersection of conjugate common secants for any two of their reciprocal curves.

Again, when the curves (4) and (6) are concentric, we have $\alpha=\alpha'$, $\beta=\beta'$, and we readily obtain from (5′) and (7)

$$\frac{M}{B^2-AC}(Cx^2-2Bxy+Ay^2)\ \frac{M'}{B'^2-A'C'}(C'x^2-2B'xy+Ay^2)=0,$$

which evidently denotes two straight lines, real or imaginary, passing through the origin; hence we have the following theorem:

The polar reciprocals of any two concentric conic sections have the centre of the director for a point of intersection of conjugate common secants.

IX.

To find the polar reciprocal of any curve of the second degree with respect to any other curve of the second degree taken as director.

Let the equation of the director be

$$A'x^2+2B'xy+C'y^2+2D'x+2E'y+F'=0 \ldots\ldots\ldots (8),$$

and that of the given curve

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0 \ldots\ldots\ldots\ldots (9);$$

then if $x'y'$ be any point on the given curve (9), and x_1y_1 the corresponding point on the required curve, the equation of the tangent applied to the given curve at the former point is

$$(Ax'+By'+D)x+(Bx'+Cy'+E)y+Dx'+Ey'+F=0,$$

and that of the polar of the latter point

$$(A'x_1+B'y_1+C')x+(B'x_1+C'y_1+E')y+D'x_1+E'y_1+F'=0;$$

and since these lines are identical, we have

$$\frac{Ax'+By'+D}{Dx'+Ey'+F}=\frac{A'x_1+B'y_1+D'}{D'x_1+E'y_1+F'},$$

$$\frac{Bx'+Cy'+E}{Dx'+Ey'+F}=\frac{B'x_1+C'y_1+E'}{D'x_1+E'y_1+F'}.$$

Now if the second members of these equations be denoted by $-X$ and $-Y$, they become identical in form to equations (a), and by following the process in (v) we should find

$$(FC-E^2)X^2+2(DE-FB)XY+(FA-D^2)Y^2$$
$$+2(CD-BE)X+2(AE-BD)Y-(B^2-AC)=0;$$

from which it is evident that the required equation is

$$(E^2-FC)(A'x+B'y+D')^2+(D^2-AF)(B'x+C'y+E')^2$$
$$-2(DE-FB)(A'x+B'y+D')(B'x+C'y+E')$$
$$+2(CD-BE)(A'x+B'y+D')(D'x+E'y+F')$$
$$+2(AE-BD)(B'x+C'y+E')(D'x+E'y+F')$$
$$+(B^2-AC)(D'x+E'y+F')^2=0 \ldots\ldots\ldots\ldots (10).$$

X.

The equation of central curves of the second degree being

$$A'x^2+C'y^2=1,$$

if the axes of the director be taken as axes of coordinates, we have $B'=0$, $D'=0$, $E'=0$, $F'=-1$, and the equation (10) of the reciprocal curve becomes

$$(E^2-FC)A'^2x^2+2(FB-DE)A'C'xy+(D^2-FA)C'^2y^2$$
$$+2(BE-CD)A'x+2(BD-AE)C'y+(B^2-AC)=0 \ldots\ldots\ldots (11);$$

and since this equation is exactly analogous to (5) we should readily find properties analogous to those developed in Nos. VI, VII, VIII; but a little more general. Thus we have seen, VII, that when the conditions $FB-DE=0$, and $D^2-AF=E^2-CF$ are satisfied the curve (9) has a focus at the origin; and if the condition $C'=\pm A'$ be added to these, the reciprocal curve (11) is a circle: hence we have the following theorem, which may be regarded as a generalization of the one obtained in No. VII.

When the director is either a circle or an equilateral hyperbola, if its centre be at a focus of the given curve the reciprocal curve is a circle.

When the reciprocal curves are referred to any system of conjugate diameters of the director, equations (9) and (11) remain unchanged, A' and C' representing the squared reciprocals of those semidiameters. Now if $D^2-AF=0$, the curve (9) touches the axis of x, and (11) is an hyberbola having an asymptote parallel to the axis of y; hence *if a diameter of the director be drawn touching one of the curves, its conjugate diameter will be parallel to an asymptote of the other curve.*

The other properties in No. VI. may be generalized in a similar manner.

XI.

If the semiparameter of any diameter of a parabola be taken as the linear unit, the equation of the parabola referred to that diameter and the tangent at its vertex is

$$y^2=2x;$$

hence when the director is a parabola we may assume $A'=0$, $B'=0$, $C'=1$, $D'=1$, $E'=0$, $F'=0$, and the equation (10) of the reciprocal curve becomes

$$(B^2-AC)x^2+2(BD-AE)xy+(D^2-AF)y^2$$
$$+2(CD-BE)x+2(DE-FB)y+E^2-FC=0 \ldots\ldots\ldots\ldots\ldots (12).$$

As this equation differs considerably from (5), while (9) coincides with (4), the properties of two curves which are polar reciprocals with respect to a parabola, are not strictly analogous to those of curves which are similarly related with respect to a circle. The following are some of the most obvious and elegant relations of parabolic polar reciprocals.

1st. When $B^2-AC=0$ the reciprocal curve has an asymptote parallel to the axis of x, hence *when one of the curves* (4) *is a parabola, the other has an asymptote parallel to the axis of the director.*

2nd. When $BD-AE=0$ the centre of the curve (9) is on the axis of x, and the curve (12) has a system of conjugate diameters parallel to the axes of x and y; hence *if a diameter of the director be drawn through the centre of one of the reciprocal curves, the other curve has a system of conjugate diameters respectively parallel to that diameter and the tangent applied to the director at its vertex.*

It is evident from this theorem that when the centre of one of the curves is on the axis of the director, the axes of the other curve are respectively parallel and perpendicular to that axis.

3rd. When $D^2-AF=0$ the curve (9) touches the axis of x, and the reciprocal curve (12) has an asymptote parallel to the axis of y; hence *if a diameter of the director be drawn touching one of the curves, the tangent applied to the director at the vertex of that diameter is parallel to an asymptote of the other curve.*

4th. When $E^2-FC=0$ the given curve (9) touches the axis of y, and (12) passes through the origin; hence *if a common tangent be applied to the given curve and the director, the reciprocal curve passes through the point in which it touches the director.*

5th. When the centre of the curve (9) is at the origin, we have $D=0$, $E=0$, and equation (12) becomes

$$(B^2-AC)x^2-F(Ay^2+2By+C)=0,$$

which denotes a conic section, the axis of y being a diameter, and that of x parallel to its conjugate diameter; hence *when the centre of the given curve is on the director, the tangent applied to the director at that point is a diameter of the reciprocal curve, and the diameter of the director which passes through the point of contact is parallel to its conjugate diameter.*

6th. When the curve (9) passes through the origin and touches the axis of y, we have $E=0$ and $F=0$, and (12) becomes

$$(B^2-AC)x^2+2BDxy+D^2y^2+2CDx=0;$$

which represents a conic section touching the axis of y at the origin; hence *when one of the reciprocal curves touches the director at any point, the other touches it at the same point.*

7th. When $A=0$ and $B=0$ equation (12) becomes

$$D^2y^2+2CDx+2DEy+E^2-FC=0, \text{ or}$$

$$\left(y+\frac{E}{D}\right)^2=\frac{C}{D}\left(\frac{F}{D}-2x\right);$$

hence *when the given curve is a parabola having its axis parallel to that of the director, the reciprocal curve is also a parabola whose axis is parallel to the axis of the director.*

8th. The equation of the diameter of (12) which bisects chords parallel to the axis of x is

$$(B^2-AC)x+(BD-AE)y+CD-BE=0,$$

and if $\alpha\beta$ denote the centre of (9) its polar with respect to the director is represented by

$$\beta y=x+\alpha;$$

and since the last two equations are evidently identical, we have the following theorem.

The polar of the centre of one of the curves coincides with the diameter of the other which bisects chords parallel to the axis of the director.

XII.

Equation (12) may evidently be written in the form

$$(B^2-AC)(x^2-2\beta xy+2\alpha x)$$
$$+(D^2-AF)y^2+2(DE-FB)y+(E^2-FC)=0 \ldots . (12'),$$

and similarly if $\alpha'\beta'$ denote the centre of the curve

$$A'x^2+2B'xy+C'y^2+2D'x+2E'y+F'=0 \ldots \ldots . . (13),$$

the equation of its parabolic polar reciprocal will be

$$(B'^2-A'C')(x^2-2\beta'xy+2\alpha'x)$$
$$+(D'^2-A'F')y^2+2(D'E'-F'B')y+E'^2-F'C'=0 \ . . (14).$$

Now if the curves (9) and (13) be concentric, we have $\alpha=\alpha'$, and $\beta=\beta'$, and we obtain by eliminating α and β from (12′) and (14)

$$(B^2-AC)\{(D'^2-A'F')y^2+2(D'E'-F'B')y+E'^2-F'C'\}$$
$$=(B'^2-A'C')\{(D^2-AF)y^2+2(DE-FB)y+(E^2-FC)\};$$

from which it is evident that the curves (12′) and (14) intersect each other on two straight lines parallel to the axis of x; hence *if two conic sections be concentric, their polar reciprocals with respect to any parabola intersect each other on two straight lines parallel to the axis of the parabola.*

Again, if the axes be rectangular, the conditions in order that the origin may be a common focus to the curves (9) and (13), are

$$E^2-FC=D^2-FA, \ DE-FB=0;$$
$$E'^2-F'C'=D'^2-F'A', \ D'E'-F'B'=0;$$

and when these are satisfied, equations (12) and (14) become

$$(B^2-AC)(x^2-2\beta xy+2\alpha x)+(D^2-AF)(y^2+1)=0,$$
$$(B'^2-A'C')(x^2-2\beta'xy+2\alpha'x)+(D'^2-A'F')(y^2+1)=0;$$

from which we obtain by eliminating y^2+1,

$$(D'^2-A'F')(B^2-AC)(x^2-2\beta xy+2\alpha x)$$
$$-(D^2-AF)(B'^2-A'C')(x^2-2\beta' xy+2\alpha' x)=0:$$

now since this may be resolved into the two equations $x=0$ and

$$D\beta B'(x-2\beta' y+2\alpha)-D'\beta' B(x-2\beta y+2\alpha)=0,$$

we have the following theorem.

If any system of conic sections have the principal vertex of a parabola for a common focus, the polar reciprocals of any two of them, taken with respect to the parabola, have a common secant which touches the parabola at that vertex.

XIII.

If any two radii vectores be drawn from a fixed point in the plane of a conic section, making a given angle with each other; to find the locus of the pole of the chord connecting their extremities.

The equation of the conic section with respect to any rectangular axes passing through the given point being

$$Ax^2+Bxy+Cy^2+2Dx+2Ey+F=0 \ldots\ldots\ldots\ldots\ldots (15),$$

let the equations

$$\alpha x+\beta y=1 \text{ and } y=mx$$

denote the chord in question and the radius vector drawn to one of its extremities. By eliminating y from these three equations we get

$$(A+2Bm+Cm^2)x^2+2(D+Em)x+F=0,$$
$$(\alpha+\beta m)x=1;$$

hence by eliminating x we have

$$A+2Bm+Cm^2+2(D+Em)(\alpha+\beta m)+F(\alpha+\beta m)^2=0,$$

or $\quad A+2D\alpha+F\alpha^2+2(B+D\beta+E\alpha+F\alpha\beta)m+(C+2E\beta+F\beta^2)m^2=0.$

The roots of this equation are evidently the indices of position of the radii vectores drawn from the given point to the extremities of the chord. Denoting them by m' and m'' we obtain by quadratics

$$\frac{m'+m''}{2}=-\frac{B+D\beta+E\alpha+F\alpha\beta}{C+2E\beta+F\beta^2},\ m'm''=\frac{A+2D\alpha+F\alpha^2}{C+2E\beta+F\beta^2};$$

and by subtracting the latter of these from the square of the former we get

$$\frac{m'-m''}{2}=\frac{\sqrt{\{(B+D\beta+E\alpha+F\alpha\beta)^2-(A+2D\alpha+F\alpha^2)(C+2E\beta+F\beta^2)\}}}{C+2E\beta+F\beta^2}:$$

but if t denote the tangent of the given angle, we have

$$t=\frac{m'-m''}{1+m'm''},$$

hence, by substitution we obtain,

$$\frac{t}{2}=\frac{\sqrt{\left\{\begin{array}{l}(E^2-CF)\alpha^2+2(BF-DE)\alpha\beta+(D^2-AF)\beta^2\\ +2(BE-CD)\alpha+2(BD-AE)\beta+B^2-AC\end{array}\right\}}}{A+C+2(D\alpha+E\beta)+F(\alpha^2+\beta^2)}\ldots\ldots\ldots (16).$$

Let $x'y'$ be the pole of the chord; then its equation is

$$(Ax'+By'+D)x+(Bx'+Cy'+E)y+Dx'+Ey'+F=0,$$

and by identifying this equation with $ax+\beta y=1$, we get

$$a=-\frac{Ax'+By'+D}{Dx'+Ey'+F}, \quad \beta=-\frac{Bx'+Cy'+E}{Dx'+Ey'+F} \dots\dots\dots (17).$$

By substituting these values of a and β in equation (16), we should obtain the equation of the required locus, which is therefore in general a curve of the fourth degree. In the following cases, however, the locus is a conic section.

1st. When the given angle is a right angle. For in this case $t=\infty$, and we obtain from (16)

$$A+C+2(Da+E\beta)+F(a^2+\beta^2)=0,$$

hence by (17) the equation of the locus is

$$\begin{aligned} &F\{(Ax'+By'+D)^2+(Bx'+Cy'+E)^2\} \\ &-2(Dx'+Ey'+F)\{D(Ax'+By'+D)+E(Bx'+Cy'+E\} \\ &\qquad +(A+C)(Dx'+Ey'+F)^2=0 \dots\dots\dots\dots (18); \end{aligned}$$

and therefore the locus is a conic section.

Since the locus of the pole of the chord is, in this case, a conic section, it follows that the envelope of the chord is also a conic section (IX), and we have the following theorem.

If two rectangular radii vectores be drawn from a fixed point in the plane of a conic section, the locus of the pole of the chord connecting their extremities is a conic section, and the envelope of the chord is also a conic section.

2nd. When the given point is on the curve. For, in this case, $F=0$, and equation (16) gives

$$\begin{aligned} \tfrac{1}{4}t^2\{A+C+2(Da+E\beta)\}^2=(E^2-CF)a^2+2(BF-DE)a\beta+(D^2-AF)\beta^2 \\ +2(BE-CD)a+2(BD-AE)\beta+B^2-AC=0, \end{aligned}$$

which is of the second degree with respect to a and β, and therefore also with respect to x' and y'; hence *if any two chords be drawn from a fixed point on a conic section making a given angle with each other, the locus of the pole of the chord subtending that angle is a conic section, having a double contact with the given conic section, and its envelope is also a conic section.*

When the given angle is a right angle the preceding equation becomes

$$A+C+2(Da+E\beta)=0,$$

hence, in this case, the locus of the pole of the chord subtending the given angle is a straight line (the chord of contact), and the chord always passes through a fixed point, the pole of that line.

3rd. When the vertex of the given angle coincides with a focus of the given curve. For, in this case, we have $E^2-CF=D^2-AF$ and $DE-BF=0$ (VII); now the last of these gives $F=\frac{DE}{B}$, and substituting this in the other two, we obtain

$$D^2-AF=\frac{D}{B}(BD-AE), \quad E^2-CF=\frac{E}{B}(BE-CD);$$

hence equation (16) becomes

$$\frac{t}{2}=\frac{\sqrt{\left\{(D^2-AF)\left(a^2+\beta^2+2\frac{B}{E}a+2\frac{B}{D}\beta\right)+B^2-AC\right.}}{A+C+F\left(2\frac{B}{E}a+2\frac{B}{D}\beta+a^2+\beta^2\right)},$$

and putting, for the sake of brevity,

$$a^2+\beta^2+2\frac{B}{E}a+2\frac{B}{D}\beta=G,$$

we shall have

$$\frac{t}{2}=\frac{\sqrt{(D^2-AF)G+(B^2-AC)}}{A+C+FG}.$$

By solving this equation for G we obtain a result of the form G = constant; hence the equation of the locus will be found by substituting the values of a and β in the equation

$$a^2+\beta^2+2\frac{B}{E}a+2\frac{B}{D}\beta=G, \text{ or}$$

$$F(a^2+\beta^2)+2(Da+E\beta)=FG;$$

and therefore the locus is a conic section.

Since the polar reciprocal of this last locus is also a conic section we have the following theorem:—*If the vertex of a moveable angle of invariable magnitude be fixed at a focus of a conic section, the locus of the pole of the chord subtending it is a conic section, and the envelope of the chord is also a conic section.*

In the general case if $u=0$ denote the equation (15) of the given curve, and $v=0$ the equation (18) of the locus when the radii vectores are rectangular, we obtain by substituting the values (17) of a and β in (16), and attending to the value of M in (5′),

$$\tfrac{1}{4}t^2v^2=Mu(Dx'+Ey'+F)^2;$$

hence the locus is in general a curve of the fourth degree, touching the given curve $u=0$ in the four points in which it cuts the conic section $v=0$, and touching the latter curve and the straight line $Dx+Ey+F=0$ at both their points of intersection.

XIV.

To find the polar reciprocal of a given curve with respect to any curve of the second degree taken as director.

Let $u=0$ be the equation of the given curve, and

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0,$$

the equation of the director; then if $x'y'$ be any point on the given curve, and x_1y_1 the corresponding point of the sought curve, the polar of the latter point is (I)

$$(Ax_1+By_1+D)x+(Bx_1+Cy_1+E)y+Dx_1+Ey_1+F=0,$$

and the tangent applied to the given curve at the former,

$$(x-x')\frac{du'}{dx'}+(y-y')\frac{du'}{dy'}=0.$$

Now, since these lines are identical (III), we have the conditions

$$\frac{\frac{du'}{dx'}}{x'\frac{du'}{dx'}+y'\frac{du'}{dy'}}=-\frac{Ax_1+By_1+D}{Dx_1+Ey_1+F}\ldots\ldots\ldots\ldots(19),$$

$$\frac{\frac{du'}{dy'}}{x'\frac{du'}{dx'}+y'\frac{du'}{dy'}}=-\frac{Bx_1+Cy_1+E}{Dx_1+Ey_1+F}\ldots\ldots\ldots\ldots(20),$$

and the equation of the required curve will be obtained by eliminating x' and y' from these equations by means of the equation $u'=0$.

Ex. 1. Let the given curve be a parabola represented by the equation

$$y^2=2x;\ \text{then}\ u'=y'^2-2x',\ \frac{du'}{dx'}=-2,\ \frac{du'}{dy'}=2y';$$

$$\text{hence}\quad x'\frac{du'}{dx'}+y'\frac{du'}{dy'}=2y'^2-2x'=2x',$$

and equations (19) and (20) become

$$-\frac{1}{x_1}=-\frac{Ax_1+By_1+D}{Dx_1+Ey_1+F},\quad \frac{y_1}{x_1}=-\frac{Bx_1+Cy_1+E}{Dx_1+Ey_1+F};$$

adding twice the former to the square of the latter we get

$$(Bx_1+Cy_1+E)^2=2(Ax_1+By_1+D)(Dx_1+Ey_1+F);$$

hence the reciprocal curve touches the straight lines

$$Ax+By+D=0\ \text{and}\ Dx+Ey+F=0,$$

at the points in which they are cut by the straight line

$$Bx+Cy+E=0,$$

and we have the following theorem.

If any tangent be applied to the given parabola, the reciprocal curve touches the polar of the point of contact and the diameter of the director which bisects chords parallel to the axis of the parabola; and the straight line joining the points of contact is the diameter of the director which bisects chords parallel to the tangent.

Ex. 2. Let the equation of the given curve be $x^{2n}+y^{2n}=1$, then

$$u'=x'^{2n}+y'^{2n}-1,\ \frac{du'}{dx'}=2nx'^{2n-1},\ \frac{du'}{dy'}=2ny'^{2n-1},\ x'\frac{du'}{dx'}+y'\frac{du'}{dy'}=2n,$$

and equations (19) and (20) become

$$x'^{2n-1}=-\frac{Ax_1+By_1+D}{Dx_1+Ey_1+F},\ y'^{2n-1}=-\frac{Bx_1+Cy_1+E}{Dx_1+Ey_1+F};$$

and by raising both sides to the power $\frac{2n}{2n-1}$, and adding, we obtain

$$(Ax_1+By_1+D)^{\frac{2n}{2n-1}}+(Bx_1+Cy_1+E)^{\frac{2n}{2n-1}}=(Dx_1+Ey_1+F)^{\frac{2n}{2n-1}},$$

which is the equation of the reciprocal curve.

MATHEMATICAL NOTES.

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I.

New method of finding the equation of a tangent to any of the conic sections.

If the origin of rectangular coordinates be taken at the vertex A, and the distance of A from the focus F be denoted by m, then, e being the eccentricity, and AF the axis of x, the equation

$$(1-e^2)x^2-2(1+e)mx+y^2=0 \ldots\ldots\ldots\ldots\ldots (1),$$

represents any of the conic sections (see any treatise).

Also the equation of a point (h, k) is

$$(1-e^2)(x-h)^2+(y-k)^2=0 \ldots\ldots\ldots\ldots\ldots (2).$$

Let this be a point in (1); then taking (2) from (1), we have

$$\{2(1-e^2)h-2(1+e)m\}x+2ky-\{(1-e^2)h^2+k^2\}=0 \ldots (3).$$

But because (h, k) is a point in (1),

$$(1-e^2)h^2-2(1+e)mh+k^2=0;$$

the equation (3) therefore becomes

$$\{(1-e^2)h-(1+e)m\}x+ky-(1+e)mh=0 \ldots\ldots (4).$$

Now if we eliminate x between this equation and (1), we shall find that the two values of y in the resulting quadratic, are *equal*. It hence follows that (4) is *tangential* to (1).

For the *parabola*, $e=1$; hence by (1) and (4), the equations of the curve and tangent to it at a point (h, k) are

$$y^2=4mx, \text{ and } ky=2m(x+h) \ldots\ldots\ldots\ldots\ldots (5).$$

For the *ellipse*, $1-e^2=\frac{b^2}{a^2}$, and $1+e=\frac{b^2}{ma}$; the equations therefore of the ellipse and tangent to it at a point (h, k), are, by (1) and (4),

$$a^2y^2=b^2(2ax-x^2), \text{ and } a^2ky+b^2(h-a)x=ab^2h \ldots\ldots\ldots (6).$$

The analogous expressions for the *hyperbola*, when $e^2-1=\frac{b^2}{a^2}$, and $e+1=\frac{b^2}{ma}$, are

$$a^2y^2=b^2(2ax+x^2), \text{ and } a^2ky-b^2(h+a)x=ab^2h \ldots\ldots\ldots (7).$$

Let us, for the ellipse and hyperbola, remove the origin to the centre, by writing $x+a$ for x and $h+a$ for h, in the former case, and $x-a$ for x and $h-a$ for h, in the latter: the results in (6) and (7) hence become

$$a^2y^2+b^2x^2=a^2b^2, \quad \text{and } a^2ky+b^2hx=a^2b^2 \ldots\ldots\ldots (6'),$$

$$a^2y^2-b^2x^2=-a^2b^2, \text{ and } a^2ky-b^2hx=-a^2b^2 \ldots\ldots\ldots (7').$$

We are thus enabled to find the equation of a tangent to a conic section, without limits, or any use of the indeterminate expression $\frac{0}{0}$, to which so much well founded objection has been raised.

II.

On the double sign in the expression for the perpendicular distance of a point from a given line.

The meaning of the double sign in the expression for the perpendicular distance of a given point from a given line, is not well explained in any of our treatises on Analytical Geometry. The following remarks may therefore be interesting to some of our readers.

Let
$$y = ax + b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
be the equation of a line AB, and (h, k) the coordinates of a point P, referred to axes which make an angle ω. Draw PB perpendicular to the given line AB; then because PB passes through the point (h, k), and is perpendicular to (1), its equation is
$$y - k = -\frac{1 + a\cos\omega}{a + \cos\omega}(x - h) \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Moreover, if we denote the point B by (x, y), and the distance PB by p, we have
$$p = \pm\sqrt{\{(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega\}} \ldots\ldots\ldots(3).$$

If, now, we eliminate x and y from these three equations,—the point (x,y) being common to all,—the resulting equation will contain p and known quantities.

Equating the values of y in (1) and (2), and subtracting ah from each side of the resulting equation, we get
$$x - h = \frac{(a+\cos\omega)(k-ah-b)}{a^2+2a\cos\omega+1}.$$

Consequently by (2),
$$y-k = -\frac{(1+a\cos\omega)(k-ah-b)}{a^2+2a\cos\omega+1}.$$

Hence by (3) the distance PB is
$$p = \pm\frac{(k-ah-b)\sin\omega}{\sqrt{(a^2+2a\cos\omega+1)}};$$
the upper or lower sign to be used according as the point P is *above* or *below* the given line AB. This interpretation of the double sign is thus explained:—

Let the ordinate PM at the point P meet AB in E; then at E the equation of AB gives (h being the abscissa at E),
$$EM = ah + b.$$

Hence when P is *above* AB, PM or k is greater than EM or $ah + b$; consequently $k-ah-b$ is *positive*, and therefore as p is essentially *positive**, the upper sign must be taken. If, however, P is *below* the line AB, PM or k is less than EM or $ah+b$, and hence $k-ah-b$ is negative; the lower sign must hence be taken to make p positive.

Similar reasoning applies wherever the point P is taken.

* As p is an absolute magnitude, it is not susceptible of opposition of direction, and hence it is essentially positive.

Sept. 27*th*, 1847.

DEVELOPMENT OF CIRCULAR FUNCTIONS BY MACLAURIN'S THEOREM.

[*Mr. Rutherford.*]

In the successive differentiation of functions of one variable, the labour of determining the expressions for the higher differential coefficients is, in many instances, almost insurmountable, and recourse is usually had to the method of indeterminate coefficients, expansion by the binomial theorem, *etc.*, to effect the development of the functions. We propose to develop one or two of these circular functions, to which Maclaurin's theorem is applicable, purely by successive differentiation, without having recourse to any of the ordinary algebraic methods of expansion, and to investigate the law by which any specified differential coefficient may be readily obtained.

1. Let $u = \tan x$.

Here $\dfrac{du}{dx} = u_1 = \sec^2 x$,

$$\frac{d^2u}{dx^2} = u_2 = 2\sec x.d\sec x = 2\sec^2 x\tan x = 2uu_1,$$

$$\frac{d^3u}{dx^3} = u_3 = 2\left(u\frac{du_1}{dx} + u_1\frac{du}{dx}\right) = 2(uu_2 + u^2{}_1),\text{ since }\frac{du}{dx} = u_1,\ etc.$$

Similarly, $\dfrac{d^4u}{dx^4} = u_4 = 2(uu_3 + 3u_1u_2)$,

$$\frac{d^5u}{dx^5} = u_5 = 2(uu_4 + 4u_1u_3 + 3u^2{}_2),$$

$$\frac{d^6u}{dx^6} = u_6 = 2(uu_5 + 5u_1u_4 + 10u_2u_3),$$

$$\frac{d^7u}{dx^7} = u_7 = 2(uu_6 + 6u_1u_5 + 15u_2u_4 + 10u^2{}_3),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{d^{n+1}u}{dx^{n+1}} = u_{n+1} = 2\left\{uu_n + nu_1u_{n-1} + \frac{n(n-1)}{1.2}u_2u_{n-2} + \frac{n(n-1)(n-2)}{1.2.3}u_3u_{n-3} + etc.\right\},$$

where the last term of the series within the brackets is

$$\frac{1}{2}\cdot\frac{n(n-1)(n-2)\ldots\ldots\{n-(\tfrac{1}{2}n-1)\}}{1.2.3\ldots\ldots\ldots\ldots\ldots\tfrac{1}{2}n}u^2{}_{\frac{1}{2}n},$$

$$\text{or}\quad \frac{n(n-1)(n-2)\ldots\ldots\{n-\tfrac{1}{2}(n-1)\}}{1.2.3\ldots\ldots\ldots\ldots\tfrac{1}{2}(n-1)}u_{\frac{1}{2}(n-1)}u_{\frac{1}{2}(n+1)};$$

according as n is an even or an odd number.

Now if U_0, U_1, U_2, *etc.* denote the values of u, and the differential coefficients u_1, u_2, *etc.* when $x = 0$, we shall have

$U_0 = 0,\quad U_1 = 1,\quad U_2 = 0,\quad U_3 = 2.1 = 2,\quad U_4 = 0,\quad U_6 = 0, etc.;$

$$\begin{aligned}
\therefore\quad U_5 &= 2.4\,U_1U_3 && =16\\
U_7 &= 2(6U_1U_5+10U^2{}_3) && =272\\
U_9 &= 2(8U_1U_7+56U_3U_5) && =7936\\
&\vdots && \vdots\\
U_{n+1} &= 2\left\{nU_1U_{n-1}+\frac{n(n-1)(n-2)}{1.2.3}U_3U_{n-3}+\right.\\
&\qquad\left.\frac{n(n-1)\,.\,.\,(n-4)}{1.2.\,.\,.\,.5}U_5U_{n-5}+etc.\right\}.
\end{aligned}$$

Hence by Maclaurin's theorem,

$$\tan x = x+\frac{x^3}{3}+\frac{2x^5}{3.5}+\frac{17x^7}{3^2.5.7}+\frac{62x^9}{3^2.5.7.9}+etc.,$$

which may be carried to any proposed extent by the method here employed.

2. Let $u=\sec x$.

Here $\dfrac{du}{dx}=u_1=\sec x\tan x=u\tan x,$

$$\frac{d^2u}{dx^2}=u_2=\frac{du}{dx}\tan x+u\sec^2x=u_1\tan x+u+u\tan^2x,$$

$$=2u_1\tan x+u,\qquad \text{since } u_1\tan x=u\tan^2x.$$

Similarly, $\dfrac{d^3u}{dx^3}=u_3=3u_2\tan x+3u_1-u\tan x,$

$$\frac{d^4u}{dx^4}=u_4=4u_3\tan x+6u_2-4u_1\tan x-u,$$

$$\frac{d^5u}{dx^5}=u_5=5u_4\tan x+10u_3-10u_2\tan x-5u_1+u\tan x,$$

$$\frac{d^6u}{dx^6}=u_6=6u_5\tan x+15u_4-20u_3\tan x-15u_2+6u_1\tan x+u,$$

$$\vdots\qquad\vdots\qquad\vdots$$

$$\frac{d^nu}{dx^n}=\left\{nu_{n-1}-\frac{n(n-1)(n-2)}{1.2.3}u_{n-3}+\frac{n(n-1)\,.\,.\,(n-4)}{1.2.\,.\,.\,.\,.\,.5}u_{n-5}-etc.\right\}\tan x$$

$$+\frac{n(n-1)}{1.2}u_{n-2}-\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}u_{n-4}+etc.$$

Here the law of the successive differential coefficients is sufficiently obvious, and when $x=0$, the values of all the even differential coefficients will be zero; hence we have $U_0=1$, $U_2=1$,

$$\begin{aligned}
U_4^1 &= 6U_2-U_0 && =5\\
U_6 &= 15U_4-15U_2+U_0 && =61\\
U_8 &= 28U_6-70U_4+28U_2-U_0 && =1385\\
U_{10} &= 45U_8-210U_6+210U_4-45U_2+U_0 && =50521,\\
&etc. \qquad\qquad etc.
\end{aligned}$$

which may be carried to any extent by the latter part of the preceding general expression for the n^{th} differential coefficient; hence

$$\sec x = 1 + \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2..6} + \frac{1385x^8}{1.2..8} + \frac{50521x^{10}}{1.2..10} + etc.$$

In the former of these developments, the successive differentiation offers no difficulty, but in the latter a slight artifice is required to transform the result of each differentiation into the proper form.

The artifice will be understood from the following illustration.

Differentiating the third differential coefficient, for example, we have

$$u_4 = 3u_3\tan x + 3u_2\sec^2 x + 3u_2 - u_1\tan x - u\sec^2 x, \text{ but}$$

$$u_3\tan x = \qquad + 3u_2\tan^2 x \qquad + 3u_1\tan x - u\tan^2 x, \text{ by 3rd diff. coeff.};$$

hence subtracting and transposing, we get at once

$$u_4 = 4u_3\tan x + 6u_2 - 4u_1\tan x - u,$$

which is the expression for the fourth differential coefficient, in the form here given; and in this manner the successive differentiation may be elegantly effected, the numerical coefficients being the same as those of the corresponding powers of a binomial expression.

Sept. 23rd, 1847.

MATHEMATICAL NOTE.

An angle and a point within it are given in position, and the major axis of a conic section in magnitude; it is required to place the curve so that its centre may be the given point, and the two lines be tangents to it.

We know that the perpendiculars from the foci on the tangents meet the tangents in the circumference of a circle described on the major axis as diameter; therefore, if with the given point as centre, and the semi-major axis as radius, we describe a circle, its intersections with the given tangents will be the feet of these perpendiculars. Hence through these points drawing lines at right angles to the tangents we get the foci by their intersections. Whence the curve is determined, for its major axis is the diameter through the foci, and if we determine on each tangent a point such that the lines to the foci may make equal angles with the tangents, these will be the points of contact.

As each perpendicular will intersect the two on the other tangent, there will in general be two solutions, namely two hyperbolas, or two ellipses, or an ellipse and an hyperbola, according as the two pairs of foci fall both out of the circle or both within it, or one within and the other without.

It may even happen that two of the intersections fall on the circumference, that is, the distance of the foci will be equal to the major axis; in which case either the straight line joining these two points or the two prolongations of this line will be the conic section, the first being an ellipse and the other an hyperbola, both curves having their minor axes equal to zero.

Cor. If any two chords are drawn in a circle, perpendiculars through their extremities will form a parallelogram whose diagonals will pass through the centre of the circle.

This corollary admits of a very simple elementary proof.

Jersey, Sept. 7th, 1847. H. G.

SOLUTIONS OF MATHEMATICAL EXERCISES.

*** The Editors regret that an elegant solution of Exercise 107, by Herr William von Holleben, pupil of Dr. August Wiegand, Halle, Germany, arrived too late for insertion in No. 6, Vol. II.

II. *By a Mathematician.*

Can the expression $(-1)^{h^{\frac{m}{n}}}$ be real for any values of h, m, and n? and if so discriminate the cases.

[Solution. *By R.*]

It appears evident that the given expression is real whenever $h^{\frac{m}{n}} = +1$, which holds when h, m, and n are each $= +1$; or when h is a finite affirmative quantity, and m the same, and n indefinitely great, or when $\frac{m}{n} = 0$, or whatever h may be, or when $h = \pm 1$, and $\frac{m}{n}$ is any even whole number, or more generally when $h^{\frac{m}{n}}$ is any whole number affirmative or negative.

Again, let $y = \cos x + \sqrt{-1} \sin x$, then differentiating, multiplying by $\sqrt{-1}$, and taking the integral, we have

$$c - x = \sqrt{-1} \log_e y = \sqrt{-1} \log_e (\cos x + \sqrt{-1} \sin x),$$

which when $x = 0$, becomes $c = \sqrt{-1} \log_e (1)$, so that

$$\sqrt{-1} \log_e (1) - x = \sqrt{-1} \log_e (\cos x + \sqrt{-1} \sin x);$$

$$\therefore -\frac{x}{\sqrt{-1}} = \log_e \left(\frac{\cos x + \sqrt{-1} \sin x}{1} \right).$$

When $x = \frac{\pi}{2}$, this equation becomes $-\frac{\pi}{2\sqrt{-1}} = \log_e \left(\frac{\sqrt{-1} \times 1}{1} \right)$,

$$\text{or } e^{-\frac{\pi}{2\sqrt{-1}}} = 1 \times \sqrt{-1}, \text{ or } e^{-\frac{\pi\sqrt{-1}}{2 \times -1}} = e^{\frac{\pi\sqrt{-1}}{2}} = 1 \times \sqrt{-1};$$

$$\therefore \quad e^{\pi\sqrt{-1}} = 1^2 \times -1; \text{ hence } e^{\pi\sqrt{-1} \times h^{\frac{m}{n}}} = (-1)^{h^{\frac{m}{n}}}.$$

Now if $h^{\frac{m}{n}} = p\pi\sqrt{-1}$, this equation becomes

$$e^{-p\pi^2} = (1)^{p\pi\sqrt{-1}}, \text{ or } \frac{1}{e^{p\pi^2}} = (-1)^{p\pi\sqrt{-1}}.$$

It thus appear that $(-1)^{h^{\frac{m}{n}}}$ is also real whenever $h^{\frac{m}{n}} = p\pi\sqrt{-1}$, when p may be any number affirmative or negative, whole or fractional.

CXI. *Mr. G. F. Lacey, Woolwich Common.*

Two perfectly smooth and hard planes, the one horizontal and the other vertical, meet each other; two equal balls whose elasticities are (e, e_1) are projected along the horizontal plane in parallel lines so as to impinge on the

vertical in the points E, E_1, then if e_1 be greater than e, their paths after reflexion will intersect in some point D. Shew that if the same balls be projected in parallel lines under any other angle of incidence, the points E,E_1 remaining the same, the locus of the successive intersections D_1,D_2, *etc.* of their paths corresponding to successive angles of incidence θ_1, θ_2, *etc.* .. will be a straight line perpendicular to the vertical plane; and that their distances on this line from the vertical plane will be to each other as $\cot\theta : \cot\theta_1 : \cot\theta_2$, *etc.*

[FIRST SOLUTION. *Mr. H. Godfray, Jersey; and similarly by Mr. W. Mortimer, Seedley, near Manchester, and Dr. Burns, Rochester.*]

From D let fall the perpendicular DH on the line EE_1, which joins the points E and E_1, and put $DEH=a$, and $DE_1H=a$. Take any points A,A_1 in the paths of the balls before they meet the vertical plane and draw A C, A_1C_1, perpendicular to EH; produce these perpendiculars till $\frac{CB}{CA}=e$, and $\frac{C_1B_1}{C_1A_1}=e_1$: then draw BE, B_1E_1, these lines produced will be the paths of the balls after reflexion, and therefore BED and B_1E_1D are straight lines.

Now $\qquad HD = EH\tan a = E_1H \tan a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$

But $EH\tan a = EH.\frac{BC}{EC} = EH.\frac{eAC}{EC} = EH.e\cot\theta.$ Similarly

$E_1H\tan a = (EH - EE_1)e_1\cot\theta.$ Substituting these in (1), we get

$$EH = EE_1\frac{e_1}{e_1-e},$$

which being constant shews that the point D will always be on the perpendicular HD.

Again, $\qquad HD = EH\tan a = EHe\cot\theta = EE_1\frac{ee_1}{e_1-e}\cot\theta;$

$\therefore$ HD varies as $\cot\theta$; *etc.*

[SECOND SOLUTION. *Messrs. Clarke, Gentleman Cadet, R. M. Academy, and Joseph Paul, Shooter's Hill; and similarly by Messrs. N. Smyth, of Galway, and J. W. Elliott, Greatham.*]

Let EE_1 be the axis of x and a perpendicular to this on the horizontal plane from E, the axis of y. Then if θ be the angle of incidence, the equations to the reflected paths will be

$$y = ex\cot\theta \ldots\ldots(1), \qquad y = e_1(x-a)\cot\theta \ldots\ldots (2).$$

a being put for EE_1.

At the point of intersection of (1) and (2) we have

$$x = \frac{e_1a}{e_1-e}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

consequently, the lines (1) and (2) intersect in a perpendicular (3) to EE_1, whatever be the value of θ.

From (1) and (3),

$$y = \frac{e_1ea}{e_1-e}\cot\theta,$$

and hence y varies as $\cot\theta$.

The truth of the theorem is therefore established.

Correct and elegant solutions were also received from the proposer, Mr. G. F. Lacey; and Mr. R. H. Wright.

CXII. *Mr. Fenwick.*

Let four points in a plane be so posited that one of them is within the triangle formed by joining the other three. Then having joined the points, taken in pairs, by six straight lines, these lines (produced if necessary) meet in three more points, which lie in an ellipse passing through the middle points of the six lines. Required proof.

*** The proposer's investigation of this exercise, both when the fourth point is within and without the triangle formed by joining the other three, is given at p. 290, Vol. II. In the latter case, it will be seen, the nine points lie in the hyperbola.

[FIRST SOLUTION. *Dr. Burns, Rochester.*]

Let A, B, C be three points (the student will readily sketch the fig.), and D a fourth point within the triangle ABC. Draw the lines as enunciated, so that E may be the intersection of AD with BC, F that of BD with AC, G that of CD with AB; let H, K, L, M, N, P be the middle points of AB, BC, CA, AD, BD and CD. Join HL, PN cutting AE in Q and R: join also NH, PL, and through M draw a line parallel to HL, meeting NH, PL in S and T.

It will be obvious from the bisected lines, that HN, AE, LP are parallel, and MQ=RE; and since N, H, M, L are points in an ellipse, the square of a diameter parallel to HL : square of a diameter parallel to HN :: MS.MT : HS.SN :: HQ.QL : MQ.QE; hence E is in the ellipse.—(*Leslie's Lines of the Second Order*, Book II. Prop. 5.)

In the same manner it may be shewn that F and G are points in the ellipse.

[SECOND SOLUTION. *Mr. Joseph Paul, Shooter's Hill; and similarly by Messrs. A. Clark, Gentleman Cadet; N. Smyth, Galway; and R. H. Wright, London.*]

Let the point D be within the triangle formed by joining the points A,B,C; join AD, BD, and CD, and produce them to meet the other lines in E, F and G; also bisect AB, BC, CA, AD, BD, and CD, in H,K,L,M,N,P.

Assuming CA and CB as axes of coordinates, and denoting the point A by $(2a,0)$, B by $(0,2b)$, and D by $(2c,2d)$, the other points will be denoted as follows:—

$$\text{H}\ (a,b);\ \text{K}\ (0,b);\ \text{L}\ (a,0);\ \text{M}\ (a+c,d);\ \text{N}\ (c,b+d);\ \text{P}\ (c,d);$$

$$\text{E}\left(0,\ \frac{2ad}{a-c}\right);\ \text{F}\left(\frac{2bc}{b-d},\ 0\right);\ \text{G}\left(\frac{2abc}{ad+bc},\ \frac{2abd}{ad+bc}\right);$$ and the equations to the lines LE and FK are

$$2d(x-a)+(a-c)y=0\ldots\ldots(1), \text{ and } (b-d)x+2c(y-b)=0\ldots\ldots(2).$$

Hence, as Professor Hearn has shewn (*Mathematician*, vol. II, p. 46), the equation to the conic section passing through the four points F,L,E,K is

$$\{2d(x-a)+(a-c)y\}.\{(b-d)x+2c(y-b)\}=kxy\ldots\ldots(3).$$

By substituting the coordinates of G in (3) the value of k is determined, and equation (3) becomes

$$d(a-x).(b-d)x-2cd(a-x).(b-y)+c(a-c).(b-y)y=0\ldots(4),$$

which is easily proved to be the equation to an ellipse when $ab > ad+bc$, that is, (as appears from a comparison of the coordinates of G and D), when D is within the triangle ABC.

An inspection of (4) will suggest that two values of x and y are (a,b) and (c,d), which are the coordinates of the points H and P; and since when c is substituted for x, $y = d$ or $b + d$, it appears that $(c,b+d)$ and $(a+c,d)$, which are the coordinates of N and M, also satisfy equation (4). Therefore the points E,F,G are in an ellipse which passes through the middle points of the six lines.

[THIRD SOLUTION. *Mr. W. Mortimer, Seedley.*]

Let the six lines and their middle points be as in the preceding solution. Then by a common property of triangles,

$$AG.BE.CF = BG.CE.AF.$$

Hence, $AH.AG.BK.BE.CF.CL = AL.AF.CE.CK.BG.BH.$

It follows therefore that an ellipse can pass through H,G,K,E,F,L.

Also, by transversals,

$$AG.BE.CD = AB.CE.GD, \text{ or}$$

$$AG.CD.BE.CG.BG.BK = AB.CE.GD.CG.BG.BK \ldots\ldots (1).$$

Join ML meeting CB, BA in U and V: then

$$\frac{CD}{CE} = \frac{UM}{UE}, \ \frac{CG}{BK} = \frac{2CG}{BC} = \frac{2UL}{UK}, \ BG = 2AH - 2AV$$

$$= 2VH, \ GD = 2VM, \ CG = 2VL, \ AG = 2VG, \ AB = 2AH.$$

Substituting these values in (1) we have

$$VG.VH.BK.BE.UL.UM = BG.BH.UK.UE.VL.VM.$$

Hence the ellipse passing through H, G, K, E, F, L, passes also through M. It may be proved in a similar way that it also passes through the points N and P.

Mr. H. W. Levy, Shalbourne, also favoured us with a good solution.

CXIII. *Mr. J. F. Heather, M.A., R.M. Academy.*

Two forces applied to a material particle, being resolved into components, which have their lines of direction along two coordinate axes, inclined to one another at an angle θ; it is required to find the position of one of these axes, so that the sum of the components along it shall be constant, whatever be the value of θ.

[SOLUTION. *Pen-and-Ink.*]

[This is a very neat porism, and as the solution of it under this aspect will somewhat illustrate the methods laid down for discussing such propositions in the first volume of this work, it is preferred to other solutions—of which several elegant ones have been received.]

Let p, q be the magnitudes of the given forces, β the angle which they make with each other, ϕ the angle made by p with the axis X, and θ the arbitrary (or porismatic) angle of ordination. Then the resolved parts of the forces in X are

$$X_p = \frac{p\sin(\theta - \phi)}{\sin\theta} = p\{\cos\phi - \sin\phi\cot\theta\},$$

$$X_q = \frac{q\sin\{\theta - (\phi+\beta)\}}{\sin\theta} = q\{\cos(\phi+\beta) - \sin(\phi+\beta)\cot\theta\}.$$

The sum of these is constant, but unknown, whatever be the value of θ: whence arranging in respect to $\cot\theta$, and equating the coefficients to zero, we have

$$q\cos(\phi+\beta)+p\cos\phi=x \ldots\ldots\ldots\ldots(1),$$
$$q\sin(\phi+\beta)+p\sin\phi=0 \ldots\ldots\ldots\ldots(2),$$

x being the unknown constant.

Add the squares of (1, 2): then

$$x^2=q^2+2pq\{\cos(\phi+\beta)\cos\phi+\sin(\phi+\beta)\sin\phi\}+p^2$$
$$=q^2+2pq\cos\beta+p^2 \ldots\ldots\ldots\ldots(3),$$

which gives for the value of x the resultant of the two forces.

Again, expand (2) and we obtain

$$\tan\phi=\frac{-q\sin\beta}{q\cos\beta+p}, \text{ or } \sin\phi=\frac{-q\sin\beta}{R}.$$

Whence the axis X coincides with the resultant in ***direction*** also; and the property is established.

Correct and elegant solutions were received from Dr. Burns, Rochester; and Messrs. Clarke, Gentleman Cadet, R. M. Academy; H. Godfray, Jersey; N. Smyth, Galway; and R. H. Wright, London.

CXVII. *Mr. Thomas Weddle, Newcastle-upon-Tyne.*

Let a closed curve be described on any surface, (the equation of the latter being $z=f(x, y)$,) and with the origin of coordinates as vertex and this curve as director, let a conical surface be described. If V denote the volume of the solid sector comprised between the two surfaces, and $x_0 y_0 z_0$ the coordinates of its centre of gravity, it is required to show that

$$V=\pm\tfrac{1}{3}\int\int(z-px-qy)dxdy,$$
$$V.x_0=\pm\tfrac{1}{4}\int\int z(z-px-qy)dxdy, \textit{ etc.}$$

where $z=f(x, y)$, and the partial differential coefficients, $p=\dfrac{dz}{dx}$, and $q=\dfrac{dz}{dy}$, are derived from $z=f(x, y)$. The integration must extend over all the elements, $dx\,dy$, of the space contained within the projection of the closed curve on the plane of xy.

[Solution. *Mr. Weddle, the proposer.*]

Let
$$w=\sqrt{1+p^2+q^2}.dx.dy \ldots\ldots\ldots\ldots(1),$$

be the element of the *base* of the sector at the point (x, y), and P the perpendicular from the origin on the tangent plane at the same point; the volume δV, of the elemental cone which has w for its base and the origin for its vertex, will be

$$\delta V=\tfrac{1}{3}P.w \ldots\ldots\ldots\ldots(2).$$

Now the equation to the tangent plane at (x, y) is

$$z'-z=p(x'-x)+q(y'-y) \ldots\ldots\ldots\ldots(3),$$

x', y' being the current coordinates. Hence

$$P=\pm\frac{z-px-qy}{\sqrt{(1+p^2+q^2)}} \ldots\ldots\ldots\ldots(4),$$

where the sign must be such that P may be positive. Hence (1, 2, 4) we have

$$\delta V = \pm \tfrac{1}{3}(z - px - qy)dx\,dy \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and integrating, $$V = \pm \tfrac{1}{3}\int\int(z - px - qy)dx\,dy \ldots\ldots\ldots\ldots(6).$$

Again, the centre of gravity of the elemental cone (2 or 5) is evidently $(\frac{3}{4}x, \frac{3}{4}y, \frac{3}{4}z)$, and hence we have $V.\overline{x} = \frac{3}{4}\int x\delta V$, *etc.*;

$$\left.\begin{array}{ll} \therefore\ (5) & V.\overline{x} = \pm \frac{1}{4}\int\int x(z-px-qy)dx\,dy \\ & V.\overline{y} = \pm \frac{1}{4}\int\int y(z-px-qy)dx\,dy \\ \text{and} & V.\overline{z} = \pm \frac{1}{4}\int\int z(z-px-qy)dx\,dy \end{array}\right\}\ldots\ldots(7).$$

It must be observed that in (6), $z-px-qy$ ought not to change sign within the limits of integration, for if so, P would also change sign (4), which is inadmissible. Now $z - px - qy$ evidently, (3), denotes the distance of the origin from the point in which the tangent plane intersects the axis of z; hence (6) will fail to be applicable when some of the tangent planes to the base of the sector intersect the axis of z on one side of the origin and some on the other. In this case we may divide the base, and consequently the sector, into two such parts that (6) may be applicable to each, and the sum of the volumes of the two parts will determine the whole volume. Similar observations are true of (7).

The determination of the volume and centre of gravity of such a solid as that here considered, will, however, be generally most conveniently effected by means of polar coordinates in the following manner.

Let (r, θ, ϕ) denote the polar coordinates of the surface, part of which constitutes the base of the solid sector. The equation to this surface may consequently be denoted by

$$r = F\ (\theta,\phi)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Here θ is the angle which r makes with a fixed straight line (OQ), drawn from the origin *within* the sector, and ϕ that which the plane of r and OQ makes with a fixed plane drawn through OQ. The equation to the curve may, since the vertex is taken as the origin, be denoted by

$$\theta = F'(\phi)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b).$$

Now if (ρ, θ, ϕ) be the coordinates of any element $\rho^2\sin\theta.d\rho.d\theta.d\phi$ of the solid sector, we shall have

$$V = \int\int\int \rho^2 \sin\theta.\,d\rho.d\theta.d\phi,$$

and integrating with respect to ρ between the limits 0 and r, we have

$$V = \tfrac{1}{3}\int\int r^3\sin\theta.d\theta.d\phi\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c).$$

Also if OQ be taken as the axis of z, and the fixed plane as that of xz, then, recollecting that $x=\rho\sin\theta.\cos\phi$, $y=\rho\sin\theta.\sin\phi$, $z=\rho\cos\theta$, we get

$$V.\overline{x} = \int\int\int \rho^3\sin^2\theta\cos\phi.d\rho.d\theta.d\phi = \tfrac{1}{4}\int\int r^4\sin^2\theta.\cos\phi.d\theta.d\phi,\ etc.;$$

$$\left.\begin{array}{l} \therefore\ V\,\overline{x} = \frac{1}{4}\int\int r^4\sin^2\theta.\cos\phi.d\theta.d\phi \\ V\,\overline{y} = \frac{1}{4}\int\int r^4\sin^2\theta.\sin\phi.d\theta.d\phi \\ V\,\overline{z} = \frac{1}{4}\int\int r^4\sin\theta.\cos\theta.d\theta.d\phi \end{array}\right\}\ldots\ldots\ldots\ldots(d).$$

r must be eliminated from (c) and (d) by means of (a); and the integration must be performed, first with respect to θ, between the limits 0 and $F'(\phi)$, and then with respect to ϕ between the limits 0 and 2π.

CXVIII. *Mr. S. Barry, City School, London.*

Given the base of a triangle, and the angle subtended by it at the centre of gravity of the triangle; to find the locus of the vertex.

[FIRST SOLUTION. *Mr. Hugh Godfray, David Place, St. Mark's Road, Jersey; and similarly by Dr. Burns, Rochester, and Mr. Nicholas Smyth, Galway.*]

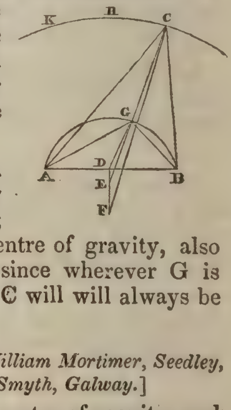
On the given base describe a segment of a circle capable of the given angle, and from the middle point of the base draw any line cutting the arc. Then taking on this line outside the arc a distance double of the inner portion, we evidently get a point of the locus.

Now if through this point we draw lines respectively parallel to those joining the extremities of the base with the point of section of the arc and line, they will, as is evident from the similar triangles, cut off, on the base produced, distances equal to the base itself, and form, with one another, an angle equal to the given angle.

Hence the required locus is the arc of a segment containing an angle equal to the given angle, and described on the line formed by producing the base a distance equal to itself, on each side.

[SECOND SOLUTION. *Messrs. Joseph Paul, Shooter's Hill; Patrick Kincella, Leighlinbridge, Ireland; and R. Harley, Mathematical Master, Mr. Hoole's Academy, Blackburn.*]

On AB the given base describe a segment of a circle AGB containing an angle equal to the given angle; through E, the centre of that segment, draw DF perpendicular to AB, and in it take DF the triple of DE; from D draw DC cutting the circumference in G and making any angle with DF, join GE, and and through F draw FC parallel to EG and meeting DC in C, the circle CHK described with the centre F, and radius FC, shall be the locus required.



For since DEG and DFC are similar triangles, and DE is one third of DF; DG and EG are respectively one third of DC and FC; also AB is bisected in D; therefore ACB is the triangle of which G is the centre of gravity, also the base AB subtends at G the given angle: and since wherever G is taken in the arc ACB, FC is triple of EG, the vertex C will will always be in the circle CHK.

[THIRD SOLUTION. *Lieutenant A. R. Clarke, R.E.; Messrs. William Mortimer, Seedley, Manchester; J. W. Elliott, Greatham; and Nicholas Smyth, Galway.*]

Let ABC be the triangle, AB the given base, G the centre of gravity, and AGB the given angle. Since AB and angle AGB are constant, it is evident that the locus of G is a segment of a circle.

Let x_1y_1 be the coordinates of G, and xy those of C; then $x=3x_1$, and $y=3y_1$. Let the equation of the locus of G be

$$(y_1+a)^2+x_1^2=r^2 \ldots\ldots\ldots\ldots (1),$$

then since $x_1=\dfrac{x}{3}$ and $y_1=\dfrac{y}{3}$, we get from (1)

$$(y+3a)^2+x^2=9r^2, \text{ or } (y+a')^2+x^2=r'^2 \ldots\ldots\ldots\ldots (2),$$

which is the equation of a circle, and it is also evident that the segment described by C is *similar* to that described by G, since a' and r' are equimultiples of a and r.

CXIX. *Mr. A. R. Clarke, Gentleman Cadet, R. M. Academy.*

Let straight lines be drawn through the angular points of a plane triangle ABC perpendicular to the lines bisecting these angles; if Δ, P be the area and perimeter of the original triangle, and Δ_1, P_1 those of the new triangle, then

$$(1)\quad 4\Delta\Delta_1 = Pabc$$

$$(2)\quad PP_1 = 4\Delta_1(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C).$$

[FIRST SOLUTION. *Messrs. Hugh Godfray, Jersey; Joseph Paul, Shooter's Hill; A. Roy, Dundee Academy; Nicholas Smyth, Galway; and Patrick Kincella, Leighlinbridge.*]

Let ABC be the original triangle, and $O_1O_2O_3$ the new triangle; then because the sides of ABC are equally inclined to those of $O_1O_2O_3$ at the points A,B,C, the angles O_3, O_2, and O_1 are equal to $\frac{1}{2}(A+B)$, $\frac{1}{2}(A+C)$ and $\frac{1}{2}(B+C)$, and the sines of these angles are equal to the cosines of $\frac{1}{2}C$, $\frac{1}{2}B$ and $\frac{1}{2}A$ respectively.

Now $\quad O_1C = a\dfrac{\cos\frac{1}{2}B}{\cos\frac{1}{2}A}$, and $O_2C = b\dfrac{\cos\frac{1}{2}A}{\cos\frac{1}{2}B}$; therefore

$$O_1O_2 = a\frac{\cos\frac{1}{2}B}{\cos\frac{1}{2}A} + b\frac{\cos\frac{1}{2}B}{\cos\frac{1}{2}A} = \frac{a\cos^2\frac{1}{2}B + b\cos^2\frac{1}{2}A}{\cos\frac{1}{2}A\cos\frac{1}{2}B};$$

but $a\cos^2\frac{1}{2}B + b\cos^2\frac{1}{2}A = \frac{1}{2}a(1+\cos B) + \frac{1}{2}b(1+\cos A)$

$$= \frac{a+b+a\cos B + b\cos A}{2} = \frac{a+b+c}{2} = \frac{P}{2};$$

$$\therefore O_1O_2 = \frac{P}{2\cos\frac{1}{2}A\cos\frac{1}{2}B}.\ \text{Sim. } O_2O_3 = \frac{P}{2\cos\frac{1}{2}B\cos\frac{1}{2}C},\ \text{and } O_1O_3 = \frac{P}{2\cos\frac{1}{2}A\cos\frac{1}{2}C};$$

$$\therefore\ 2\Delta_1 = O_1O_2.O_1O_3\sin O_2O_1O_3 = \frac{P^2}{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} = \frac{Pabc}{2\Delta};$$

and consequently, $\quad 4\Delta\Delta_1 = Pabc \ldots\ldots\ldots\ldots\ldots\ldots(1).$

Again,

$$P_1 = O_1O_2 + O_2O_3 + O_3O_1 = \frac{P}{2}\left\{\frac{1}{\cos\frac{1}{2}A\cos\frac{1}{2}B} + \frac{1}{\cos\frac{1}{2}B\cos\frac{1}{2}C} + \frac{1}{\cos\frac{1}{2}A\cos\frac{1}{2}C}\right\}$$

$$= \frac{P}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}\left\{\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C\right\}$$

$$= \frac{4\Delta_1}{P}\left(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C\right):$$

$$\therefore\ PP_1 = 4\Delta_1(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C)\ldots\ldots\ldots\ldots(2).$$

[SECOND SOLUTION. *Messrs. R. Harley, Blackburn; J. W. Elliott, Greatham; William Mortimer, Seedley; and Myconius, Till-side.*]

The angular points of the new triangles are known to be the centres of the escribed circles of the original triangle. Hence denoting the radii of the circumscribed and inscribed circles of the triangle ABC by R and r respectively, we have, (Diary for 1842, page 88, and known properties,)

$$r\Delta_1 = 2R\Delta;\quad \tfrac{1}{2}P.r = \Delta;\ \text{ and }\ 4R\Delta = abc.$$

Whence eliminating R and r, we obtain at once

$$4\Delta.\Delta_1 = P.abc\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again let as usual O_1, O_2, O_3 denote the centres, and r_1, r_2, r_3 the corresponding radii of the circles escribed to the triangle ABC, then (Diary for 1845, page 79, eq. 151,)

$$O_2O_3 = 4R\cos\tfrac{1}{2}A\,; \quad O_1O_3 = 4R\cos\tfrac{1}{2}B\,; \quad O_1O_2 = 4R\cos\tfrac{1}{2}C\,;$$

$$\therefore\ P_1 = 4R(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C)\,;\ \text{but } 4R = \frac{abc}{\Delta} = (\text{by } 1)\,\frac{4\Delta_1}{P}\,;\ \text{hence}$$

$$PP_1 = 4\Delta_1(\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C).$$

Solutions were sent by Mr. D. Maver, and Lieutenant A. R. Clarke, R. E.

CXX. *Mr. John Taylor, Woolwich Common.*

Let ABC be a plane triangle; DEF the triangle formed by joining the points where perpendiculars from the angles A, B, C meet the opposite sides; O the centre of the circumscribed circle, and R its radius; P the intersection of the perpendiculars; and ρ the radius of the circle inscribed in DEF; then there exists the following relation,

$$AP^2 + BP^2 + CP^2 + OP^2 = 5R^2 \pm 8R\rho,$$

the upper or lower sign being taken according as the triangle ABC is obtuse or acute.

[FIRST SOLUTION. *Messrs. R. Harley, Blackburn; William Mortimer, Seedley; and Myconius, Till-Side.*]

Let Δ and Δ_1 denote respectively the areas of the triangles ABC and DEF. Then if we suppose the triangle ABC to be acute, we shall have (Horæ Geom. XXVIII and XXXI, Diary for 1842; and page 70, "Mathematician, Vol. I.),

$$\Delta_1 = \frac{\rho}{R}\,\Delta\,; \text{ and } \frac{4\Delta_1}{\Delta} = \frac{R^2 - PO^2}{R^2}\,;$$

$$\therefore \quad PO^2 = R^2 - 4R\rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also (Horæ Geom. XVIII, Diary for 1835),

$$AP^2 + BP^2 + CP^2 = 4R^2 - 4R\rho \ldots\ldots\ldots\ldots (2).$$

Now when the triangle ABC is obtuse, the point P will obviously be *without* the triangle, and consequently the circle, centre P, will be *escribed* to the triangle DEF. Hence in this case the radius ρ will have a contrary sign to that of the inscribed circle, and equations (1) and (2) become respectively

$$PO^2 = R^2 + 4R\rho\,;$$

and $$AP^2 + BP^2 + CP^2 = 4R^2 + 4R\rho.$$

Therefore, by addition, we obtain the relation in question: viz.

$$AP^2 + BP^2 + CP^2 + OP^2 = 5R^2 \pm 8R\rho.$$

Cor. By subtraction we get

$$AP^2 + BP^2 + CP^2 - OP^2 = 3R^2.$$

[SECOND SOLUTION. *Mr. J. W. Elliott, Greatham.*]

Let the triangle ABC be acute: then the point P is *within* ABC, and the points A, B, C and P are the centres of the circles of contact of the triangle DEF; also R is equal to the diameter of the circle circumscribing DEF, (Diary for 1842, p. 80).

Now we have the well known properties,

$$AB^2+BC^2+CA^2=8R^2+4R\rho \ldots\ldots\ldots\ldots (1)$$

$$AP^2+BP^2+CP^2=4R^2-4R\rho \ldots\ldots\ldots\ldots (2)$$

$$AB^2+BC^2+CA^2+OP^2=9R^2. \ldots\ldots\ldots\ldots (3).$$

Taking (1) from (3): then

$$OP^2=R^2-4R\rho \ldots\ldots\ldots\ldots (4).$$

Hence, adding (2) and (4),

$$AP^2+BP^2+CP^2+OP^2=5R^2-8R\rho,$$

which proves the first case.

When the triangle ABC is obtuse, C being the obtuse angle: then the point P is *without* the triangle ABC, and C is the point of intersection of perpendiculars of the triangle ABP; hence the following properties,

$$AP^2+BP^2-CP^2=4R^2+4R\rho,$$

$$2CP^2+OP^2= R^2+4R\rho,$$

which being added, proves the second case.

Solutions were sent by Messrs. Godfray, Maver, Paul, Roy, Smyth, and Lieutenant Clarke, R.E.

CXXI. *Mr. J. J. Walker, Trinity College, Dublin.*

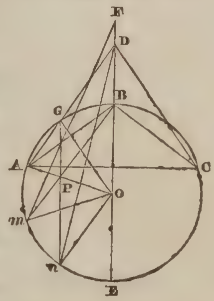
Let ABC, ADC be two isosceles triangles on the base AC, and on the same side of it, having their vertical angles as 3 : 2. Let O be the centre of the circle circumscribing the triangle ABC: draw the diameter BOE; in OBD produced take BF = BO; and let the arcs Am, An be respectively one third and two thirds of the arc AE: prove that

$$Dn = Fm.$$

[First Solution. *Mr. Hugh Godfray, Jersey.*]

Join Bm, Om, On and OA, and let OA cut Bm in P; then the two angles EBm and EDA are each two-thirds of EBA; therefore Bm is parallel to DA. Also On is parallel to Bm. Now the angle PmO = mBO=nOE=mOA; hence PmO is an isosceles triangle, and therefore similar to the isosceles triangle mOB, the angle OmB being common to both. Hence

$$OP : Om :: Om : Bm;$$



but on account of the parallels BP and DA,

$$OP : OA :: OB : OD;$$

therefore OD=Bm, since Om = OA = OB. We have also On = BF, and the angle nOD = FBm; therefore Dn = Fm.

[Second Solution. *Mr. David Maver, Bon-Accord School, Aberdeen.*]

Let AD cut the circle in G, and join Gn, Bm, GO, AO, mO, nO.

Since 3ADE = 2ABE = AOE =3nOE; therefore nO is parallel to AD. And since ADE = nOE = $\frac{1}{2}m$OE = mBE; therefore mB is parallel to AG, and GB = Am = nE; hence Gn is parallel to DE, and nGDO is a parallelogram;

$$\therefore \quad OG^2+Dn^2 = Gn^2+GD^2+DO^2+On^2 = 2nG^2+2nO^2$$
$$= 2mB^2+2BO^2, \text{ (because } nG=mB\text{, being the chords of equal arcs)}$$
$$= Fm^2+mO^2, \text{ since } FB = BO \text{ by hypothesis;}$$
$$\therefore \quad Dn^2 = Fm^2, \text{ and } Dn = Fm.$$

[THIRD SOLUTION. *Mr. William Mortimer, Seedley; and similarly by Mr. Joseph Paul, Shooter's Hill.*]

Let DA intersect the circle in G; join OG, Om, On, Bm and Gn; then angle ABE=BDA+DAB=$\frac{3}{2}$BDA; therefore DAB $=\frac{1}{2}$BDA $=\frac{1}{3}$ABE= ABm; hence Bm is parallel to AD. And since ADB $=\frac{2}{3}$ABE =AGn; therefore Gn is parallel to DE; hence it follows that angle DGn = FBm. And because GOB = 2GAB = GDO; therefore GD = GO = BF, and also Bm=Gn, since arc Bm = arc Gn; hence the triangles BmF and DGn are equal in all respects; therefore Dn = Fm.

Dr. BURNS, of Rochester, gives the following demonstration.

Since AGn $=\frac{2}{3}$ABE = ADE =nOE; therefore nG=OD=mB; hence DO, On and the angle DOn are respectively equal to mB, BF and the angle mBF; therefore Dn = Fm.

Solutions were sent by Messrs. R. Harley; Nicholas Smyth; A. Roy, Dundee Academy; and Myconius, Till-side.

CXXII. *Mr. John Riddle, Nautical School, Greenwich.*

In a spherical triangle ABC, right angled at B, given the sum of the two sides a and c equal to a quadrant, and the measure of the angle C equal to twice one of these sides, to determine the triangle.

[FIRST SOLUTION. *Mr. Joseph Paul, Shooter's Hill.*]

Since $\quad C = 2a$ and $c = 90° - a$,

$$\therefore \quad \sin a (=\cot C \tan c) = \cot 2a \cot a = \frac{1-\tan^2 a}{2\tan^2 a} = \frac{1-2\sin^2 a}{2\sin^2 a};$$

$$\therefore \quad 2\sin^3 a + 2\sin^2 a = 1.$$

The solution of which equation gives $\sin a = \cdot 56519772$;

$$\therefore \quad a = 34° \ 24' \ 57\cdot 7'',$$
$$c = 55° \ 35' \ 2\cdot 3'',$$
$$\text{and } \angle C = 68° \ 49' \ 55\cdot 4'',$$

from which data we obtain by the usual method

$$\angle A = 39° \ 42' \ 34\cdot 4'',$$
$$\text{and} \quad b = 62° \ 12' \ 28\cdot 9''.$$

[SECOND SOLUTION. *Mr. R. Harley, Blackburn.*]

Let the angle $C = 2a$, then we shall have

$$\sin a = \cot C \tan c = \cot 2a \tan\left(\frac{\pi}{2} - a\right);$$

$$\text{or } \sin a = \cot 2a \cot a = \frac{\cos 2a}{\sin 2a} \cdot \frac{\cos a}{\sin a};$$

$$\sin a = \frac{1-2\sin 2a}{2\sin a \cos a} \cdot \frac{\cos a}{\sin a} = \frac{1-2\sin^2 a}{2\sin^2 a};$$

hence $\sin^3 a = \frac{1}{2} - \sin^2 a$; or, $\sin^3 a + \sin^2 a = \frac{1}{2}$; whence we find

$$a = \sin^{-1} \cdot 565193 = 34° \ 24' \ 56'';$$

hence $\quad c = \frac{\pi}{2} - a = 55° \ 35' \ 4''$; and $C = 2a = 68° \ 49' \ 52''$.

[THIRD SOLUTION. *Mr. John Riddle, the proposer.*]

$$\sin a = \tan c . \cot C,$$

$$\therefore \quad \sin a = \cot a . \cot 2a$$

$$= \frac{\cos a}{\sin a} . \frac{\cos 2a}{\sin 2a}$$

$$= \frac{\cos a . \cos 2a}{\sin a . 2 \sin a . \cos a}$$

$$= \frac{\cos 2a}{2 \sin^2 a};$$

$$\therefore \quad 2 \sin^3 a = \cos 2a = 1 - 2 \sin^2 a,$$

$$2 \sin^3 a + 2 \sin^2 a = 1, \text{ or } \sin^3 a + \sin^2 a = \tfrac{1}{2}.$$

Let $\sin a = x$, then $x^3 + x^2 = \frac{1}{2}$, and substituting $y - \frac{1}{3}$ for x we have

$$y^3 - \frac{y}{3} = \frac{23}{54}.$$

Comparing this with $y^3 + py = q$, we have $p = -\frac{1}{3}$ and $q = \dfrac{23}{54}$.

Now putting $\left\{\dfrac{q}{2} + \sqrt{\left(\dfrac{q}{2}\right)^2 + \left(\dfrac{p}{3}\right)^3}\right\}^{\frac{1}{3}} = n$, we have by Cardan's rule,

$$y = n - \frac{p}{3n} = n + \frac{1}{9n}.$$

Making the proper substitutions, we readily get

$$n = \tfrac{1}{3}\left\{\tfrac{1}{4}(23 + 3\sqrt{57})\right\}^{\frac{1}{3}}.$$

$$\text{Hence } y = \tfrac{1}{3}\left\{\tfrac{1}{4}(23 + 3\sqrt{57})\right\}^{\frac{1}{3}} + \frac{1}{3\left\{\frac{1}{4}(23 + 3\sqrt{57})\right\}^{\frac{1}{3}}}.$$

Computation of y by logs.

log 57 .... 2(1·7558749

7·549835 .... ·8779374

3

22·649505

23

4)45·649505

11·4123762 log .... 3)1·0573759

3) 2·251430 .... ·3524586

$n =$ ·750477

3 log 1....0·

6·754290 log .... ·8295797

$\dfrac{1}{9n} =$ ·148054 $-$1·1704203

$y =$ ·898531 $= x + \frac{1}{3} = x +$ ·333...... ;

$\therefore\ x =$ ·565198 $=$ nat. sin a.

$$\begin{aligned} &\text{Whence} & \log\sin a &= 9{\cdot}7522006, \\ &\text{and} & a &= 34^\circ\ 24'\ 57{\cdot}7'', \\ &\text{and} & c = 90^\circ - a &= 55^\circ\ 35'\ \ 2{\cdot}3''. \end{aligned}$$

For proof of the correctness of these values of a and C, it only remains to see if they will give such a value of C as will fulfil the equation $C = 2a$.

$$\begin{aligned} \sin a &= \tan c.\cot C, \\ \text{or} \qquad \sin a &= \cot a.\cot C; \\ \therefore \qquad \cot C &= \sin a.\tan a. \\ \log\sin a &= 9{\cdot}7522006 \\ \log\tan a &= \underline{9{\cdot}8357696} \\ \angle C\ 68^\circ\ 49'\ 55{\cdot}4''\cot\ & \ \ 9{\cdot}5879702 \end{aligned}$$

Which is $= 2a$ to the tenth of a second.

For b we have $\cos b = \cos a.\cos c = \cos a.\sin a$.

$$\begin{aligned} \log\sin a &= 9{\cdot}7522006 \\ \cos a &= \underline{9{\cdot}9164305} \\ b = 62^\circ\ 12'\ 28{\cdot}8''\ldots\ldots\ & \ \ 9{\cdot}6686311 \end{aligned}$$

For A we have $\sin a.\sin C = \sin c.\sin A$,

$$\begin{aligned} \sin a.\sin 2a &= \cos a.\sin A, \\ \sin A &= \frac{\sin a.\sin 2a}{\cos a} = 2\sin^2 a; \\ \log\sin a &= \underline{9{\cdot}7522006} \\ \log\sin^2 a &= 9{\cdot}5044012 \\ \log 2 &= \underline{\ \ 3010300} \\ A = 39^\circ\ 42'\ 34{\cdot}8''\ldots\ & \ \ 9{\cdot}8054312 \end{aligned}$$

Good solutions were also received from Dr. Burns, and Messrs. W. Mortimer, A. Roy, and N. Smyth.

CXXIII. *Mr. Hugh Godfray, St. Mark's Road, Jersey.*

If $\frac{N}{N'}, \frac{P}{P'}, \frac{Q}{Q'}$ be three consecutive convergents, q the quotient corresponding to $\frac{Q}{Q'}$, , and $\frac{1}{x}$ the value of the remaining portion $\frac{1}{r} + \frac{1}{s} +$ *etc.* of the continued fraction, and if we form the $q - 1$ secondary convergents between $\frac{N}{N'}$ and $\frac{Q}{Q'}$, namely,

$$\frac{N+P}{N'+P'},\ \frac{N+2P}{N'+2P'},\ \frac{N+3P}{N'+3P'},\ldots\ldots\ \frac{N+(q-1)P}{N'+(q-1)P'},$$

it is required to determine which of these secondaries approach nearer to the total value of the continued fraction than the simpler principal convergent $\frac{P}{P'}$.

[SOLUTION. *Mr. Hugh Godfray, the proposer.*]

Let $\quad X = a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \ldots . \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} + etc.;$

then by the theory of convergents,

$$\frac{Q}{Q'} = \frac{Pq + N}{P'q + N'}, \quad \text{and } X = \frac{P\left(q + \frac{1}{x}\right) + N}{P'\left(q + \frac{1}{x}\right) + N'}, \text{ and } X - \frac{P}{P'} =$$

$$\frac{P\left(q + \frac{1}{x}\right) + N}{P'\left(q + \frac{1}{x}\right) + N'} - \frac{P}{P'} = \frac{NP' - PN'}{P'\left[P'\left(q + \frac{1}{x}\right) + N'\right]} = \frac{\pm 1}{P'\left[P'\left(q + \frac{1}{x}\right) + N'\right]} = \Delta.$$

Now taking any one of the secondary fractions $\frac{N + Ph}{N' + P'h}$, h being less than q we have

$$\frac{N + Ph}{N' + P'h} - X = \frac{N + Ph}{N' + P'h} - \frac{P\left(q + \frac{1}{x}\right) + N}{P'\left(q + \frac{1}{x}\right) + N'}$$

$$= \frac{\left(NP' - PN'\right)\left(q + \frac{1}{x} - h\right)}{\left(N' + P'h\right)\left[P'\left(q + \frac{1}{x}\right) + N'\right]} = \frac{q + \frac{1}{x} - h}{N' + P'h} P'\Delta = \Delta.h;$$

$\therefore \Delta h < = > \Delta$ according as $N' + P'h > = < P'\left(q + \frac{1}{x} - h\right)$, or according as $2h > = < q + \frac{1}{x} - \frac{N'}{P'}$.

Now $\frac{1}{x}$ and $\frac{N'}{P'}$ are both proper fractions; their difference is therefore also a proper fraction: hence when q is even, Δh will be $<$ or $>$ Δ according as $h >$ or $< \frac{1}{2}q$: when $h = \frac{1}{2}q$, Δh will be less or greater than Δ according as $\frac{1}{x}$ is less or greater than $\frac{N'}{P'}$. When q is odd, the conditions are the same, but h cannot then be equal to $\frac{1}{2}q$, since h must be an integer.

Hence we see that in all cases, the first half of the secondary fractions (leaving out the middle one when their number is odd) will be nearer to, and the second half farther from the true value of the whole fraction than the simpler principal convergent $\frac{P}{P'}$.

As to the middle secondary when the number is odd it being more or less approximative than $\frac{P}{P'}$, depends as we have seen on $\frac{1}{x}$ being less or greater

than $\frac{N'}{P'}$, and as $\frac{1}{x}$ is comprised between $\frac{1}{r}$ and $\frac{1}{r+1}$, it will generally be sufficient to examine whether $\frac{N'}{P'} > \frac{1}{r}$ or $< \frac{1}{r+1}$. Should $\frac{N'}{P'}$ fall within these limits, we must go one step further, and compare $\frac{N'}{P'}$ with $\frac{1}{r} + \frac{1}{s}$ and $\frac{1}{r} + \frac{1}{s+1}$, and so on.

It may even happen that $\frac{N'}{P'} = \frac{1}{x}$, in which case $\frac{P}{P'}$ and the middle secondary fraction will be equally approximative, one being greater and the other less than the true value.*

The following theorem and remarks, which are intimately connected with the above, will perhaps be properly placed here.

THEOREM.

If we write out in the order of their denominators the whole series of fractions, both principal and secondary (with the exception of those secondaries which are to be cancelled by the preceding rule,) then will each of these fractions express the quantity X more correctly than any other fraction whatever whose denominator is less than that of the *succeeding* fraction in the series.

For it will be seen that the series is composed of sets of fractions alternately greater and less than the true value, each set being terminated by a principal convergent, thus

$$\frac{N}{N'}\Bigg|^{g} \cdots\cdots \frac{P}{P'}\Bigg| \frac{N+Ph}{N'+P'h}, \frac{N+P(h+1)}{N'+P'(h+1)}, \frac{N+P(h+2)}{N'+P'(h+2)}, \cdots \frac{Q}{Q'}\Bigg| \textit{etc.}$$

Let us firstly consider two consecutive fractions of one set $\frac{N+Pr}{N'+P'r}$, and $\frac{N+P(r+1)}{N'+P'(r+1)}$ for instance. Let $\frac{m}{n}$ be any fraction approximating more nearly than $\frac{N+Pr}{N'+P'r}$. Now this last fraction being one of the secondaries to be kept is nearer to X than $\frac{P}{P'}$, which is moreover situated in value on the other side of X; therefore $\frac{m}{n}$ must fall between $\frac{N+Pr}{N'+P'r}$ and $\frac{P}{P'}$, or, dividing this interval into two parts, $\frac{m}{n}$ must fall between $\frac{N+Pr}{N'+P'r}$ and $\frac{N+P(r+1)}{N'+P'(r+1)}$, or between $\frac{N+P(r+1)}{N'+P'(r+1)}$ and $\frac{P}{P'}$. But to fall in the

* Bonnycastle has therefore committed an oversight in his Algebra, Vol. I., Page 293, where he says, speaking of the secondary fractions, "but each of them is nearer the truth than any other fraction *whatever* that can be expressed in more simple terms;" and again, Vol. II., Page 220, "Each of which approaches nearer to the true value of X than any other fraction having a less denominator."

first interval it must differ from the first fraction by a quantity smaller than that by which the two fractions differ from each other; that is, we must have

$$\frac{m}{n} - \frac{N+Pr}{N'+P'r} < \frac{N+P(r+1)}{N'+P'(r+1)} - \frac{N+Pr}{N'+P'r},$$

or $$\frac{m(N'+P'r)-n(N+Pr)}{n(N'+P'r)} < \frac{1}{\{N'+P'(r+1)\}(N'+P'r)};$$

which inequality can be verified only* when $n \gt N'+P'(r+1)$.

Similarly if $\frac{m}{n}$ falls between $\frac{N+P(r+1)}{N'+P'(r+1)}$ and $\frac{P}{P'}$, we must have

$$\frac{P}{P'} - \frac{m}{n} < \frac{P}{P'} - \frac{N+P(r+1)}{N'+P'(r+1)},$$

or $$\frac{nP-mP'}{nP'} < \frac{1}{P'\{N'+P'(r+1)\}};$$

which again † requires n to be greater than $N'+P'(r+1)$, therefore the proposition is proved in the case of two consecutive fractions on the same side of X.

Next consider the last fraction of one set and the first of the next, $\frac{P}{P'}$ and $\frac{N+Ph}{N'+P'h}$, for instance, which are on opposite sides of X; and let $\frac{m}{n}$ be a fraction approximating more closely to X than $\frac{P}{P'}$. Then $\frac{m}{n}$ must fall between $\frac{P}{P'}$ and $\frac{N+P(h-1)}{N'+P'(h-1)}$, this latter fraction having been cancelled as not being so approximative as $\frac{P}{P'}$. This interval we may again divide into two, namely, from $\frac{N+P(h-1)}{N'+P'(h-1)}$ to $\frac{N+Ph}{N'+P'h}$, and from $\frac{N+Ph}{N'+P'h}$ to $\frac{P}{P'}$, and a precisely similar investigation to the preceding will

* We may, it is true, have $m(N'+P'r)-n(N+Pr)=0$, but this will merely give $\frac{m}{n}=\frac{N+Pr}{N'+P'r}$, and not nearer to X. Nor need we suppose the numerator $m(N'+P'r)-n(N+Pr)$ to be negative, for by writing $\frac{m}{n}-\frac{N+Pr}{N'+P'r}$, we have tacitly assumed the two fractions to belong to a set less than X, and therefore increasing. Should this not be the case, we have merely to write $\frac{N+Pr}{N'+P'r}-\frac{m}{n} < \frac{N+Pr}{N'+P'r}-\frac{N+P(r+1)}{N'+P'(r+1)}$, and the same conclusion will follow.

† Here also we may have $nP-mP'=0$, which gives $\frac{m}{n}=\frac{P}{P'}$, and therefore less approximative than $\frac{N+Pr}{N'+P'r}$.

shew that no fraction can approximate more nearly than $\frac{P}{P'}$, if its denominator is less than $N'+P'h$.

Hence in all cases the proposition is true.

It follows evidently from what precedes, that the numerator and denominator of any two consecutive fractions, cross multiplied, will give results differing by unity, and that this property holds for the last fraction of one set and any one of the fractions of the following set.

Scholium. A continued fraction being given, we are now able to write out a series of convergents such that no fraction *whatever*, not belonging to the series, can express the true value of the quantity more closely than some one *simpler* fraction of the series.

It is to be hoped that this investigation will remove all objection * to the introduction of secondary convergents in our class books, where even the principal convergents have hitherto, with difficulty, obtained a place.

For the sake of illustration we subjoin the following example of the application of these rules :

Let it be required to find *all* the fractions converging to the fraction $\frac{1103}{256}$, and such that any fraction not belonging to the series may be less approximative than some fraction of the series expressed in simpler terms.

We have by the ordinary method,

$$\frac{1103}{256}=4+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{3},$$

which fraction gives the series of principal convergents,

$$\begin{array}{cccccc} & 4 & 3 & 4 & 6 & 3 \\ \frac{1}{0}, & \frac{4}{1}, & \frac{13}{3}, & \frac{56}{13}, & \frac{349}{81}, & \frac{1103}{256}. \end{array}$$

Now inserting the secondaries, we have between $\frac{1}{0}$ and $\frac{13}{3}$ two secondaries, and therefore only one to be kept, viz. $\frac{4(2)+1}{1(2)+0}=\frac{9}{2}$. Between $\frac{4}{1}$ and $\frac{56}{13}$ we have $4-1$, or three secondaries, of which the middle one $\frac{13(2)+4}{3(2)+1}$ must be kept on account of $\frac{1}{3} > \frac{1}{6}$. In the next interval between $\frac{13}{3}$ and $\frac{349}{81}$ we have five secondaries, but the middle one must be neglected, because $\frac{3}{13} < \frac{1}{3+1}$; there will consequently be but two fractions

* Lagrange, in his additions to Euler's Algebra (Hewlett's translation, 3rd edit.) p. 487, says, " It may also happen that one of the *intermediate* fractions of one series does not approximate towards the given fraction so nearly as one of the fractions of the other series, although expressed in terms less simple than the former ; for this reason, it is not proper to employ intermediate fractions, except we wish to have the fractions sought either all less or all greater than the given fractions."

to keep on this interval, namely, $\frac{56(4)+13}{13(4)+3}$ and $\frac{56(5)+13}{13(5)+3}$. Proceeding in the same manner to the end we find, collecting the results and drawing a vertical line after each principal fraction, for the sake of distinction, the following series, in which the fractions between two vertical lines are on the same side of the true value.

$$\frac{1}{0}\ \Big|\ \frac{4}{1}\ \Big|\ \frac{9}{2}\ \ \frac{13}{3}\ \Big|\ \frac{30}{7}\ \ \frac{43}{10}\ \ \frac{56}{13}\ \Big|\ \frac{237}{55}\ \ \frac{293}{68}\ \ \frac{349}{81}\ \Big|\ \frac{754}{175}\ \ \frac{1103}{256}.$$

Messrs. W. Mortimer, Seedley; Joseph Paul, Shooter's Hill; and A. Roy, Dundee Academy, favoured us with good solutions.

MATHEMATICAL EXERCISES—*(continued.)*

129. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*

Let CP and CP′ be tangents to a circle whose centre is O, and CD a perpendicular from C to a radius OB; then if PB, P′B be drawn to meet CD in Q,Q′ respectively, the four points P, P′, Q, Q′ lie in the circumference of a circle whose centre is C.

130. *Dr. Burns, Rochester.*

AB is the double ordinate to the axis of a given parabola, BR a diameter, AR any line cutting the curve in Q; then if AP be taken in AR equal to QR, the locus of P is a parabola: required a demonstration.

131. *Mr. Hugh Godfray, Jersey.*

Two points being given, it is required to find ten other points which with these two will form the angular points of a regular dodecagon, using the compasses only.

132. *By A. R.*

From the extremity A of a diameter of a given circle as centre, a circular arc is described cutting the diameter in the point B, and the circumference in the point C; to find the values of the angle ABC when (BD) the perpendicular upon AC, the chord BC, the arc BH, and the segment BFCB are severally the greatest possible.

133. *Mr. R. H, Wright, M. A., London.*

Let a hemispheroid rest upon two planes inclined to each other ϕ°, the respective distances of the points of contact to the point in which the planes meet being h and k; then, if the line passing through the two points of contact pass also through the focus of the elliptic section in that plane which is perpendicular to each of the given inclined planes, and is parallel to the horizon, W the weight of the hemispheroid, T its resistance at a point h distant from the intersection of the two planes, r and r' as usual the focal distances from the points of contact, $8m = 8\sqrt{a^2-b^2} - 3a$, where a and b are the axes major and minor of the elliptic section: shew that there will be equilibrium when

$$\mathrm{W}a^2m.\tan\theta.\tan\left\{\frac{\theta}{2}+\tan^{-1}\left(\frac{m-r'}{m+r'}\cot\frac{\theta}{2}\right)\right\}$$
$$+\mathrm{T}hk\sqrt{a^2-b^2}.\cos\theta.\sin\phi = 0,$$

where θ is the supplement of the angle which the axis major of the elliptic section makes with the horizon.

134. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*

Find the value of $\phi(x,y) = \frac{(x+y)^n}{x^n+y^n}$, when $x=0, y=0$.

135. *Mr. Thomas Weddle, at Mr. Stoton's Academy, Wimbledon.*

Let E, E_1, E_2, E_3 denote the spherical excesses of any system of associated spherical triangles, and E′, E'_1, E'_2, E'_3 those of the polar associated system: prove that

$$\frac{2}{n} = \cot \tfrac{1}{2} E + \cot \tfrac{1}{2} E_1 + \cot \tfrac{1}{2} E_2 + \cot \tfrac{1}{2} E_3$$

and $$\frac{2}{N} = \cot \tfrac{1}{2} E' + \cot \tfrac{1}{2} E'_1 + \cot \tfrac{1}{2} E'_2 + \cot \tfrac{1}{2} E'_3,$$

where, as usual, $$n^2 = \sin s \sin(s-a) \sin(s-b) \sin(s-c),$$
and $$N^2 = -\cos S \cos(S-A) \cos(S-B) \cos(S-C).$$

136. *Pen-and-Ink.*

Three lines in the same plane which meet in a point are given, and likewise two other points in that plane; it is required to describe a triangle having its vertices on the given lines, and two of its sides passing through the given points; and likewise be such that its perimeter shall be a minimum.

137. *James Cockle, Esq., M.A., Barrister-at-Law.*

Let $U=0$ be the equation of a surface of the second degree referred to the axes X, Y, Z, of which axes let P represent one; let $u=0$ be the equation of the same surface referred to any other system of axes x, y, z, of which let p be one; also let $V=0$, $W=0$, be the equations of planes referred to the former axes, and $v=0$, $w=0$, those of planes referred to the latter. Then, $F(P)=0$, and $f(p)=0$ denoting quadratics (in P and p respectively) with real and unequal roots, it is required to show that if $U=0$ can be put under the form

$$VW + F(P) = 0,$$

then $u=0$ can also be put under the form

$$vw + f(p) = 0.$$

138. *Mr. Thomas Weddle, at Mr. Stoton's Academy, Wimbledon.*

Let $F(x,y,z)=0$, or simply, $F=0$ be the equation of any surface, and put

$$\frac{dF}{dx}=U, \frac{dF}{dy}=V, \frac{dF}{dz}=W, \frac{d^2F}{dx^2}=u, \frac{d^2F}{dy^2}=v, \frac{d^2F}{dz^2}=w,$$
$$\frac{d^2F}{dydz}=u', \frac{d^2F}{dxdz}=v', \text{ and } \frac{d^2F}{dxdy}=w',$$

then according as the expression

$$(vw-u'^2)U^2+(uw-v'^2)V^2+(uv-w'^2)W^2+2(v'w'-uu')VW$$
$$+2(u'w'-vv')UW+2(u'v'-ww')UV$$

is positive or negative, the surface in the immediate vicinity of the point (x,y,z) will be situated entirely on one side of the tangent plane at that point, or partly on one side and partly on the other.

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ON PLANE AND CIRCULAR SECTIONS OF SURFACES OF THE SECOND DEGREE.

[*Mr. Thomas Weddle, Wimbledon.*]

I. ON THE SPECIES OF PLANE SECTIONS.

Let the surface of the second degree whose equation is

$$Px^2 + P_1y^2 + P_2z^2 + 2Qx + 2Q_1y + 2Q_2z + R = 0 \quad \ldots\ldots(1)$$

be cut by the plane

$$lx + my + nz = p \ldots\ldots(2),$$

l, m, n being the directing cosines of the normal to the plane, it is required to find the *species* of the section, that is, whether it is an ellipse, an hyperbola, or a parabola.

Let us change the direction of the coordinate planes (without displacing the origin), so that the axis of x may coincide with the normal to the plane (2), and let $(l_1\ m_1\ n_1)$ and $(l_2\ m_2\ n_2)$ be the directing cosines of the new axes of y and z; hence if x', y', z' denote the new coordinates, we have

$$\left.\begin{aligned} x &= lx' + l_1y' + l_2z' \\ y &= mx' + m_1y' + m_2z' \\ z &= nx' + n_1y' + n_2z' \end{aligned}\right\} \ldots\ldots(3);$$

also, $$x' = lx + my + nz \ldots\ldots(4);$$

and, (2, 4), the equation to the cutting plane, is

$$x' = p \ldots\ldots(5).$$

Hence if (3) be substituted in (1), and p be written for x' in the resulting equation, we shall get the equation to the section, which will thus be found to be

$$Ay'^2 + 2By'z' + Cz'^2 + 2Dy' + 2Ez' + F = 0 \ldots\ldots(6),$$

where $$\left.\begin{aligned} A &= Pl_1^2 + P_1m_1^2 + P_2n_1^2 \\ C &= Pl_2^2 + P_1m_2^2 + P_2n_2^2 \\ B &= Pl_1l_2 + P_1m_1m_2 + P_2n_1n_2, \text{ etc.} \end{aligned}\right\} \ldots\ldots(7).$$

$$\therefore\ AC - B^2 = P_1P_2(m_1n_2 - m_2n_1)^2 + PP_2(l_2n_1 - l_1n_2)^2 + PP_1(l_1m_2 - l_2m_1)^2;$$

and this by Lagrange's formulas (Gregory's Solid Geom., p. 51) reduces to

$$P_1P_2l^2 + PP_2m^2 + PP_1n^2 \ldots\ldots(8).$$

Now according as $AC - B^2$ is positive, negative, or zero, the section (6) will be an ellipse, an hyperbola, or a parabola; and hence

The curve in which the surface (1) *is cut by the plane* (2) *will be an ellipse, an hyperbola, or a parabola, according as* (8) *is positive, negative, or zero.*

By means of the very simple criterion (8) the species of the plane sections of the different surfaces of the second degree may easily be ascertained, as the following brief discussion will show.

In the central surfaces P, P_1, P_2 are all different from zero, and for an ellipsoid they are all positive, hence (8) being always positive and never $=0$ *, all the sections are ellipses. For the two hyperboloids and the cone, either one (as P), or two (as P_1, P_2) of the coefficients P, P_1, P_2 are negative, and hence one of the terms of (8) being positive and two negative, (8) may be either positive, negative, or zero; consequently any of these surfaces may be cut by a plane in either an ellipse, an hyperbola, or a parabola.

With respect to the paraboloids and the elliptic and hyperbolic cylinders, one (and one only) of the coefficients P, P_1, P_2 is zero; suppose $P=0$, then (8) becomes

$$P_1P_2l^2 \ldots\ldots\ldots\ldots (8').$$

Now if P_1, P_2 have the same sign, (8′) will be positive, but if different signs it will be negative, except $l=0$, in which case (8′) will vanish. Hence every section of any of these surfaces made by a plane parallel to the axis of x is a parabola †, while every other plane section of the elliptic paraboloid and elliptic cylinder is an ellipse, but of the hyperbolic paraboloid and hyperbolic cylinder, an hyperbola.

The only remaining surface of the second degree is the parabolic cylinder, for which two of the coefficients P, P_1, P_2 are zero; hence (7) will always vanish, and the plane sections are all parabolas, or a system of two parallel straight lines.

If the equation to the surface be given under the general form,

$$Ax^2+A_1y^2+A_2z^2+2Byz+2B_1xz+2B_2xy+2Cx+2C_1y+2C_2z+D=0 \ldots (9)$$

the species of the section made by any plane (2) may still be ascertained. We have only to treat (9) in the same manner as we have just treated (1), and it will readily be found that the criterion here is

$$(A_1A_2-B^2)l^2+(AA_2-B_1^2)m^2+(AA_1-B_2^2)n^2+2(B_1B_2-AB)mn$$
$$+2(BB_2-A_1B_1)ln+2(BB_1-A_2B_2)lm \ldots\ldots (10),$$

hence the section will be an ellipse, an hyperbola, or a parabola, according as (10) is positive, negative, or zero.

It may be observed that if the equation of the cutting plane be

$$Lx+My+Nz=P'$$

where L, M, N are not, as in (2), the directing cosines, l, m, n, of the normal, the species of the section may be determined just as easily as before, for l, m, n being proportional to L, M, N, the latter may be used instead of the former in (8) and (10).

* For l, m, n, on account of the relation $l^2+m^2+n^2=1$, cannot be *all* equal to zero.

† In the case of the cylinders, the section evidently consists of two parallel straight lines, —*a variety* of the parabola.

Note.—Though the criteria (8) and (10) have been here established only when the axes are rectangular, they are nevertheless true when the axes are oblique; l, m, n continuing to be the cosines of the angles which the normal to the cutting plane makes with the axes, and for l,m,n we may, as above, substitute the coefficients L, M, N of x, y, z in the equation $Lx + My + Nz = P'$ of this plane. The investigations, however, though not differing greatly from the preceding, cannot be conveniently entered upon at present, especially as it would be necessary previously to establish certain formulas which are analogous to, and inclusive of, those of Lagrange employed above, but which, so far as I am aware, have not been yet published.

II. DETERMINATION OF THE CIRCULAR SECTIONS.

Let it now be proposed to determine the circular sections of surfaces of the second degree. A symmetrical transformation of coordinates similar to that already employed in the first section of this paper, will be found to be equally effective here.

The equation to the surface may, as before, be denoted by (1), the coordinate axes being parallel to the principal axes of the surface. If we change the direction of the rectangular coordinate planes so that the plane yz may be parallel to a circular section of the surface (1), the sections made by planes parallel to this coordinate plane will be circles, hence in the transformed equation the coefficients of y^2 and z^2 must be equal, and that of yz must be zero. Now the transformation here spoken of will be effected by eliminating x, y, z from (1) by means of (3); lmn; $l_1m_1n_1$; $l_2m_2n_2$, denoting the directing cosines of the axes of x', y', z'; and it must be recollected that l,m,n are also the directing cosines of the normal to the plane of a circular section. Hence if p denote the equal coefficients of y'^2 and z'^2, we shall, after what has been said above, evidently have the following equations,

$$Pl_1^2+P_1m_1^2+P_2n_1^2=p \ldots\ldots (11),$$

$$Pl_2^2+P_1m_2^2+P_2n_2^2=p \ldots\ldots (12),$$

$$Pl_1l_2+P_1m_1m_2+P_2n_1n_2=0 \ldots\ldots (13).$$

Since $l_1^2+l_2^2 = 1 - l^2$, *etc.*, if we add (11, 12), there results

$$Pl^2+P_1m^2+P_2n^2 = P+P_1+P_2-2p \ldots\ldots (14).$$

Multiply (11) by (12), and deduct the square of (13) from the product,

$$\therefore\ P_1P_2(m_1n_2-m_2n_1)^2+PP_2(l_2n_1-l_1n_2)^2+PP_1(l_1m_2-l_2m_1)^2=p^2,$$

and this by Lagrange's formulas (Gregory's Solid Geom., p. 51) reduces to

$$P_1P_2l^2+PP_2m^2+PP_1n^2=p^2 \ldots\ldots (15).$$

Also, $$l^2+m^2+n^2=1 \ldots\ldots (16).$$

Take $P(P_1+P_2)$ times (16) from the sum of (15), and P times (14),

$$\therefore\ l^2=\frac{(P-p)^2}{(P-P_1)(P-P_2)} \ldots\ldots (17).$$

Similarly, $$m^2=\frac{(P_1-p)^2}{(P_1-P)(P_1-P_2)} \ldots\ldots (18),$$

and $$n^2=\frac{(P_2-p)^2}{(P_2-P)(P_2-P_1)} \ldots\ldots (19).$$

Let P_1 be the *mean* of the quantities P, P_1, P_2, that is, let either $P < P_1 < P_2$ or $P > P_1 > P_2$ (here the signs of PP_1P_2 must be attended to), then will (18) give an imaginary value of m, unless $p = P_1$; substitute therefore P_1 for p in (17, 18, 19), and we have

$$l^2 = \frac{P_1 - P}{P_2 - P}, \quad m^2 = 0, \quad n^2 = \frac{P_2 - P_1}{P_2 - P} \ldots\ldots\ldots (20).$$

Consequently there are in general two series of circular sections, which are made by planes respectively parallel to the planes

$$(P_1 - P)^{\frac{1}{2}} x + (P_2 - P_1)^{\frac{1}{2}} z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

and, $$(P_1 - P)^{\frac{1}{2}} x - (P_2 - P_1)^{\frac{1}{2}} z = 0 \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

It must be remarked, however, that these circular sections will not really exist unless p, that is, P_1, be different from zero; for if $p=0$, the coefficients of y'^2 and z'^2 will, (11, 12), vanish in the transformed equation mentioned above, and hence each section parallel to (21) or (22) will be a straight line, not a circle. We are hence entitled to infer that the hyperbolic paraboloid and the hyperbolic and parabolic cylinders, having $P_1 = 0$, admit of no circular sections, but they have instead of them two series of plane sections, such that each section consists of a *single* straight line, (this may however be considered a circle, the radius of which is infinite). The equations of these surfaces must have either P or P_2 negative; suppose the former, then these two series of *rectilinear* sections will be respectively parallel to the planes

$$(-P)^{\frac{1}{2}} x + P_2^{\frac{1}{2}} z = 0 \ldots\ldots (21') \text{ and } (-P)^{\frac{1}{2}} x - P_2^{\frac{1}{2}} z = 0 \ldots\ldots (22').$$

The remaining surfaces of the second degree, including the ellipsoid, the two hyperboloids, the cone, the elliptic paraboloid, and the elliptic cylinder have, in general, two series of circular sections made by planes parallel to (21) and (22) respectively. The planes (21, 22) however, and hence the two series of sections, coincide if either $P_1 = P$ or $P_1 = P_2$, the surface (1) then evidently being one of revolution, and they become indeterminate if $P = P_1 = P_2$, as indeed they ought, seeing that the surface is then a sphere.

Wimbledon, Surrey, Sept. 11th, 1847.

GENERALISATION OF THE LEADING OPERATIONS OF ARITHMETIC IN REFERENCE TO GEOMETRY.

[*Dr. August Wiegand, Oberlehrer und Mathematikus an der Realschule zu Halle, der preussischen Provinz, Sachsen.*]

It is the province of geometry to represent by graphical construction the relations of magnitude, while it belongs to arithmetic to exhibit the same relations by means of abstract ideas or symbols. The attributes by which magnitudes are brought under mathematical consideration, are, (1) *quantity*, (2) *form*, and (3) *direction*. Geometry can represent all these, while arithmetic still imperfectly represents the part which is assigned to it in reference to the third of these properties, viz. *direction*. It is true that arithmetic, by its positive and negative numbers, supplies the corresponding affections of sign for two lines drawn in opposite directions from one common origin: but for the infinitely various directions, which

radiant lines from the same point may assume, arithmetic has hitherto failed to deal with these satisfactorily, and has accordingly been unable to exhibit the relations of magnitude in all their generality; and as the gradual development of the various quantitative relations of magnitudes must necessarily involve the idea of direction, we are able *à priori* to draw the conclusion that this essential property will manifest itself unambiguously in the numerical interpretations of arithmetic. The indications of quantity and direction are so independent of each other, that a change in the one does not involve a simultaneous alteration in the other. If a denotes the *length* of a line, the direction cannot be determined without the introduction of another element. For two opposite directions, we have, as already mentioned, the plus and minus signs, and we know that it is perfectly indifferent which direction is taken as the positive one, but when the one or the other direction is fixed upon for this, it must remain so. For the variations of the intermediate directions geometry has a determinant in the *angle*, while the appropriate arithmetical element still seems to be wanting. If then arithmetic is capable of representing magnitudes in the case of lines drawn in opposite directions from one common origin, should it not at the same time be capable of showing, by ideas, or symbols, the gradual passage of a line revolving from the one direction to the other, i. e. from the direction of the positive magnitudes to that of the negative, and conversely? All sudden transitions, as it were by leaps or jumps, are unnatural, and consequently illogical.

The inadequacy of the ideas of positive numbers for the representation of all formative laws of magnitude shows itself in a striking manner, if we consider the series of the successive powers of a negative quantity:

$$(-a)^0 \quad (-a)^{\frac{1}{4}} \quad (-a)^{\frac{1}{2}} \quad (-a)^{\frac{3}{4}} \quad (-a)^1 \quad (-a)^2 \quad (-a)^3 \quad (-a)^4 \, .. etc.$$

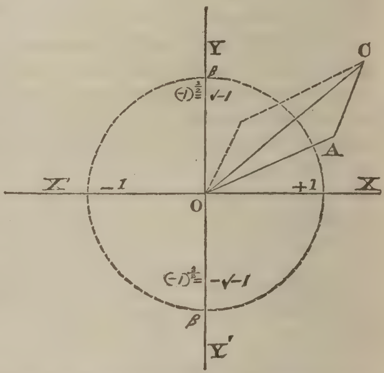
that is,

$$1 \quad \sqrt[4]{-a} \quad \sqrt{-a} \quad \sqrt[4]{-a^3} \quad -a \qquad a^2 \qquad -a^3 \qquad a^4 \, .. etc.$$

If the exponents of these powers increase continuously, then the arithmetical progress through the separate members of the above series will correspond with the geometrical notion of the constant movement of a point, which always forms the extremity of a line introduced by the arithmetical expression. If we begin, for instance, with $(-a)^0 = 1$, we observe that in the first instant of the movement the point will be situated at a distance equal to the linear unit on the positive side from the origin; when we have come to the term $(-a)^1 = -a$, we see that the point must have moved from its first position to a distance a upon the negative side of the origin: subsequently it will be found at different distances alternately upon the positive and negative side. Every position, which the point in its movement occupies, will now evidently be represented by a member of the above series. As now some of the members of it are imaginary, the question is what notion we are to form in respect to this, of the describing motion of the point. We readily perceive, that geometry produces the negative lines from the positive by a semi-revolution of the former round the origin of the system; by another semi-revolution the positive are reproduced.

Arithmetic, on the contrary, produces the negative numbers from the positive by multiplication by -1, we hence see that a semi-revolution in geometry corresponds to a multiplication by (-1) in arithmetic, and to an entire revolution in geometry there corresponds a multiplication by $(-1)^2$ or $+1$ in arithmetic.

If this correlation is systematically followed up, we must conclude that if to a quarter revolution in geometry there correspond in arithmetic a multiplication by the magnitude a, this symbol must stand to the symbol -1 in such a relation, that by repeated multiplication by the same, consequently by multiplication by a^2, the negative numbers must arise, because their representatives are produced from the foregoing by a repeated quarter revolution, and so on. This consideration leads to the conclusion, that by the arithmetical expression $(-1)^{\frac{1}{2}}a$, where a is an absolute magnitude, in a geometrical point of view, a line of the absolute length a is expressed, which stands at right angles upon the real axis, and $(-1)^{\frac{3}{2}}a$ represents a line of the same length a, which lies in the above perpendicular produced. In the figure, OX may be called the positive real, and OX′ the negative real axis; on the other hand, OY the positive imaginary, and OY′ the negative imaginary axis. It is now, without going further, manifest that if β be the sign by which the positive magnitudes are to be multiplied in order to produce magnitudes, which geometry produces by a revolution round the n^{th} part of $180°$, we have to put $\beta^n = -1$, consequently $\beta = (-1)^{\frac{1}{n}}$, and hence it follows that in general by the multiplication of positive magnitudes by the factor $\beta^m = (-1)^{\frac{m}{n}}$, such magnitudes are produced, as geometry produces by a revolution round the angle $\frac{m}{n}\,180°$, or $\frac{m}{n}\,\pi$, reckoned in the positive direction. The magnitude $(-1)^{\frac{m}{n}}$ may be termed the *direction coefficient* of the magnitudes by which it is affected.



The application of this formative law in converse order leads evidently to *direction coefficients* with negative exponents. It is furthermore manifest, that the general laws of exponents cannot be applied to the exponents of the direction coefficients, and consequently that, for instance, $(-1)^{\frac{2}{n}}$ cannot be considered equal to $\left((-1)^2\right)^{\frac{1}{n}}$. It is equally manifest that $(-1)^{-m} = (-1)^{2k-m}$ where k is any positive whole number, and thus all direction magnitudes possible in a plane can be expressed by the general symbol $(-1)^m a$, where m is an absolute number.

From these investigations it now follows that, for example, the terms of the series

$$\ldots (-a)^{-\frac{3}{n}} \quad (-a)^{-\frac{2}{n}} \quad (-a)^{-\frac{1}{n}} \quad (-a)^{0} \quad (-a)^{\frac{1}{n}} \quad (-a)^{\frac{2}{n}} \ldots$$

have for geometrical representatives the radii vectores of a logarithmic spiral, winding infinitely on two opposite sides round the origin.

The above considerations lead us now to a general notion of the fundamental arithmetical operations. If two numbers are to be added, then the general idea of addition is comprehended in this, that *proceeding from the extremity of the one additive magnitude in the direction and about the quantity or absolute length of the second magnitude*, we view the numbers as lines. Hence it immediately follows that in the case of like direction, $(-1)^m a + (-1)^m b = (-1)^m (a + b)$. In the case of unlike direction, if the lines OA and AC are represented respectively by the arithmetical expressions $(-1)^m a$ and $(-1)^m b$, then $OC = OA + AC$ both in quantity and direction. If now the angle $XOC = S\pi$, then $(-1)^m a + (-1)^m b = (-1)^s c$, where c denotes the absolute length of OC. *In this consideration at the same time is found the natural solution of the problem of the parallelogram of forces*, and it is reasonable to expect, that this general comprehension of the fundamental arithmetical operations may exercise a due influence upon the calculations applicable in mechanics, inasmuch as it renders unnecesary the peculiar axioms, which this branch of mathematics is constrained to adopt, and thus removes the isolation in which the applied mathematics at present stand in reference to pure mathematics.

By subtraction we forthwith convince ourselves that this operation reduces itself to an addition of the subtrahends taken in the opposite direction. In order to exhibit $(-1)^m a \times (-1)^n b$, the quantity of the multiplicand $(-1)^m a$ must be taken b times, and we have also to turn the absolute length a round $n\pi$, whereby consequently we get $(-1)^m a \times (-1)^n b = (-1)^{m+n} ab$, which result completely corresponds with the rules of multiplication of powers. That here also the factors admit of being interchanged will at the same time be manifest without further observation.

The operation of division is easily comprehended from that which precedes, thus we obtain

$$(-1)^m a \div (-1)^n b = (-1)^{\frac{m}{n}} \frac{a}{b}.$$

The involution with a positive or negative whole number extends to a repeated multiplication, and we therefore obtain, keeping in sight what has been already shewn,

$$\left[(-1)^m a\right]^r = (-1)^{mr} a^r = (-1)^{2k+mr} a^r,$$

$$\left[(-1)^m a\right]^{-r} = (-1)^{-mr} a^{-r} = (-1)^{2k-mr} a^{-r}.$$

If the exponents be fractional,

$$\left[(-1)^m a\right]^{\frac{r}{s}} = (-1)^{\frac{mr}{s}} a^{\frac{r}{s}} = (-1)^{2k+\frac{mr}{s}} a^{\frac{r}{s}},$$

$$\left[(-1)^m a\right]^{-\frac{r}{s}} = (-1)^{-\frac{mr}{s}} a^{-\frac{r}{s}} = (-1)^{2k-\frac{mr}{s}} a^{-\frac{r}{s}},$$

where the operation of involution requires that the quantity of the root first of all raised, according to the rules for the absolute numbers, consequently divides itself into s equal factors, and each of such factors is multiplied r times by itself; that further, the angle of inclination $m\pi$ of the root is

divided towards the positive axes into s equal parts, and each such part is taken r times. For the case in which the exponent is negative, the quantity $\frac{1}{a^{\frac{r}{s}}}$ or $\frac{mr}{s}\pi$ must be made to revolve in a backward direction from the positive position. (See H. SCHEFFLER upon the relation of Arithmetic to Geometry. Brunswick, 1846).

NOTE ON A PROPERTY OF STATICAL MOMENTS.

[*Mr. Fenwick.*]

The following analytical investigation of a property of statical moments relative to the equilibrium of three forces in the same plane, may interest some of our readers.

Let (h, k) be any point in the plane of the parallelogram ABCD. Take the adjacent sides AB, AD for axes of coordinates, and put $AB=a$, $AD=b$, $AC=d$, and $\angle DAB=\omega$; then if p_1, p_2, be the perpendiculars from (h, k) on the lines AB, AD, we evidently have

$$ap_1 = ak \sin \omega \ldots\ldots\ldots(1), \text{ and } bp_2 = bh \sin \omega \ldots\ldots\ldots(2).$$

Again, as the line AC or d passes through the origin and the point $C(a, b)$, its equation is

$$y = \frac{b}{a}x.$$

Hence the perpendicular (p_3) on this line from (h, k), is

$$p_3 = \frac{\left(k - \frac{b}{a}h\right)\sin\omega}{\sqrt{\left(\frac{b^2}{a^2} + 2\frac{b}{a}\cos\omega + 1\right)}} = \frac{(ak - bh)\sin\omega}{\sqrt{(b^2 + 2ab\cos\omega + a^2)}}$$

$$= \frac{(ak - bh)\sin\omega}{d}.$$

Consequently, $$dp_3 = (ak - bh)\sin\omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now if the point (h, k) be *within* the angle formed by BA and AD, or the angle formed by these lines produced, h and k will have the *same sign*; and hence (3) will be equal to the *difference* of (1) and (2). If, however, (h, k) be *without* this angle, h and k will have *different signs*, and hence (3) will be equal to the *sum* of (1) and (2). Assuming, therefore, the parallelogram of forces, we have this general theorem :—

If three forces which act in the same plane be in equilibrium ; and if from any point in that plane perpendiculars be drawn to the directions of the three forces : then the rectangle contained by the resultant and its perpendicular will be equal to the sum or difference of the rectangles contained by the component forces and their respective perpendiculars, according as the point is without or within the angle formed by the directions of the component forces.

This theorem is incompletely stated in several works.

Royal Military Academy,
6*th Nov.* 1847.

FORMULA TO DETERMINE THE ASTRONOMICAL REFRACTION.

[*W. Galbraith, M.A., F.R.A.S., Edinburgh.*]

The following formula is well adapted to the determination of the astronomical refraction as far as 85° :—

$$r'' = 14{\cdot}5\mathrm{R}''a\tan(\delta - 3{\cdot}375\mathrm{R}''a\tan\delta)\frac{b}{\mathrm{T}}\ldots\ldots\ldots\ldots\ldots(1),$$

in which $\mathrm{R}'' = 206264''{\cdot}8$, the arc in seconds equal to radius,

$a = 0{\cdot}000283$, the coefficient of refraction at 45°, when $b = 30$ inches, and $t = 50°$ Fahrenheit,

$\delta = \theta$, or the apparent zenith distance,

$b =$ the barometer reduced to 50° Fahrenheit,

$\tau =$ the temperature of the mercury,

$t =$ the temperature of the air,

$b' =$ the observed height of the barometer,

$\mathrm{T} = 385° + t$,

$$b = b'\left(1 + \frac{50° - t}{10000}\right).$$

1st constant log = log $- 3{\cdot}375\mathrm{R}''a$ .... $2{\cdot}29449-$

2nd constant log = log $+ 14{\cdot}5\mathrm{R}''a$ .... $2{\cdot}92758+$

When the zenith distance does not exceed 40° or 50°,

$$r'' = 14{\cdot}5\,\mathrm{R}''a\tan\delta\,\frac{b}{\mathrm{T}},\ \text{very nearly}\ldots\ldots\ldots\ldots\ldots(2).$$

Ex.—Let $\theta = 78°\,25'\,30''$, $b' = \overset{\text{m.m.}}{773{\cdot}5}$, $\tau = 18°$ Cent., $t = 20°$ Cent.

$b' = \overset{\text{inches.}}{30{\cdot}454}$, $\tau = 64°{\cdot}4$ Ft. $t = 68°{\cdot}9$ Faht.

$b = \overset{\text{in.}}{30{\cdot}41}$ $\mathrm{T} = 385° + 68°{\cdot}9 = 453{\cdot}9$

$50° - \tau = 50° - 64°{\cdot}4 = -14°{\cdot}4$ Faht.

1st const. log	$= 2{\cdot}29449-$	2nd const. log........	$2{\cdot}92758$
$\theta = 78°\,25'\,30''$ tan	$0{\cdot}68864$	$b = 30{\cdot}41$ inches log..	$1{\cdot}48302$
(1) = — 16 2 log	$\overline{2{\cdot}98313}-$	$\mathrm{T} = 453{\cdot}9$ ar.co. log..	$7{\cdot}34304$
(2) = $\overline{78\ \ 9\ \ 28}$ tan	....		0.67844
$r'' = 4'\,30''{\cdot}46$ log	....		$\overline{2{\cdot}43208}$

Comparison of Galbraith's Formula with Ivory's Table.

	Formula	Table	Δ
$\theta = 70°\,46'\,30''$	$r'' = 2'\,46''{\cdot}48$	$r'' = 2'\,46''{\cdot}35$	$+0''{\cdot}13$
$\theta = 75\ \ 57\ \ 54$	3 54 ·52	3 53 ·60	+ 0 ·92
$\theta = 78\ \ 25\ \ 30$	4 30 ·46	4 30 ·74	— 0 ·28
$\theta = 81\ \ 27\ \ 18{\cdot}6$	6 4 ·17	6 3 ·42	+ 0 ·75
$\theta = 85\ \ 23\ \ 18{\cdot}7$	10 45 ·88	10 53 ·68	— 7 ·80
Laplace's Formula	10 43 ·20		

At such great zenith distances, the differences, except the last, are very small.

Formula (1) may be reduced to the following,

$$r'' = 846''{\cdot}41\tan(\delta - 197''{\cdot}01\tan\delta)\frac{b}{385° + t}\ldots\ldots\ldots(3).$$

ON THE EXPANSION OF IMAGINARY EXPONENTIALS.

[*Mr. Davies.*]

In the first number of the Mathematician the following question was proposed :—

" Can the expression $(-1)^{h^{\frac{m}{n}}}$ be real for any values of h, m, and n? and if so, discriminate the cases."

The sketch of a solution which I drew up at the time was accidentally mislaid, and I had not then leisure sufficient to resume the subject. Though at this distance of time it might seem that the subject had lost its interest as a problem, yet as a neat subject of inquiry it may be worth while to print the notes which were then made on the expression.

PROP. I.

The expression $k^{\sqrt{-1}}$ is always imaginary when k is real, and real when k is imaginary.

For, put $\sqrt{-1}$ for x, in the exponential theorem; and let K be the coefficient of x in the expansion : then we have

$$k^{\sqrt{-1}} = 1 + \frac{K\sqrt{-1}}{1} - \frac{K^2}{1.2} + \frac{K^3\sqrt{-1}}{1..3} + \frac{K^4}{1...4} + \frac{K^5\sqrt{-1}}{1....5} - ..$$

$$= \left\{1 - \frac{K^2}{1.2} + \frac{K^4}{1...4} - \frac{K^6}{1...6} + ..\right\}$$

$$+ K\sqrt{-1}.\left\{\frac{1}{1} - \frac{K^2}{1...3} + \frac{K^4}{1...5} - ..\right\}$$

the law of which is sufficiently manifest.

Now the first bracketed part of this value of $k^{\sqrt{-1}}$ is real under all circumstances; as is likewise the second. The reality of the entire value turns, then, upon that of the factor $K\sqrt{-1}$; and the inference is as stated in the proposition.

PROP II.

The expression $(k)^a$ is real or imaginary under the following circumstances :—

(1) *When k is positive and a any real number whatever, the expression is real*;

(2) *When k is negative and a any integer whatever or any fraction (in its lowest terms) whose denominator is an odd number, the expression is real;*

(3) *When k is negative, a as in the preceding, but with an even denominator, the expression is imaginary.*

All these cases are too familiarly known to render more than the statement of them necessary.

(4) *When k is negative and a not a finite number (that is, such as results from extraction or any inverse operation, which gives rise to an*

infinite series of terms) the expression will be sometimes real and sometimes imaginary, depending on the value *of* a.

In this case we have

$$(-k)^a = (-k)^{p + \frac{p'}{10} + \frac{p''}{100} + \cdots}$$

$$= (-k)^p (-k)^{\frac{p'}{10}} (-k)^{\frac{p''}{100}} \cdots\cdots$$

As $p', p'', p''', \ldots$ are single digits, it is evident that only the two first of the indices of $-k$ can be fractions with odd denominators. All the factors, therefore, after the *first three* are necessarily imaginary. The conclusion will therefore under these circumstances depend upon whether we assume an infinite number to be an odd or even number of terms. This determination is, however, too arbitrary and assumptive to be admitted into science: for odd and even are both excluded from our idea of infinity. If, however, any values of a however extended be given us, irrespective of the generating function from which they were derived, we can always assign whether the expression be real or imaginary from a consideration of the mere number of places of figures, and the values of the two first fractional terms $\frac{p'}{10}$ and $\frac{p''}{100}$.

When the generating function which gave the number a is also given, then the question will require some special considerations. These will be given hereafter.

PROP. III.

When k is real, the expression $k^{a+\beta\sqrt{-1}}$ is sometimes real and sometimes imaginary.

For the expression may be written

$$k^a\, k^{\beta\sqrt{-1}}. \quad \text{Put } k^\beta = h: \text{ then it becomes } h^{\frac{a}{\beta}}\, h^{\sqrt{-1}}.$$

(1) If h and $h^{\frac{a}{\beta}}$ be real: then the factor $h^{\sqrt{-1}}$ is imaginary, and the expression imaginary.

(2) If h be real and $h^{\frac{a}{\beta}}$ imaginary, the factors are both imaginary. Let them be denoted by $(a+b\sqrt{-1})$ and $(a_1+b_1\sqrt{-1})$; and the expression will become $aa_1 - bb_1 + (ab_1+a_1b)\sqrt{-1}$. This will be real or not according as ab_1+a_1b is equal to zero or not.

(3) If $h^{\frac{a}{\beta}}$ be real, the expression $h^{\frac{a}{\beta}}\, h^{\sqrt{-1}}$ or $h^{a+\beta\sqrt{-1}}$, will be essentially imaginary, since one single factor of it is so.

PROP. IV.

The expression $(a+b\sqrt{-1})^m$ *is essentially imaginary, m being real.*

This follows obviously from the form of its expansion as a binomial.

PROP. V.

To find when the expression $(a+b\sqrt{-1})^{h\sqrt{-1}}$ *is real and when imaginary.*

Put $\frac{b}{a}=\tan\theta$: then $\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$, and $\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$; and the expression is converted into

$$(a^2+b^2)^{\frac{h}{2}\sqrt{-1}}(\cos\theta+\sqrt{-1}\,\sin\theta)^{h\sqrt{-1}},\text{ or into } e^{h\theta}(a^2+b^2)^{\frac{h}{2}\sqrt{-1}}.$$

The question therefore turns upon h; and may be thus discussed.

(1) If h be real the second factor is essentially imaginary by *prop. i.*, whilst the first is real: whence the expression is imaginary.

(2) Let h be an imaginary of the form $k\sqrt{-1}$: then the first factor becomes imaginary and the second real, and the same conclusion follows.

(3) Let h be of the form $\alpha+\beta\sqrt{-1}$: then the transformed expression becomes

$$e^{\alpha}e^{\beta\sqrt{-1}}(a^2+b^2)^{\frac{\alpha}{2}\sqrt{-1}}(a^2+b^2)^{-\frac{\beta}{2}}.$$

The first and last of these are real; and the second and third imaginary by *prop. i.* Put them

$$(a_1+b_1\sqrt{-1})\,(a_2+b_2\sqrt{-1});$$

then the condition of reality is

$$a_1b_2+a_2b_1=0,$$

a condition which obviously can, under no circumstances, be fulfilled, except $(a^2+b^2)^{\frac{\alpha}{2}}=e^{\beta}$.

The same conclusion might have been deduced as to the general character of the expression, by actually expanding it by the binomial theorem.

PROP VI.

The expression $(a+b\sqrt{-1})^{\alpha+\beta\sqrt{-1}}$ *is generally imaginary.*

For it is equivalent to

$$(a+b\sqrt{-1})^{\alpha}\,(a+b\sqrt{-1})^{\beta\sqrt{-1}}.$$

In all cases the first factor is imaginary by *prop. iv.*; and by *prop. v.* the second is real or imaginary. When the second is real and the first imaginary the total expression is imaginary. When the second is imaginary, the expression takes the form $(a+b\sqrt{-1})\,(a_1+b_1\sqrt{-1})$, to which the remarks on the preceding proposition apply.

PROP. VII.

The expression $(h\sqrt{-1})^{k\sqrt{-1}}$ *is generally imaginary.*

If h and k be each equal to unity, it was shewn by Euler that the expression

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{c}{4}} = \cdot 207879457... = \omega,$$

where c denotes the circumference of a circle to diameter unity.*

The general expression proposed for consideration may therefore be written thus :—

$$h^{k\sqrt{-1}} \cdot (\sqrt{-1})^{k\sqrt{-1}} = \left(h^k\right)^{\sqrt{-1}} (\sqrt{-1})^{\sqrt{-1}.k} = \left(h^k\right)^{\sqrt{-1}} . \omega^k .$$

If k be real the first factor is imaginary, and the second real; and if k be imaginary the first may be real or imaginary, and the second essentially imaginary. The expression in the first case is imaginary; and the preceding criterion will shew in any individual case whether the corresponding expression be real or imaginary.

* The following neat investigation of this is extracted from a paper "on the necessary truth of certain conclusions obtained by means of imaginary quantities," by the late Mr. Woodhouse. *Phil. Trans.*, 1801.

Put $a^2 + b^2 = r^2, \frac{a}{r} = \cos\theta$, and consequently $\frac{b}{r} = \sin\theta$: then by known properties of these functions we have $(a+b\sqrt{-1})^{m+n\sqrt{-1}}$ converted into

$$r^m e^{-nx} \left\{ \cos(mx + n\log r) + \sqrt{-1}.\ \sin(mx + n\log r) \right\}.$$

Putting $a = 0$, $m = 0$, $b = 1$, and $n = 1$, and making familiar substitutions, the conclusion follows.

(To be continued.)

NOTE ON THE CHORD OF CONTACT IN REFERENCE TO CURVES OF THE SECOND ORDER.

[*Mr. Fenwick.*]

The equation in general of curves of the second order is

$$ay^2 + 2bxy + cx^2 + 2dy + 2ex + f = 0 \dots\dots\dots\dots\dots (1);$$

and the equation of a tangent at a point (x,y),

$$\mathrm{Y} - y = -\frac{by + cx + e}{ay + bx + d}(\mathrm{X} - x).$$

Let this tangent pass through the point (α,β); its equation then becomes by reduction

$$(\beta a + \alpha b + d)y + (\beta b + \alpha c + e)x + \beta d + \alpha e + f = 0 \dots\dots (2).$$

Now every locus whose equation is formed by the combination of (1) and (2), *in any way whatever*, must contain the points common to these. Hence the curve

$$ay^2 + 2bxy + cx^2 + 2dy + 2ex$$
$$= (\beta a + \alpha b + d)y + (\beta b + \alpha c + e)x + \beta d + \alpha e \dots\dots\dots (3),$$

passes through the points of contact.

Again, since (1) and (3) both contain the points of contact, any line which results from their combination, must pass through these points. Hence taking (1) from (3) we have

$$(\beta a + ab + d)y + (\beta b + ac + e)x + \beta d + ae + f = 0 \ldots\ldots (4),$$

for the equation of the chord of contact corresponding to the point (a,β) in reference to (1).

This method of deducing the chord of contact is new so far as I know.

In the case of the *circle*, let

$$x^2 + y^2 = r^2, \text{ and } ax + \beta y = r^2,$$

be the equations corresponding to (1) and (2),—the axes being rectangular and originating at the centre —; then the curve that contains the points of contact is

$$x^2 + y^2 - ax - \beta y = 0, \text{ or } (x - \tfrac{1}{2}a)^2 + (y - \tfrac{1}{2}\beta)^2 = \tfrac{1}{4}(a^2 + \beta^2),$$

which is a circle, the centre and radius of which are

$$(\tfrac{1}{2}a, \tfrac{1}{2}\beta) \text{ and } \tfrac{1}{2}\sqrt{(a^2 + \beta^2)}.$$

Consequently, *the centre of this circle is in the middle of the line which joins the given point and the centre of the given circle.* Hence the geometrical construction of a pair of tangents from a given point to a given circle is obvious.

For the *ellipse*, let

$$a^2y^2 + b^2x^2 = a^2b^2 \text{ and } a^2\beta y + b^2 ax = a^2b^2,$$

be the curve and tangent; then the curve corresponding to (3) is

$$a^2y^2 + b^2x^2 - a^2\beta y - b^2ax = 0.$$

In this, remove the origin to the point $(\frac{1}{2}a, \frac{1}{2}\beta)$, by writing $x + \frac{1}{2}a$ for x and $y + \frac{1}{2}\beta$ for y; we then get by obvious reductions

$$a^2y^2 + b^2x^2 = \tfrac{1}{4}(a^2\beta^2 + b^2a^2).$$

Hence, *the points of contact of two tangents drawn from a given point to a given ellipse, lie in a similar ellipse, which passes through the given point and the centre of the given ellipse, and whose centre is in the middle of the line which joins the given point and the centre of the given ellipse.*

A similar property obviously belongs to the hyperbola.

I am not aware that these properties of the ellipse and hyperbola have been noticed.

Dec. 2, 1847.

THE EQUATION OF THE COMMON TANGENT TO TWO CIRCLES.

[*Pen-and-Ink.*]

Let the circles be denoted by

$$(x - h_1)^2 + (y - k_1)^2 = \rho_1^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$(x - h_2)^2 + (y - k_2)^2 = \rho_2^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Put x_1y_1 and x_2y_2 for the (at present *unknown*) points of contact: then the tangents at x_1y_1 and x_2y_2 will be

$$y(y_1 - k_1) + x(x_1 - h_1) = y_1(y_1 - k_1) + x_1(x_1 - h_1) \ldots\ldots (3),$$

$$y(y_2 - k_2) + x(x_2 - h_2) = y_2(y_2 - k_2) + x_2(x_2 - h_2) \ldots\ldots (4).$$

These two equations are required by the problem to represent an identical line; and hence they can only differ by a constant factor λ. Wherefore

$$y_2 - k_2 = \lambda(y_1 - k_1) \quad \ldots\ldots (5),$$

$$x_2 - h_2 = \lambda(x_1 - h_1) \quad \ldots\ldots (6),$$

$$y_2(y_2 - k_2) + x_2(x_2 - h_2) = \lambda\{y_1(y_1 - k_1) + x_1(x_1 - h_1)\} \ldots (7).$$

Also, since x_1y_1 is in (1) and x_2y_2 is in (2), we have

$$(x_1 - h_1)^2 + (y_1 - k_1)^2 = \rho_1^2 \quad \ldots\ldots (8),$$

$$(x_2 - h_2)^2 + (y_2 - k_2)^2 = \rho_2^2 \quad \ldots\ldots (9).$$

Add the squares of (5, 6) and reduce by (8, 9): then

$$(x_2 - h_2)^2 + (y_2 - k_2)^2 = \lambda^2\{(x_1 - h_1)^2 + (y_1 - k_1)^2\}, \text{ or}$$

$$\lambda = \pm \frac{\rho_2}{\rho_1} \quad \ldots\ldots (10).$$

For the sake of simplifying the expression we shall work with only the *upper sign* of (10), and give, in the final result, the double sign to ρ_2.

By means of (10) we shall have (5, 6, 7) changed into

$$\rho_1(y_2 - k_2) = \rho_2(y_1 - k_1) \quad \ldots\ldots (11),$$

$$\rho_1(x_2 - h_2) = \rho_2(x_1 - h_1) \quad \ldots\ldots (12),$$

$$\rho_1\{y_2(y_2 - k_2) + x_2(x_2 - h_2)\} = \rho_2\{y_1(y_1 - k_1) + x_1(x_1 - h_1)\} \ldots (13).$$

The solution will require us to find the values of x_1, y_1, x_2 and y_2.

Now, we have

$$y_1(y_1 - k_1) + x_1(x_1 - h_1) = (y_1 - k_1)^2 + (x_1 - h_1)^2 + k_1(y_1 - k_1) + h_1(x_1 - h_1)$$

$$= \rho_1^2 + k_1(y_1 - k_1) + h_1(x_1 - h_1); \text{ and similarly}$$

$$y_2(y_2 - k_2) + x_2(x_2 - h_2) = \rho_2^2 + k_2(y_2 - k_2) + h_2(x_2 - h_2).$$

Insert these in (13), and we shall obtain

$$\rho_1\{\rho_2^2 + k_2(y_2 - k_2) + h_2(x_2 - h_2)\} = \rho_2\{\rho_1^2 + k_1(y_1 - k_1) + h_1(x_1 - h_1)\} \ldots (14).$$

Substitute (11, 12) in (14): then we get

$$\rho_1\rho_2^2 + k_2\rho_2(y_1 - k_1) + h_2\rho_2(x_1 - h_1) = \rho_2\rho_1^2 + \rho_2k_1(y_1 - k_1) + \rho_2h_1(x_1 - h_1), \text{ or}$$

$$\rho_1\rho_2(\rho_2 - \rho_1) = \rho_2\{(k_1 - k_2)(y_1 - k_1) + (h_1 - h_2)(x_1 - h_1)\}.$$

In this put

$$\rho_1 - \rho_2 = \rho, \quad h_1 - h_2 = h, \quad k_1 - k_2 = k;$$

and dividing by ρ_2, we shall have

$$-\rho\rho_1 = k(y_1 - k_1) + h(x_1 - h_1), \text{ or}$$

$$y_1 - k_1 = -\frac{\rho\rho_1 + h(x_1 - h_1)}{k} \quad \ldots\ldots (15).$$

Insert (15) in (8); and likewise put in the result

$$h^2 + k^2 = d^2, \text{ and } d^2 - \rho^2 = \mathrm{R}^2.$$

Then,

$$(x_1 - h_1)^2 + \frac{\{\rho\rho_1 + h(x_1 - h_1)\}^2}{k^2} = \rho_1^2; \text{ or, again,}$$

$$(h^2 + k^2)(x_1 - h_1)^2 + 2\rho\rho_1h(x_1 - h_1) = (k^2 - \rho^2)\rho_1^2, \text{ and hence substituting}$$

and completing the Hindu square, we have

$$\begin{aligned} d^4(x_1 - h_1)^2 + 2d^2\rho\rho_1 h(x_1 - h_1) + \rho^2\rho_1^2h^2 &= \rho^2\rho_1^2h^2 + d^2(k^2 - \rho^2)\rho_1^2 \\ &= \rho_1^2\{\rho^2h^2 + k^2d^2 - d^2\rho^2\} \\ &= \rho_1^2\{k^2d^2 - (d^2 - h^2)\rho^2\} \\ &= \rho_1^2\{k^2d^2 - k^2\rho^2\} \\ &= k^2\rho_1^2\{d^2 - \rho^2\} \\ &= k^2\rho_1^2\mathrm{R}^2. \end{aligned}$$

Hence, extracting, we have, considering for the present only $+\mathrm{R}$,

$$x_1 - h_1 = -\frac{\rho_1(h\rho - k\mathrm{R})}{d^2} \qquad \text{(16)}.$$

Again, $$y_1 - k_1 = -\frac{\rho\rho_1 + h(x_1 - h_1)}{k}$$

$$= -\frac{\rho\rho_1 - \dfrac{\rho_1(h\rho - k\mathrm{R})h}{d^2}}{k}$$

$$= -\frac{\rho_1\{d^2\rho + hk\mathrm{R} - h^2\rho\}}{d^2k} = -\frac{\rho_1(k\rho + h\mathrm{R})}{d^2} \qquad \text{(17)}.$$

From (11, 17) and (12, 16) we also have

$$y_2 - k_2 = -\frac{\rho_2(k\rho + h\mathrm{R})}{d^2} \qquad \text{(18)},$$

$$x_2 - h_2 = -\frac{\rho_2(h\rho - k\mathrm{R})}{d^2} \qquad \text{(19)}.$$

Put for the moment,

$$x_1 - h_1 = a_1, \qquad x_2 - h_2 = a_2,$$
$$y_1 - k_1 = b_1, \qquad y_2 - k_2 = b_2.$$

Then the two equations for the tangent (3, 4) are changed to

$$a_1x + b_1y = a_1(a_1 + h_1) + b_1(b_1 + k_1) \qquad \text{(20)},$$
$$a_2x + b_2y = a_2(a_2 + h_2) + b_2(b_2 + k_2) \qquad \text{(21)}.$$

Now since these two equations represent the same line, their sum or difference will also represent that line. Let us, therefore, for securing symmetrical results, take the difference: then we have

$$\begin{aligned} (a_1 - a_2)x + (b_1 - b_2)y &= \{a_1(a_1 + h_1) - a_2(a_2 + h_2)\} + \{b_1(b_1 + k_1) - b_2(b_2 + k_2)\} \\ &= (a_1^2 + b_1^2) - (a_2^2 + b_2^2) + a_1h_1 - a_2h_2 + b_1k_1 - b_2k_2 \qquad \text{(22)}. \end{aligned}$$

Substituting the values of a_1, a_2, b_1, b_2, we have the following series of equations:—

$$a_1 - a_2 = -\frac{(\rho_1 - \rho_2)(h\rho - k\mathrm{R})}{d^2} = -\frac{\rho(h\rho - k\mathrm{R})}{d^2},$$

$$b_1 - b_2 = -\frac{(\rho_1 - \rho_2)(k\rho + h\mathrm{R})}{d^2} = -\frac{\rho(k\rho + h\mathrm{R})}{d^2},$$

$$a_1h_1 - a_2h_2 = -\frac{(\rho_1h_1 - \rho_2h_2)(h\rho - k\mathrm{R})}{d^2},$$

$$b_1k_1 - b_2k_2 = -\frac{(\rho_1k_1 - \rho_2k_2)(k\rho + h\mathrm{R})}{d^2},$$

$$a_1^2 + b_1^2 = (x_1 - h_1)^2 + (y_1 - k_1)^2 = \rho_1^2,$$
$$a_2^2 + b_2^2 = (x_2 - h_2)^2 + (y_2 - k_2)^2 = \rho_2^2.$$

Inserting these values in (22) we have

$$\rho(h\rho - k\mathrm{R})x + \rho(k\rho + h\mathrm{R})y = -d^2(\rho_1{}^2 - \rho_2{}^2) + (\rho_1 k_1 - \rho_2 k_2)(k\rho + h\mathrm{R}) + (\rho_1 h_1 - \rho_2 h_2)(h\rho - k\mathrm{R}) \ldots\ldots (23).$$

Since ρ and R have each two values, this expression contains four distinct equations, and hence there are (as, geometrically, we know there generally should be) four lines which can each be drawn to touch the two given circles.

CASES.

1. If the two circles intersect each other, $\rho_1 + \rho_2$ is greater than d, whilst $\rho_1 - \rho_2$ is less than d. Under such circumstances, there will be two cases real; viz. those which depend on $\mathrm{R}^2 = d^2 - (\rho_1 - \rho_2)^2$; whilst the other two dependent on $\mathrm{R}^2 = d^2 - (\rho_1 + \rho_2)^2$ will be imaginary. There can hence be only two tangents in this case.

2. If one circle lie *wholly within* the other; then both $\rho_1 + \rho_2$ and $\rho_1 - \rho_2$ are greater than d, and R is imaginary: whence in this case no common tangent can be drawn to the two circles.

3. If the circles touch *externally*, we shall have $\rho_1 + \rho_2 = d$, and d greater than $\rho_1 - \rho_2$. Wherefore the cases corresponding to $\rho_1 - \rho_2$ are *possible*, and two tangents correspond to them.

Also, $d^2 - (\rho_1 + \rho_2)^2 = 0$ gives a *single equation* by the disappearance of the radical; or the two other tangents coalesce and form a single one, at the point of contact of the circles.

4. If one circle lie within and touch the other, we shall have $\mathrm{R}^2 = d^2 - (\rho_1 - \rho_2)^2 = 0$, and there will be one tangent expressed by it as in the last case.

At the same time $\mathrm{R}^2 = d^2 - (\rho_1 + \rho_2)^2$ is negative and R imaginary: so that there is only the single tangent possible in the case just supposed.

Further Discussions.

Let us take the same value of λ with the opposite values of R.

1st. Let $\lambda = +$, or ρ_2 retain the sign of (23).

Multiply out the terms of (23) on the right side, and subtract the one resulting equation from the other: then dividing all by R, we get

$$\rho(hy - kx) = (\rho_1 k_1 - \rho_2 k_2)h - (\rho_1 h_1 - \rho_2 h_2)k,$$

or restoring $h_1 - h_2$ and $k_1 - k_2$, the values of h and k, the whole becomes divisible by $\rho - \rho_2$ and we have

$$(h_1 - h_2)\,y - (k_1 - k_2)\,x = h_1 k_2 - h_2 k_1 \ldots\ldots\ldots\ldots (24),$$

which is evidently the line through $(h_1 k_1)$, $(h_2 k_2)$ *the centres of the two circles.*

Instead of subtracting, let us add the two equations; then we get

$$\rho\,(hx + ky) = -d^2(\rho_1 + \rho_2) + (k_1\rho_1 - k_2\rho_2)\,k + (h_1\rho_1 - h_2\rho_2)h \;..(25),$$

which passes through the same point as the two tangents, and is perpendicular to the line (24) *joining the centres.*

(It passes through the intersection of the tangents from its simultaniety with them; and is perpendicular to the alleged line by the comparison of the left sides of 24, 25.)

2nd. Let $\lambda = -$, and hence the sign of ρ_2 is changed in (23); whilst R takes its two values. Then $\rho = \rho_1 + \rho_2$ in this case, and the reduced equation is again divisible by $\rho_1 + \rho_2$; giving a repetition of equation (24). *The intersection therefore, of this pair of tangents is in the line joining the centres.*

Again, by adding we have, $\rho = \rho_1 + \rho_2$, and hence

$$\rho\,(hx + ky) = -d^2(\rho_1 - \rho_2) + k\,(k_1\rho_1 + k_2\rho_2) + h(h_1\rho_1 + h_2\rho_2) \ldots . (26),$$

which is the line through the intersection of the tangents perpendicular to the line joining the centres, or to that determined in (24).*

It still remains to consider, whilst R is alike *in sign* in two equations, $\lambda = +$ and $\lambda = -$; and the results of such substitution in eq. (23). We shall thus get the equations of tangents of the two species simultaneously combined. The investigation however is long and intricate, and the result very complex, on account of the two resulting values of R^2, viz. $d^2-(\rho_1-\rho_2)^2$ and $d^2-(\rho_1+\rho_2)^2$ rendering it apparently impossible to simplify the expressions.

[*Another discussion of the same problem. By* γ.]

Take the origin of rectangular coordinates at the centre of the greater circle, and let the line passing through the centres of the two circles be the axis of x. Then if R and r be the radii, and d the distance of the centres, the equations of the circles will be

$$x^2 + y^2 = R^2 \ldots\ldots . (1), \qquad (d - x)^2 + y^2 = r^2 \ldots\ldots . (2).$$

Let

$$y = ax + b \ldots\ldots\ldots\ldots\ldots\ldots . (3),$$

be the common tangent, of which the parameters a and b are to be determined.

Eliminating y between (1) and (3) we have

$$(1 + a^2)\,x^2 + 2\,abx + b^2 - R^2 = 0 \ldots\ldots\ldots\ldots . (4).$$

In order that (3) may be a tangent to (1), (4) must be a complete square, and this will be the case when

$$(1 + a^2)\,(b^2 - R^2) = a^2b^2, \text{ or } b^2 = R^2\,(1 + a^2) \ldots\ldots\ldots (5).$$

We get in a similar way from (2, 3),

$$(a\,d + b)^2 = r^2\,(1 + a^2) \ldots\ldots\ldots\ldots\ldots\ldots . (6).$$

Let us now find the point or points in which the common tangent cuts the axis of x.

In (3) put $y = 0$; then $x = -\dfrac{b}{a}$. Hence, dividing (6) by (5), we have

$$\frac{ad+b}{b} = \pm\,\frac{r}{R}, \text{ which gives } -\frac{b}{a} = \frac{dR}{R \mp r} = x \ldots\ldots . (7).$$

We have therefore the following proportions :—

$$R - r : R :: d : x, \text{ and } R + r : R :: d : x.$$

* The preceding cases refer to the tangents which are *symmetrically related* to each other and to the circles: viz. those which touch the circle in the same manner. The case of $\lambda = +$ refers to the tangents passing between the centres; and $\lambda = -$ to those which do not pass between the centres. The points of intersection are what, in modern terminology, are called respectively the *external and internal poles of similitude.*

Hence, *to construct geometrically the abscissas, take any radius* AD *(centre* A*) of the greater circle and from this cut off* DE $= r$, *and also produce this radius till* DF $= r$*; draw to the centre* B *of the less circle, the lines* EB, FB, *and from* D *draw lines parallel to these to meet* AB *in* C *and* C′ : *these will be the points in which the common tangents meet the line which joins the centres.* The problem is now reduced to that of "drawing from a given point a tangent to a given circle."

To complete the solution, we will now find the value of a.

Eliminating b between (5) and (7),

$$a = \pm \frac{R \mp r}{\sqrt{\{d^2 - (R \mp r)^2\}}} \quad \ldots\ldots\ldots\ldots\ \ldots.(8).$$

As the value of a is *quadruple*, the problem in general admits of *four* solutions, as indeed is evident from the preceding, for two pairs of tangents can be drawn from C and C′.

Now if we make $r = 0$ in (8) and call a' the corresponding value of a, we have

$$a' = \pm \frac{R}{\sqrt{(d^2 - R^2)}}\,;$$

an expression which determines the trigonometrical tangent of the angle made with the axis of x by the tangent drawn from the point B to the circle whose centre is A; for the less circle or that whose centre is B, is, in this case a *point*. But the value of a is deduced from that of a', by changing R into $R - r$, or $R + r$; consequently, the value of a is the tangent of the angle made with the axis of x by the tangent drawn from the centre B to the circle described from A with radius $R - r$, or $R + r$. This tangent, therefore, from the point B, is parallel to the required one from C or C′, and hence we can construct geometrically each pair of tangents.

HISTORICAL NOTICES RESPECTING AN ANCIENT PROBLEM.

[*Mr. Davies.*]

A short time ago, certain difficulties connected with the theory of the PORISM were proposed to me by a distinguished foreigner—who unquestionably stands at the head of living geometers. His objections to the views which we, in this country, take of the character of this peculiar proposition go to "the root of the matter;" and I look forward to the day as not very distant when the difficulties which he has pointed out (not in a captious, but in the most philosophic spirit) will be explained and removed without any public discussion. Meanwhile, it is probable that a summary of the notes made during a somewhat extended examination of the *uses* of one of Euclid's Porisms (made specially for another purpose, though still with the principal object in view) will be of sufficient interest to the readers of the *Mathematician,* to justify its insertion in that work.

PORISM.*

If from two given points A, B *straight lines* AC, BC *be drawn to any point* C *in the given circle* CDE, *meeting the circle again in* D, E *respectively, and* DE *be drawn : then it will contain a* given angle *with*

* Pappi Coll. Math. Ven. 1689, sig. 161; Simsoni de Porismatibus Tractatus, p. 472.

a line drawn from one of its extremities to a given point; *or it will be parallel to a* given line; *or it will converge to a* given point.

One very convenient practice used by Mr. Noble, of the Royal Military College, Sandhurst,* has been adopted here: viz. that of printing in a different type the parts which it is the essence of the porism to affirm to be given, in the sense of Euclid's Data. It is, moreover necessary to remark that the porismatic data affirmed above, will be, in general, different for the two extremities of the line DE.

This porism is sufficiently familiar to all well-read English geometers, and need not be discussed here: but it may be well to remark that the *analysis* given by Noble in the place referred to, is much more simple and concise than that given by Simson. At the same time I confess that I have been hitherto unable to seize upon *any new principle* in Mr. Noble's papers (either in "Repository" or "Companion") which tends to simplify the analysis of porisms, such as his preliminary remark seems to intimate his possessing. His actual analyses of the porisms given in his papers are undoubtedly improvements upon those given by Simson, and in some cases great improvements; but, as far as I can discover, the *principle* that runs through all these analyses is the same. Whilst, too, Noble enunciates the porism for a polygon of $n+1$ sides, n of them passing through n given points, he only analyses it for $n=2$, as enunciated above; and as far as proof goes, he has only investigated in another (though a better) form, the proposition as it stands in Pappus and Simson. The first investigation of the porism in its generalized form is due, I believe, to Mr. Lowry, of the Royal Military College:† but the most complete and general one (though not in the complete *porismatic form)* is given by the late Mr. J. H. Swale of Liverpool.‡

There is too much disposition amongst modern geometers to suppose that we know all, or nearly all, the classes of research upon which the ancients had exercised their powers. An instance in point, occurs in respect to this porism viewed in connection with Prop. 117. book vii. of Pappus. The *principal* use of this porism, as far as has yet been discovered, is to solve the problem:—*to inscribe a triangle in a given circle, such that its sides (produced if necessary) shall pass through three given points.* This porism furnished the process at once. A particular case of it, however, constitutes the 117th prop. above referred to in Pappus:—viz. where the three given points are in a straight line. This modification is proposed and constructed as a *problem;* and it has hence been inferred that the solution of the general problem, when the three points had any positions whatever, transcended the powers of the Greek geometers! Before the publication of Simson's Restoration of the Porisms, this might have been a pardonable error; but this opinion, first uttered by Castillon, (before I think he could have seen Simson's *Opera Reliqua* ‖) has been absurdly repeated by all succeeding writers, as though it were not open to a single doubt. Yet who, with this porism before him, can doubt for a moment that Euclid was in possession of the construction of its immediately dependent problem? He who does so doubt, should, at least, tell us to what other and more obvious pur-

* Leyb. Repos. NS. vol. i. p. 35 pt. ii.

† Leyb. Repos., vol. ii. p. 189.

‡ Liverpool Appollonius, pp. 41—52.

‖ Castillon's construction was published in the Berlin *Memoires* for 1776, the same year that Lord Stanhope published the posthumous works of Simson. I shall have occasion to revert to this topic hereafter.

pose it can be applied. Were this, indeed, a suitable place, I think I could render it highly probable that the construction of the problem was antecedent to the discovery of the porism :—nay more, that such is the real history of this entire class of the ancient geometrical propositions. Dr. Simson himself had in 1731 accomplished a construction of the problem* in which there are no traces of the influence of the porism upon his method; although he is considered by his pupil and biographer, Dr. Traill, to have discovered the essential principle of the porisms in 1722.† He viewed it then, simply as an extension of the 117th of Pappus; and we are hence entitled to infer that he had not at that time taken into his consideration the last porism (the one in question) of Euclid's third book. It is, indeed, probable that the construction of the problem would itself direct his attention to the corresponding porism; the more especially as the truth of the porism may be very easily deduced (though not exactly, perhaps, in the form to which he ultimately restored it) from the analysis, composition and demonstration which he there employed. As it was Simson's practice to transcribe into a series of books all those of his researches upon which he set any value, and as these books ("*Adversaria*") are still preserved in the library of his college, it would be a very interesting task to examine them in detail, so as to trace by means of the entries, the real history of his own mind in the execution of one of the most wonderful intellectual undertakings ever executed by man. It is a task, however, which can only be performed by a geometer of leisure and judgment, and one who is resident on the spot. A treasure like Simson's *Adversaria* should not be entrusted to private keeping, even for a single hour.

The insertion of the particular case amongst the "Mathematical Collections" has often been a source of perplexity to geometers, and has not by any of them, obvious as the reason is, been explicitly accounted for. It is simply this: that Pappus was enumerating the *lemmas* that Apollonius had employed as subservient to the constructions of the different cases of the problem of "Tangencies." He had *professedly* that sole object in view; and of course it would have been going out of his way to make the slightest allusion to any collateral questions or to have spoken of other problems, to which this lemma might have been related either by affinity of subject or subordination of case. It is simply given as the "thirtieth lemma." How far the Taction-problem has been "restored" may be judged from the fact that in no attempt at such restoration, has this "lemma" been employed. The conditions of Apollonius have not been fulfilled; and though it may be urged that we are in possession of solutions as elegant (or perhaps even more elegant) as that of the illustrious Greek geometer, we are not compelled to admit the position, till we have at least obtained a solution which can with more propriety be called Apollonian than any that we yet possess. Till that is done, no comparison can be fairly instituted between the ancient and the modern solutions.

I am indebted to a friend, who has paid long and close attention to this and the other Apollonian problems, for directing my mind to this circum-

* The original Analysis and Composition of Simson is given literally by Dr. Traill at p. 97 of the "Life;" and it is preceded by a succinct history of the problem itself by Playfair—the most accurate as far as it goes of all the histories of the problem that I have met with, except that of Mr. Potts in his "Appendix to his Euclid," p. 97, and that of M. Chasles in his *Aperçue Historique*, p. 328.

† Traill's Life of Simson, p. 19, note.

stance; and he has since pointed out to me that Dr. Simson* has actually applied the 117th prop. of Pappus to the case of "OO.", and he has himself applied it also to "OOO" and "OO—." I am not without a hope that the very complete restoration which he has effected of the entire problem will one day be given to the public; and I regret that he does not permit me to further describe his labours on this occasion.

I am not able to say whether this problem was, in any of its forms, considered by the great continental geometers of the 16th and 17th centuries; as I have now no means of reference to their writings, and when I had such, the history of the problem had not engaged my attention. From the silence of Montucla on this head (whilst he mentions the problem) it is certain that it had not, at all events, attracted much attention; and from the uniform reference to Cramer as its author, by the continental writers, we may conclude that it had not been discussed at all during the periods in question. Nor, I feel tolerably certain, had any discussion of it been given by English geometers, antecedently to the publication of Simson's Posthumous Works in 1776: although, as before remarked, Simson himself had constructed it for the triangle, as far back as 1731.

* Opera Reliqua, Appendix, p. 21. The solution of this case by Simson is preceded by the following remarks, which tend to shew that he was sensible of the imperfection of all attempts that had then been made to restore this general problem, as well as impressed with its difficulties; Simson's Latin is not easy to translate literally: but the departure here made from it, is as slight as a distinct enunciation of his meaning rendered possible. Dr. Traill, indeed, remarks upon the difference between Simson's earlier and his later writings in this respect. "Life," p. 96.

"In the Seventh Book of the Mathematical Collections of Pappus Alexandrinus, (every admirer of the ancient geometrical analysis ought to rejoice that this work has been preserved to our times) among the lemmas which that most elegant writer has handed down, there exists a problem for one of the tangencies of Apollonius, namely, in Prop. 117, B. VII; in which it is required, when a circle being given by position and three points in a straight line, to inflect from two of the points two lines meeting in the circumference so as to make the two points in which they intersect the circle and the third given point in the same straight line. It is not difficult to investigate the rest of the lemmas which are subsidiary to the Problems on the Tangencies; and some of these Vieta has used in his Apollonius Gallus: but to what problem the aforesaid lemma could be subsidiary, neither Vieta nor any other Geometer has attempted to conjecture. Often indeed have I revolved the subject in my mind, but I have never succeeded in arriving at any satisfactory conclusion; except that the lemma, by no uncertain marks, appeared to be necessary for the following Problem:—two circles and a point being given by position, it is required to describe a third circle which shall touch the given circles and pass through the given point. In what manner, however, the lemma might be subsidiary to this problem I did by no means perceive. I have directed my attention to the solutions of Vieta and others, hoping that by chance I might hit upon the analysis requiring this lemma; but in vain until this day, after various trials, I discovered the true analysis of Apollonius,—to which indeed both this Prop. 117 of Pappus as well as Props. 116 and 118 are manifestly subsidiary. Feb. 9, 1734."

It may be as well here to notice another circumstance connected with this subject. It is, that Pappus, in his fourth book, has solved three particular cases of the Problem of Tangencies: viz. props. 8, 9, 10; and for assisting the particular determination of a case of 10 the props. 11, 12 are added. It was from the prop. 10 as given by Pappus, that Vieta derived his construction of the general case. Had the seventh book of Pappus been lost, we should no doubt have been told by modern writers that the Greek geometers had never succeeded in describing a circle to touch three given circles, except when the three given circles touched each other! This inference, at any rate, would have been as justifiable as that upon which the comment in the text is made. The fact is, that in the fourth book, the particular cases constructed are *merely subsidiary* to an investigation of the properties of the arbelon; just as vii. 117 is to the general problem of tangencies.

I am indebted to the friend before alluded to, for pointing out the facts upon which this remark is founded.

In the ages following the discovery of the calculus, it became extremely rare to find a continental mathematician who was in any degree skilled in the ancient geometry, or even moderately informed respecting its most common-place truths. Its forms of reasoning were abandoned for the Cartesian coordinates, and if an elementary truth were incidentally required in any special research, the ancient methods were *never* had recourse to for obtaining it. This particular problem had, however, been passed about "from hand to hand" (as Castillon expresses it) till, in 1742, it was proposed by Cramer of Geneva to Castillon. It had been proposed to Cramer himself "in his youth" by some one else; and this is, I believe, all that is really known respecting its origin. It appears to have been looked upon as a puzzle to discourage any young man from pursuing the ancient geometry. As Cramer was born in 1704, we must, if we consider the word *jeunesse* as to be taken literally, and the exactness of Castillon's memory of a single phrase after thirty-four years to be unquestionable, be obliged to abandon the conjecture that I am about to propose. Yet I think it deserves some attention.

It was the custom of that age for one mathematician to propose difficult questions to another (in imitation of which practice as conducted in the "Leipsic Acts," the "Ladies' Diary" and so many other of our best English periodicals have been instituted); and this problem was more likely to have been proposed by a geometer than by an analyst,—especially as the condition of solution is pointedly stated by Castillon, to have been, that the ancient geometrical methods alone shall be employed. This class of geometers was at that time exclusively English. With whom, then, was it so likely to originate as with Halley or Simson? In 1710 Simson visited London, and subsequently became a regular correspondent of Halley, and of Jurin, then secretary of the Royal Society. As he had, at that time (1710) made the ancient geometers his especial study, and had been virtually appointed Professor of Mathematics in the University of Glasgow, what is more probable than that such a problem as this should have become the topic of conversation and discussion? Very little of Simson's correspondence is preserved; and I am not aware whether his *Adversaria* commences at so early a date. It is possible, however, that some conclusive documentary evidence may yet be found. Moreover there is a remarkable degree of similarity between the leading parts of the notation, and in the particular form of the *figures*, in Simson's restoration of the porism of Euclid and the solution of Castillon, which is scarcely reconcileable with the *independent origin* of the problem with both; whilst had Castillon been in any way indebted for his *solution* to Simson he would at least have thrown over it the disguise which a change of lettering would have produced. That either they both had a common origin (and that a lettered diagram accompanied the enunciation of the problem) or that Castillon's was *at first* derived from Simson's, is thus rendered highly probable. There is certainly, on Castillon's own shewing, not the least pretence for calling it "Cramer's problem," as so many writers, following Montucla, have adopted the habit of calling it.

Be this as it may, there is no question of the fact that whilst Simson's is the earliest authenticated construction of the problem, Castillon's is the earliest published one. It has become a fashion to say that Castillon's is a complex and laborious solution; a fashion which only proves that those who adopt the expression either have not studied, or did not understand the character of the paper which contained it. It is thus that fallacies and mis-

takes " pass from hand to hand" in respect to matters of mathematical as well as of other histories. Castillon indeed enters into considerable detail respecting the different *cases* of the problem, which, if brevity had been his object, he might have passed over in silence: but he was one of the very few continental geometers (perhaps the only one) of his time, who entered fully into the spirit and manner of the ancient geometry; and this character of his solution has been given by the Cartesian geometers, who treat the discussion of the separate cases as a matter unworthy of their attention.* It is, indeed, a remarkable circumstance that the solution given by Mr. Potts (App. to Euclid, p. 97) which is professedly formed in imitation of Noble's analysis of Euclid's porism, is identical in all essential particulars with Castillon's construction in the Berlin *Mémoires*. If so able a geometer as Mr. Potts consider this solution so superior, it ill becomes the mere Cartesian to cavil with it as a specimen of geometrical complication. My own opinion is most decidedly that as far as the triangle alone is concerned, no solution superior to Castillon's has ever appeared. Had it not have been that Mr. Pott's solution, to which it is so very similar, is accessible to all my readers, it should have been introduced here, in proof of the conclusion I have formed respecting the value of it.

The day after Castillon had read his paper to the Berlin Academy, Lagrange put into his hand a trigonometrical solution of the same problem; apparently for the purpose of shewing that an analyst could solve in a single evening a problem that had confessedly required from a geometer of no mean powers more than thirty years of attention! This, in the language of Lhuilier,† was an act of " condescension" on the part of the great analyst towards geometry. Low, indeed, must geometry have fallen in the estimation of mathematicians to give rise to language like this, even in reference to the illustrious Lagrange! Instead, however, of describing Lagrange's process, I shall here give it entire: which is deemed the more necessary by the exaggerated praises bestowed upon it, by many writers who have given the best proofs of their never having read it. I have, indeed, never met with a single person who possessed a distinct idea of Lagrange's method. The solution itself forms the initial part of a second paper by Castillon in the same volume of the Berlin *Mémoires;* the paper itself containing some interesting properties of the conic sections having a close alliance with the transversal method—or, more properly speaking, forming a part of that method of investigation.

Being given in magnitude and position the circle RMNP, to inscribe a triangle MNP of which the three sides NM, PM, PN shall pass through the three given points A, B, C.

Algebraic Solution.

Draw the lines AO, BO, CO to the centre O of the circle; and denote them by a, b, c; and put AOB $= m$, and AOC $= n$. These magnitudes are given by the problem.

Let MNP be the required triangle, and draw MO, NO, PO the three radii, and denote each of these by r.

Make AOM $= x$, AON $= y$, and AOP $= z$: these are the three unknowns to be determined.

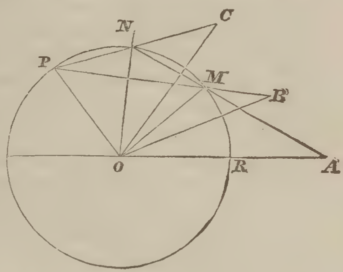
* Some pointed remarks on the Euclidean and Cartesian Geometries are given in the Mechanics' Magazine for Dec. 18, and Dec. 25, which deserve the attention of teachers and students.

† Elémens d'Analyse Géométrique et Algébrique, (1809) p. 280.

Consider in the first place the isosceles triangle NOM, the angle of which at the centre is $y - x$; and hence the angle

$$\text{ONM} = \frac{180° - y + x}{2} = 90° - \frac{y - x}{2}.$$

In the next place we have the triangle AON, the angle at the centre O of which is y, and $\text{ONM} = 90° - \frac{y-x}{2}$: then,



$$A = 180° - \left(90° - \frac{y-x}{2}\right) - y = 90° - \frac{y+x}{2}.$$

Whence we have

AO : NO : : sin ONA : sin OAN, or

$$a : r \; : : \sin\left(90° - \frac{y-x}{2}\right) : \sin\left(90° - \frac{y+x}{2}\right) : : \cos\frac{y-x}{2} : \cos\frac{y+x}{2},$$

$$\text{or} \qquad a \cos\frac{y+x}{2} = r \cos\frac{y-x}{2}.$$

This equation easily reduces to

$$a\left(\cos\frac{y}{2}\cos\frac{x}{2} - \sin\frac{y}{2}\sin\frac{x}{2}\right) = r\left(\cos\frac{y}{2}\cos\frac{x}{2} + \sin\frac{y}{2}\sin\frac{x}{2}\right),$$

$$\text{or,} \quad (a - r)\cos\frac{y}{2}\cos\frac{x}{2} = (a + r)\sin\frac{y}{2}\sin\frac{x}{2};$$

$$\text{or, again,} \qquad \tan\frac{y}{2}\tan\frac{x}{2} = \frac{a - r}{a + r} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We shall find a similar equation by considering in the first place the isosceles triangle POM and then the triangle POB; and without repeating the process anew, it will be sufficient to substitute the line OB for OA and the radius OP for ON. Then we have b instead of a, MOB instead of MOA ($x - m$ instead of x), and instead of y (NOA) we have $z - m$ (POB).

The new equation becomes

$$\tan\frac{x - m}{2} \cdot \tan\frac{z - m}{2} = \frac{b - r}{b + r} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Finally, we find a third similar equation from considering the isosceles triangle PON and the triangle POC; and for this purpose it is only necessary to put in the equation (1) for AC (a) the line OC (c); for the angle MOA (x) the angle NOC ($y - n$); and for the angle NOA (y) the angle POC ($z - n$). Then we have

$$\tan\frac{y - n}{2} \cdot \tan\frac{z - n}{2} = \frac{c - r}{c + r} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

These three equations are sufficient for the determination of x, y, z.

Put for simplification,

$$\tan\frac{x}{2} = s\,;\ \tan\frac{y}{2} = t\,;\ \text{and}\ \tan\frac{z}{2} = u:$$

$$\tan\frac{m}{2} = p;\ \tan\frac{n}{2} = q:$$

$$\frac{a-r}{a+r} = \mathrm{A}\,;\ \frac{b-r}{b+r} = \mathrm{B}\,;\ \text{and}\ \frac{c-r}{c+r} = \mathrm{C}.$$

Then by a known property of tangents, the three equations are reduced to

$$st = \mathrm{A};$$

$$\frac{s-p}{1+ps}\cdot\frac{u-p}{1+pu} = \mathrm{B};$$

$$\frac{t-q}{1+tq}\cdot\frac{u-q}{1+qu} = \mathrm{C}.$$

From the first of these we get $t = \dfrac{\mathrm{A}}{s}$; and

. . . . . second . . . . $u = \dfrac{\mathrm{B}-p^2+(1+\mathrm{B})ps}{-(1+\mathrm{B})p+(1-\mathrm{B}p^2)s}$.

Substitute these in the third; from which results

$$\frac{\mathrm{A}-qs}{\mathrm{A}q+s}\cdot\frac{\mathrm{B}-p^2+(1+\mathrm{B})pq+\{(1+\mathrm{B})p-(1+\mathrm{B}p^2)q\}s}{-(1+\mathrm{B})p+(\mathrm{B}-p^2)q+\{1-\mathrm{B}p^2+(1+\mathrm{B})pq\}s} = \mathrm{C}\,;$$

which being arranged in reference to the unknown s, rises only to the second degree, and can hence be constructed with the ruler and compasses.

Put for the sake of further abbreviation,

$$\mathrm{F} = \mathrm{B}-p^2+(1+\mathrm{B})pq,$$
$$\mathrm{G} = (1+\mathrm{B})p-(1-\mathrm{B}p^2)q,$$
$$\mathrm{H} = -(1+\mathrm{B})+(\mathrm{B}-p^2)q,$$
$$\mathrm{K} = 1-\mathrm{B}p^2+(1+\mathrm{B})pq.$$

Then the equation becomes $\dfrac{\mathrm{A}-qs}{\mathrm{A}q+s}\cdot\dfrac{\mathrm{F}+\mathrm{G}s}{\mathrm{H}+\mathrm{K}s} = \mathrm{C}$, or,

$$(\mathrm{CK}+\mathrm{G}q)s^2+\{\mathrm{CH}-\mathrm{AG}+(\mathrm{CKA}-\mathrm{F})q\}s = \mathrm{A}(\mathrm{F}-\mathrm{CH}q),$$

from which we readily get s; and thence again t and u from the foregoing formulæ. (*Berlin Mémoires*, 1776, p. 284).

For the sake of arrangement, it appears preferable to give, consecutively, an account of all the solutions that have been attempted both on the principle of Lagrange and by any trigonometrical artifices whatever; omitting, till the concluding part of these notices, all further reference to the geometrical and semi-geometrical methods of other writers, as well as to some very recent researches made in this country by the aid of coordinate processes. This brings us at once to Lexell,* who undertook, at the request of Euler, to attempt the construction of Lagrange's result.

*Acta Academiæ Petrop. 1780, pt. ii.

It will be unnecessary to say, that no one would attempt the construction of Lagrange's *final* equation. Lexell, therefore, by eliminating u from the two last equations, and by very ingenious transformations, arrived at the two,

$$st = \mathrm{A}, \text{ and } \alpha s + \beta t = \gamma;$$

where α, β, γ are complex functions of the data of the problem. He then gave constructions by which these may be exhibited: but though managed with much skill, the constructive process is very operose. This, however, effected, the construction of the problem becomes reduced to a case of the general problem known in England as the "Lineal Section," and offers no serious difficulty; though it must be confessed that Lexell has been less happy in its construction than the familiarly known process amongst ourselves* would have led him to be. It is now, however, nearly seventy years since Lexell wrote. There is, still, sufficient evidence of Lexell's ability to cope with this class of questions, to justify the inference that had the *system* upon which Lagrange proceeded been capable of extension to other polygons, Lexell would have been able to so extend it. He attempted to so treat the inscribed *quadrilateral*, but confesses that he failed. He still discovered a property of that quadrilateral, which is worthy of notice: but his demonstration seems to me to be unnecessarily cramped by his use of the trigonometrical calculus in a case to which (as Ottajano has shewn) geometry much better applies. Lexell's choice of this method was, no doubt, in some degree influenced by his respect for the "calculus of sines" which was so fully developed by his great master, Euler† (a pardonable, and even a graceful error), and partly by the general inaptitude which prevailed upon the continent at that period for the use of the ancient geometry in any shape, or for any purpose whatever. This property of the inscribed quadrilateral hereafter will be given, either with Ottajano's demonstration or an original one.

Euler and Fuss had discussed the problem, the latter geometrically and the former by a mixture of geometry and trigonometry, anteriorly to the appearance of Lexell's *Mémoire;* and subsequently, Ottajano and Malfatti had extended the process of Fuss to the general problem where any polygon is substituted for the triangle. Leaving these solutions till we come to the geometrical portion of the history, we are brought to the paper published by Lhuilier in the Berlin *Mémoires* for 1796. This *Mémoire*, like most of those which the author has published is very full and detailed in the explication of its processes—at least as far as the principal problem is concerned. He takes (as had been done geometrically by the Italian geometers) the general case of the polygon of n sides. He also extends his method by *projective considerations* to the conic sections, and to the sphere; and under a varied form of the enunciation to a problem concerning a segment of the sphere or of the conoids, and n points any how situated in space. This form of the problem will be reproposed amongst the questions in the present number of the "Mathematician" for the consideration of English geometers.

* See the Liverpool "Student" (1798-1800) where all the cases are constructed with great elegance. The analysis and synthesis of the most difficult of them was given in the Mechanics' Magazine for Dec. 5, 1846, vol. iv., p. 533; and the analysis of the whole series is promised in that work.

† The introduction of the functional symbols sin x, cos x, *etc.* has probably created a greater change in the character of the mixed sciences dependent on algebra and geometry than any circumstance so simple, obvious, and natural, ever produced on any science—even than writing x^2 for xx. The original author of it, however, was not Euler but Simpson, who first used it in his Tracts: but the full development of the system is due to Euler. This is not the place, however, to dilate on such a subject.

He commences with the following theorem:—

Let ACA′ *be an isosceles triangle having its sides* AC, CA′ *equal, and from the vertex draw lines* CB, CB′ *cutting respectively the base and the base produced in* B *and* B′ : *then,*

$$\mathrm{CA} + \mathrm{CB} : \mathrm{CA} - \mathrm{CB} :: 1 : \tan \tfrac{1}{2}\,\mathrm{ACB} \tan \tfrac{1}{2}\,\mathrm{A'CB},$$
$$\mathrm{CB'} + \mathrm{CA} : \mathrm{CB'} - \mathrm{CA} :: 1 : \tan \tfrac{1}{2}\,\mathrm{ACB'} \tan \tfrac{1}{2}\,\mathrm{A'CB'}.$$

The proofs of these are too simple to need recapitulation here. Lhuilier correctly states that these equations constitute the "basis" of Lagrange's solution. The trigonometrical methods (as far as has been shewn by writers who have discussed the problem) do not seem capable of being founded on any other "basis."

Lhuilier's next proposition is the following problem :—

The sum or difference of two arcs, and the product of their tangents being given, to find them :—

Let ϕ be the given sum or difference, and x, x' the arcs themselves, such that $\tan x \tan x' = p$: then the solution is readily found to be derivable from the equation,

$$\frac{\cos(x - x')}{\cos(x + x')} = \frac{1 + p}{1 - p};$$

as whether the sum or difference be given, the other is found from this, and thence the arcs x, x' separately.

He then comes to a proposition, which it is difficult to express in words, and in which he has himself failed to give a very intelligible verbal enunciation.* It may be thus expressed by the aid of symbols :—

Let there be given any number of arcs $\phi, \phi_1, \phi_2, \ldots \phi_n$ and let it be conceived that each is divided into two parts, of which the first (estimated in order round the circle in which the arcs are taken) are $x, x_1, x_2, \ldots x_n$; also let $p, p_1, p_2, \ldots p_n$ be any given numbers whatever : then it is required to find $x, x_1, x_2, \ldots x_n$ so as to fulfil the following scale of conditions :—

$$\begin{aligned}
\tan x \quad & \tan(\phi_1 - x_1) = p \\
\tan x_1 \quad & \tan(\phi_2 - x_2) = p_1 \\
\tan x_2 \quad & \tan(\phi_3 - x_3) = p_2 \\
& \vdots \qquad\qquad\quad \vdots \\
\tan x_{n-1} \quad & \tan(\phi_n - x_n) = p_{n-1} \\
\tan x_n \quad & \tan(\phi - x) \;\, = p_n .
\end{aligned}$$

Lhuilier's investigation is simple and inartificial. He resolves the problem *by actual expansion* for the cases of $n = 2$, $n = 3$, and $n = 4$; and finds the result to be always of the *form*

$$\frac{\mathrm{R} - \mathrm{S} \tan x}{\mathrm{R'} - \mathrm{S'} \tan x} \cdot \tan x = p;$$

where R, S, R′, S′ are determinable (*but complex*) functions of the data of

* I give, in proof, the enunciation of Lhuilier, p. 99. "*Problème.* Soient des arcs en nombre quelconques données de grandeur. Les couper chacun en deux parties, de manière que les produits des tangentes de ces parties deux-à-deux soient donnés sous les conditions suivantes, Aucun de ces produits n'a facteurs les tangentes d'un même section; et le nombre des arcs donnés étant plus grand que deux, prenant le produit fait avec les tangentes d'une partie d'un des arcs et d'une partie d'un autre arc; il n'est aucun autre produit dont les deux facteurs soient les tangentes des deux parties restantes de ces arcs."

the problem. He then proceeds by the process which is often called the "inductive syllogism," to shew that if this formula be true for n arcs, it must be true for $n+1$ arcs. The principle of this method is so frequently employed by English writers, in proving the binomial and some other theorems, as to render further account of Lhuilier's reduction unnecessary.

The determination of $x, x_1, x_2, \ldots . x_n$ is therefore always possible, since any one of them is found from an equation of the second degree, and all the others by the given series of equations, which are thereby successively reduced to the first degree. We see, also, that each of them will have two values, and only two; although in certain cases the problem may give imaginary roots and be insoluble, or it may reduce to the first degree in tan x, and give but one solution for each x. These are mere points of minute detail upon which it is unnecessary to dwell.

It is obvious from the formation of the respective numerators and denominators that the successive terms of both constitute recurring series: but Lhuilier shews that the several entire fractions also constitute a recurring series, and are consequently expressible by a continued fraction. No especial use is made of this circumstance in the method of finding R, S, R′, S′ by calculation; although the form of the fraction is elegant, and it might, possibly, be yet rendered subservient to the purpose.

Lhuilier's solution of the problem itself takes the following form.

Let C be the centre of the circle, and r its radius; let $A_1, A_2, A_3, \ldots A_n$ be the points given in position, and their distances from the centre CA, CA_1, CA_2, ...CA_n be $a, a_1, a_2, \ldots a_n$; and put*

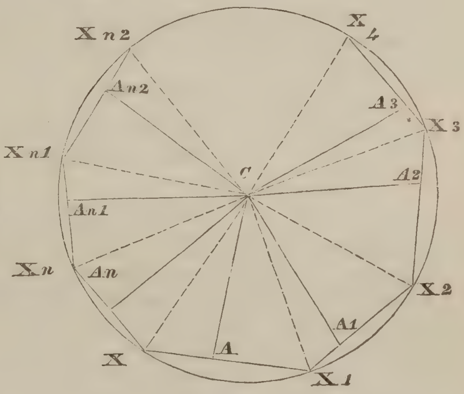
$$\frac{r-a}{r+a}=p, \frac{r-a_1}{r+a_1}=p_1, \frac{r-a_2}{r+a_2}=p_2, \ldots \frac{r-a_n}{r+a_n}=p_n:$$

also denote the angles thus

$$A_n\,CA=\phi,$$
$$ACA_1=\phi_1,$$
$$A_1\,CA_2=\phi_2,$$
$$\vdots$$
$$A_{n-1}\,CA_n=\phi_n;$$

($XX_1, X_1X_2, \ldots X_{n-1}X_n, X_nX$ being the sides of the polygon through $A, A_1, A_2, \ldots A_{n-1}, A_n$)

$$XCA=x,$$
$$X_1CA_1=x_1,$$
$$X_2CA_2=x_2,$$
$$\vdots$$
$$X_nCA_n=x_n;$$



and hence,

$$A_nCX=\phi-x,\ ACX_1=\phi_1-x_1,\ A_1CX_2=\phi_2-x_2, \ldots X_{n-1}CX_n=\phi_n-x_n.$$

Then we have by the first theorem quoted above the following equations expressing the conditions of the problem.

* These are taken on the hypothesis of $A, A_1, A_2, \ldots . A_n$ being *within* the circle: where this is not the case, Lhuilier directs the sign to be changed, or the expression to be taken $\frac{a_m-r}{a_m+r}$; and when the point is in the circumference, this vanishes, or $p_m=0$.

$$\begin{array}{ll}
\tan \tfrac{1}{2} x & \tan \tfrac{1}{2} (\phi_1 - x_1) = p \\
\tan \tfrac{1}{2} x_1 & \tan \tfrac{1}{2} (\phi_2 - x_2) = p_1 \\
\tan \tfrac{1}{2} x_2 & \tan \tfrac{1}{2} (\phi_3 - x_3) = p_2 \\
\vdots & \qquad\qquad\qquad \vdots \\
\tan \tfrac{1}{2} x_{n-1} & \tan \tfrac{1}{2} (\phi_n - x_n) = p_{n-1} \\
\tan \tfrac{1}{2} x_n & \tan \tfrac{1}{2} (\phi - x) = p_n.
\end{array}$$

These equations being identical with those already described, the solution of the problem is reduced to the resolution of those equations; and Lhuilier's view is sufficiently expressed by his *finale* to the first case—"donc le problème est résolu." It proves one thing at least, that the problem is *always possible as a strictly geometrical one*; and had it been given *anteriorly to the geometrical constructions* of Ottajano and Malfatti, it would have been valuable in this sense. No one, however, has given the algebraical resolution of these equations, so as to furnish separately the values of $x, x_1, \ldots x_n$. In fact, I am led to fear (from various trials of my own, and making use of every expedient for transformation and simplification which I could devise) that it is a hopeless task. I would speak however, with becoming hesitation as to the *possibility* of accomplishing a neat and symmetrical resolution of these equations: for their symmetry does not seem to be inconsistent with their being compounded from a set of symmetrical values—but rather the reverse. It is an open question, and a fitting exercise for the analyst. I can venture to say, on the part of the Editors, that if any of my readers shall discover such a solution, the "Mathematician" will be open to its publication. One suggestion, however, may be added here, by which a still more complete symmetry may be obtained.

Denote the several arcs by $2\omega, 2\omega_1, \ldots 2\omega_n$, and the unknown differences of their segments by $2\delta, 2\delta_1, \ldots 2\delta_n$; then the equations for solution will take the form

$$\begin{array}{l}
\tan(\omega + \delta)\tan(\omega_1 - \delta_1) = p \\
\tan(\omega_1 + \delta_1)\tan(\omega_2 - \delta_2) = p_1 \\
\tan(\omega_2 + \delta_2)\tan(\omega_3 - \delta_3) = p_2 \\
\qquad\vdots \qquad\qquad\qquad \vdots \\
\tan(\omega_n + \delta_n)\tan(\omega - \delta) = p_n.
\end{array}$$

The substitution of $\cot a, \cot a_1, \ldots$ for $p, p_1, \ldots$ and the conversion of the entire series of expressions into sines and cosines (though it initially complicate the equations) has, from several experiments upon the effect, led me to think that if the problem be soluble at all, it will be effected by this method. It is possible, however, that some hitherto undetected property of the series of tangents involved in the equations above, may lead to a solution. Still I wish these remarks to be received only as conjectural; and they may, after all, turn out to be valueless.

There is nothing extraordinary in the idea of extending this solution to the conic sections and the sphere. Indeed Euler had previously done the latter for the triangle in a scholium to his paper in the Petersburgh Acts, sixteen years before. Nor is there any point of material importance in the remaining part of Lhuilier's *Mémoire* to require mention here; and in fact, the latter part is only composed of sketchy remarks that are perfectly obvious to any one conversant with this class of subjects.

The next solution on trigonometrical principles is that of Carnot, published in his *Geometrie de Position*, 1803, p. 383. He seems to have taken up the subject incidentally; and he merely inserts a solution of this, as one of a considerable number of problems with which he illustrates his somewhat peculiar views. His notice of its history is casual and very incomplete; but what is of most consequence in this inquiry, he never alludes to Lhuilier's paper. It was, therefore, we are bound to believe, unknown to him; for his character for integrity is above all suspicion, in science as in political and personal morality. This statement is not unnecessary: for *Carnot's solution (though a little different in its manner) is yet very strikingly a reproduction of Lhuilier's*. It is, indeed, more neatly drawn up, and the steps of the process very easily followed; but still it differs in no particular of the least importance from the preceding one.

No account of Carnot's solution further than this can be required. Yet as English writers, following Mr. Bonnycastle,* have referred to this solution as the one, and only, improvement on Lagrange's, and the name of Lhuilier in connection with the problem has been *systematically* omitted by both French and English writers;† I have deemed it necessary to distinctly shew what that geometer had done, and to urge that hereafter his name should be substituted for that of Carnot in all notices of this problem.

* Geometry, page 348, 6th ed.

† M. Chasles is, I think, the only French writer who has mentioned Lhuilier's paper, and Dr. Traill and Mr. Potts the only English ones. Vid. Chasles, *Aperçu Historique des Méthodes en Géométrie*, page 328; Traill's Life of Simson, p. 96; and Pott's *Appendix to Euclid*, p. 97.

(To be continued.)

NOTE ON A VARIATION OF COTES'S THEOREM.

[*Pen-and-Ink.*]

The case of $x^{2n} - 2x^n \cos a + 1 = 0$ has been often discussed and is well known.

Let $x^{2n} + 2x^n \cos a + 1 = 0$, be proposed.

Resolving the quadratic, we have

$$x = \{-\cos a \pm \sin a \sqrt{-1}\}^{\frac{1}{n}}$$

$$= (-1)^{\frac{1}{n}} \{\cos a \pm \sin a \sqrt{-1}\}^{\frac{1}{n}}, \text{ or Demoivre's theorem,}$$

$$= (-1)^{\frac{1}{n}} \left\{ \cos \frac{2p\pi + a}{n} \pm \sin \frac{2p\pi + a}{n} \sqrt{-1} \right\},$$

in which p takes all values in succession from 0 to $n-1$, and the given function is the product of all these.

The quadratic resulting from the corresponding pair of factors is

$$x^2 - 2(-1)^{\frac{1}{n}}\, x \cos \frac{2p\pi + a}{n} + 1 = 0.$$

Now when n is even this is *imaginary*, and the resolution into real quadratic factors cannot be made by this process.

But when n is odd, the factor becomes

$$x^2 + 2\,x \cos \frac{2\,p\,\pi + a}{n} + 1 = 0.$$

This subject seems to require further investigation.

How are the roots of (-1) combined with the other factor? Do they not tend to increase the number of values (of the second side) of x? If not how are we to combine them so as to shew that the number is really not affected by the double multiplicity of the apparent roots? It is therefore recommended to the readers of the Mathematician, to examine how the roots of unity are connected in this with the old form of the property as usually given.

ON THE THEORY OF SURFACES.

[*By James Cockle, Esq., M.A., Barrister-at-Law.*]

A surface is referred to a system of rectangular axes $(x\,y\,z)$; I propose to pass from this to a system of angular coordinates $(\Psi\,\phi\,\theta)$ of which the signification will be seen from what follows:

The original meridian being in the plane of $x\,z$, the equator in that of $x\,y$, and the pole in the axis of z, Ψ is the longitude and ϕ the latitude of any point (P) of the surface. Next, take a point (A) in the axis of z and at a distance a from the origin, and let θ be the angle made with that axis by a line drawn from this point to P; θ may be termed the *pro-parallax*, and when θ, ϕ and Ψ are given, P is completely determined.

Let O be the origin of coordinates and let $\mathrm{O\,P} = r$; then since θ and $90° - \phi$ are two of the angles of the triangle O A P, the third angle is $90° - (\theta - \phi)$, and we have

$$\frac{r}{a} = \frac{\mathrm{OP}}{\mathrm{OA}} = \frac{\sin \mathrm{O\,A\,P}}{\sin \mathrm{A\,P\,O}} = \frac{\sin \theta}{\cos(\theta - \phi)},$$

$$\text{or, } r = \frac{a \sin \theta}{\cos(\theta - \phi)},$$

and the formulæ for passing from the linear to the angular coordinates are

$$x = \frac{a \sin \theta \cos \phi \cos \Psi}{\cos(\theta - \phi)}$$

$$y = \frac{a \sin \theta \cos \phi \sin \Psi}{\cos(\theta - \phi)},$$

$$z = \frac{a \sin \theta \sin \phi}{\cos(\theta - \phi)},$$

and those for passing from the angular to the linear are

$$\tan \Psi = \frac{y}{x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\tan^2 \phi = \frac{z^2}{x^2 + y^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\tan^2 \theta = \frac{x^2 + y^2}{(a - z)^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

or such modifications of them as may be found desirable. The above are by no means the most general transformations of this class. Every surface has probably a characteristic transformation by which its equation may be

materially simplified. An instance will be afforded by the surface whose equation referred to rectangular coordinates is

$$r^2 x^2 (x^2 + y^2) - \{x^2 z^2 + r^2 (x^2 - y^2)\} (r^2 - 2az + a^2) = 0.$$

I shall first show that this surface has circular sections which cut one another at right angles, and then confirm this result by means of the transformation above discussed.

In the given equation suppose

$$x = x_1 \cos a,\ y = x_1 \sin a,$$

the result will be, after dividing by $\cos^2 a$, reducing, *etc.*,

$$(x_1^2 + z^2)\, x_1^4 - x_1^2 z^2 \{x_1^2 + (z-a)^2\} + \mathrm{F}\,(x_1, z).(1 - \tan^2 a) = 0.$$

Let $\tan^2 a = 1$, then this last equation may be reduced to

$$x_1^2 \{x_1^4 - z^2 (z-a)^2\} = 0\,;$$

the second factor will vanish provided that

$$x_1^2 + z\,(z-a) = x_1^2 + (z - \frac{a}{2})^2 - \left(\frac{a}{2}\right)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots (a),$$

the other solution is unfitted for our purpose, but (a), being the equation to a circle, shows that there are planes which intersect the surface in a circle. Again, since $a = 45°$ or $-45°$ we see that if the plane of $x\,z$ were made to rotate round the axis of z through half a right angle in either direction, its two positions would coincide with those planes. And the circular sections, having the same radii, will meet at right angles.

The same thing may be shown as follows. Divide the given equation by

$$r^2 x^2 (r^2 - 2\,a\,z + a^2),$$

and it becomes

$$\frac{x^2 + y^2}{r^2 - 2\,a\,z + a^2} - \frac{z^2}{r^2} + \frac{y^2}{x^2} - 1 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

but by means of (1) and by slightly changing the forms of (2) and (3) we see that (b) may be put under the form

$$\sin^2\theta - \sin^2\phi + \tan^2\Psi - 1 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (c).$$

Let $\tan^2\Psi = 1$ and consequently $\Psi = 45°$ or $-45°$ then (c) becomes

$$\sin^2\theta - \sin^2\phi = 0,$$

of which one factor is

$$\sin\theta - \sin\phi = 0.$$

But this represents a circle and since the two values of Ψ differ by 90° from each other the given surface has two circular sections in planes at right angles to one another, and these sections intersect, since their centres coincide and their radii are each equal to $\frac{1}{2}a$.

2, *Church Yard Court, Temple*, 18*th Dec.*, 1847.

THEOREMS ON FACTORIALS ESTABLISHED BY THE THEORY OF PROBABILITIES.

[*Mr. Hugh Godfray.*]

Let there be a heap of 2A balls, A white and A black, then the chance of drawing a white ball is $\frac{1}{2}$. Now let there be any number m taken promiscuously out of the heap; then the chance of drawing a white ball out of

these m evidently remains $\frac{1}{2}$. But the probability may be put under a different form as follows:

The probability that these m balls will consist of

$$\begin{array}{llll}
m & \text{white balls and } 0 \text{ black ones} & = & \dfrac{A(A-1)(A-2)\ldots\ldots(A-m+1)}{2A(2A-1)\ldots\ldots(2A-m+1)} \\[2ex]
m-1 & \ldots\ldots\ 1 \ldots\ldots & = & \dfrac{A(A-1)\ldots\ldots(A-m+2).A}{2A(2A-1)\ldots\ldots(2A-m+1)}\cdot\dfrac{m}{1} \\[2ex]
m-2 & \ldots\ldots\ 2 \ldots\ldots & = & \dfrac{A(A-1)\ldots(A-m+3).A(A-1)}{2A(2A-1)\ldots\ldots(2A-m+1)}\cdot\dfrac{m(m-1)}{1.2} \\
\vdots & \quad\ \vdots & & \vdots \\
m-r & \ldots\ldots\ r \ldots\ldots & = & \dfrac{A(A-1)..(A-m+r+1).A(A-1)..(A-r+1)}{2A(2A-1)\ldots\ldots\ldots\ldots(2A-m+1)}\cdot\dfrac{m(m-1)..(m-r+1)}{1.2\ldots\ldots r} \\
\vdots & \quad\ \vdots & & \vdots \\
1 & \ldots\ldots\ m-1 \ldots\ldots & = & \dfrac{A.A(A-1)\ldots(A-m+2)}{2A(2A-1)\ldots(2A-m+1)}\cdot\dfrac{m(m-1)\ldots 2}{1.2\ldots\ldots(m-1)} \\[2ex]
0 & \ldots\ldots\ m \ldots\ldots & = & \dfrac{A(A-1)\ldots\ldots(A-m+1)}{2A(2A-1)\ldots(2A-m+1)}\cdot\dfrac{m(m-1)\ldots 1}{1.2\ldots\ldots m}.
\end{array}$$

Now the probability of drawing a white ball when they are all white is certainty or 1, which multiplied by the chance of their being all white

gives $\dfrac{A(A-1)\ ..\ (A-m+1)}{2A(2A-1)\ ..\ (2A-m+1)}$ for the value of that chance.

The chance of drawing a white ball when $m-1$ are white and 1 black is $\dfrac{m-1}{m}$, which multiplied by the chance of their being in that ratio, gives

$$\frac{A(A-1)\ ..\ (A-m+2).A}{2A(2A-1)..\ (2A-m+1)}.\ \frac{m-1}{1}$$ for the value of that chance.

And in general the chance of drawing a white ball when there are $m-r$ white and r black ones is $\dfrac{m-r}{m}$ which multiplied by the chance (determined above) of their being in that ratio gives

$$\frac{A\,(A-1)..(A-m+r+1)A(A-1)..(A-r+1)}{2A(2A-1)......................(2A-m+1)}.\ \frac{(m-1)(m-2)..(m-r)}{1.2.............r},$$

and the sum of all these probabilities must give the total probability $\frac{1}{2}$ as before, whence we deduce the following theorem, after dividing all through by A.

THEOREM I.

$$\Big[(A-1)(A-2)..(A-m+1)\Big] + \Big[(A-1)..(A-m+2)\frac{m-1}{1}.\,A\Big]+..$$

$$..+\Big[(A-1)(A-2)..(A-m+r+1)\frac{(m-1)(m-2)..(m-r)}{1.2......r}A..(A-r+1)\Big]+..$$

$$..+\Big[\frac{(m-1)}{1}\frac{(m-2)}{2}\ ...\ \frac{1}{m-1}.A.(A-1)..(A-m+2)\Big]$$

$$=(2A-1)\ (2A-2)..(2A-m+1),$$

wherein we may assign to m and A any whole and positive values we please, provided A is not less than $\frac{1}{2}\,m$.

Corollary. Let $A=m$ and transpose the first term of the left hand member in the preceding equality, then we get

$$\frac{m-1}{1}.2.3.4...m+\frac{(m-1)^2}{1}.\ \frac{(m-2)}{2}.3.4..m$$

$$+\frac{(m-1)^2}{1}.\ \frac{(m-2)^2}{2}.\ \frac{(m-3)}{3}.4.5..m+..$$

$$..+\frac{(m-1)^2}{1}.\ \frac{(m-2)^2}{2}.\ \frac{(m-3)^2}{3}.....\frac{2}{(m-2)}.\ \frac{1}{(m-1)}.m$$

$$=(2m-1)(2m-2)..(m+1)-(m-1)(m-2)..3.2.1.$$

Thus if $m=6$ we have

$$\frac{5}{1}.2.3.4.5.6+\frac{25}{1}.\frac{4}{2}.3.4.5.6+\frac{25}{1}.\frac{16}{2}.\frac{3}{3}.4.5.6$$

$$+\frac{25}{1}.\frac{16}{2}.\frac{9}{3}.\frac{2}{4}.5.6+\frac{25}{1}.\frac{16}{2}.\frac{9}{3}.\frac{4}{4}.\frac{1}{5}.6=11.10.9.8.7-5.4.3.2.1.$$

In general let there be A white balls and B black balls mixed together, then the probability of drawing firstly a white and then b black from among them is

$$\frac{A(A-1)(A-2)\dots(A-a+1)B(B-1)\dots(B-b+1)}{(A+B)(A+B-1)\dots\dots\dots\dots(A+B-a-b+1)}.$$

Now let m of the $A+B$ balls be taken promiscuously and set aside; the probability of drawing firstly a white and then b black from these m will remain the same as before.

The à priori probability that these m balls contain r white and $m-r$ black ones, is

$$=\frac{A(A-1)\dots(A-r+1)B(B-1)\dots(B-m+r+1)}{(A+B)(A+B-1)\dots\dots(A+B-m+1)}\cdot\frac{1.2\dots\dots m}{1.2\dots r.1.2.(m-r)}.$$

The probability of drawing firstly a white and then b black out of these, is

$$=\frac{r(r-1)\dots(r-a+1)(m-r)(m-r-1)\dots(m-r-b+1)}{m(m-1)\dots\dots\dots\dots\dots\dots(m-a-b+1)}.$$

Multiplying these together, taking the sum of all similar results and equating this to the former value of the probability, we have

THEOREM II.

$$\Sigma\frac{A(A-1)\dots(A-r+1)B(B-1)\dots(B-m+r+1)}{(A+B)(A+B-1)\dots\dots\dots(A+B-m+1)}\cdot\frac{1.2\dots\dots m}{1.2\dots r.1.2\dots(m-r)}.$$
$$\frac{(r-a+1)\dots r(m-r-b+1)\dots(m-r)}{(m-a-b+1)\dots\dots\dots\dots\dots m}=$$
$$\frac{A(A-1)\dots(A-a+1)B(B-1)\dots(B-b+1)}{(A+B)(A+B-1)\dots\dots\dots(A+B-a-b+1)},$$

where r must take all the integer values from a to $m-b$ inclusively, and Σ stands for the sum of all similar values.

Corol. 1. If we calculate the particular values for $r=a$ and $r=m-b$, we find $\frac{A(A-1)\dots(A-a+1)B(B-1)\dots(B-m+a+1)}{(A+B)(A+B-1)\dots\dots\dots(A+B-m+1)}$ and

$\frac{A(A-1)\dots(A-m+b+1)B(B-1)\dots(B-b+1)}{(A+B)(A+B-1)\dots\dots\dots\dots(A+B-m+1)}$; transposing these and taking the sum of the remaining values, we have after multiplying both sides by $\frac{(A+B)(A+B-1)\dots(A+B-m+1)}{A(A-1)\dots\dots(A-a+1)B(B-1)\dots\dots(B-b+1)}$, and reducing

$$\Sigma(A-a)(A-a-1)\dots(A-r+1)(B-b)(B-b-1)\dots(B-m+r+1)$$
$$\times\frac{1.2\dots\dots\dots\dots(m-a-b)}{1\cdot 2\dots(r-a)1.2\dots(m-r-b)}=$$
$$\Big[(A+B-a-b)(A+B-a-b-1)\dots(A+B-m+1)\Big]-$$
$$\Big[(A-a)(A-a-1)\dots(A-m+b+1)\Big]-\Big[(B-b)(B-b-1)\dots(B-m+a+1)\Big],$$

where r must take all the integer values between a and $m-b$ exclusively.

Corol. 2. The last result may be put under a somewhat different form as follows: Let $A-a=H$, $B-b=K$, $m-a-b=c$, and $r-a=v$, then the equality becomes

$$\Sigma \frac{H(H-1)..(H-v+1)}{1.2..........v} \cdot \frac{K(K-1)...(K-(c-v)+1)}{1.2...... ... (c-v)} =$$
$$\frac{(H+K)(H+K-1)......(H+K-c+1)}{1.2.......................c} -$$
$$\frac{H(H-1)....(H-c+1)}{1.2............c} - \frac{K(K-1).... (K-c+1)}{1.2..........c},$$

wherein H, K and c are constants and v must take all the integer values from 1 to $c-1$ inclusively.

Corol. 3. If in corollary (2) we make $K=m$, $H=c=m-1$, the result becomes after transposing the last term of the right hand member and multiplying both sides by $1.2...m$,

$$2.3..m+\Sigma \frac{(m-1)^2}{1} \cdot \frac{(m-2)^2}{2} \cdots \frac{(m-v)^2}{v} (v+2)(v+3)....m$$
$$=(2m-1)(2m-2)..(m+1)-(m-1)(m-2)..2.1,$$

where v must vary from 1 to $m-2$ inclusively.

These values of H, K and c correspond to $A=m$, $B=m$, $a=1$, and $b=0$, and therefore reduce the general case to that of the corollary to Theorem I, and it will be found on inspection that the results are the same, for the second members of the equations are identical, and the terms of the first will become so, by reversing their order in one of the equations and cancelling or introducing common factors in numerators and denominators as will be found necessary.

Corol. 4. In cor. (2) make $H = K = m$ and $c = m - p$, the equation becomes

$$\Sigma \frac{m(m-1)....(m-v+1)}{1.2............v} \cdot \frac{m(m-1)....(p+v+1)}{1.2....\{m-(p+v)\}} =$$
$$\frac{2m(2m-1)..(m+p+1)}{1.2......(m-p)} - \frac{m(m-1)..(p+1)}{1.2......(m-p)} - \frac{m(m-1)..(p+1)}{1.2......(m-p)};$$

now introduce the factors $1.2..m$ in numerator and denominator of the first member and in the last term of the second member, we get

$$\Sigma \frac{m(m-1)..(m-v+1)}{1.2..........v} \cdot \frac{m(m-1)..(p+v+1)}{1.2....\{m-(p+v)\}} \cdot \frac{1.2.3..m}{1.2.3..m} =$$
$$\frac{2m(2m-1)..(m+p+1)}{1.2.3......(m-p)} - \frac{m(m-1)..(p+1)}{1.2.... ...(m-p)} -$$
$$\frac{m(m-1)....(p+1)}{1.2........(m-p)} \cdot \frac{1.2.3..m}{1.2.3..m};$$

or by reducing,

$$\Sigma \frac{m(m-1)..(m-v+1)}{1.2... ....v} \cdot \frac{m(m-1)....\{m-(p+v)+1\}}{1.2...... (p+v)} =$$
$$\frac{2m(2m-1)..(m+p+1)}{1.2..........(m-p)} - \frac{m(m-1)..(p+1)}{1.2....(m-p)} - \frac{m(m-1)...(m-p+1)}{1.2............p}.$$

Now, if we represent by M_0, M_1, M_2, ... M_m the coefficients in the expansion of $(x+y)^m$, we shall have, remembering that M_0 and M_m are each $=1$, and transposing the last two terms of the second side

$$M_0\, M_p + \Sigma[M_v \,.\, M_{v+p}] + M_{m-p}M_m = \frac{2m(2m-1)\ldots.(m+p+1)}{1.2.\ldots\ldots.(m-p)},$$

where v must take all the integer values from 1 to $m-1$ and p is constant.

This is a known relation (Hind's Algebra, 3rd ed., page 462,) which when $p=0$ gives the sum of the squares of the coefficients in a binomial expansion

$$1^2 + \left(\frac{m}{1}\right)^2 + \left(\frac{m(m-1)}{1\cdot 2}\right)^2 + \ldots etc. \ldots = \frac{2m(2m-1)\ldots\ldots(m+1)}{1\,2\ldots\ldots\ldots\ldots\ m}$$

$$= 2^m \,.\, \frac{1.3\,5.7.\ \ldots.\ (2m-1)}{1.2.3.4\ \ldots\ldots.\ m}.$$

Many other theorems of great generality may be obtained by considering the heap to consist of A white, B black, C red, *etc.*, balls; and other general theorems again, by supposing, firstly, m balls to be set aside, then n of these to be again set aside, and so on.

In the Annales (Nouvelles) de Mathématiques, tom 1, page 179, the following problem is solved, in my opinion, erroneously.

An urn contains m balls of two colours, black and white, in an unknown ratio, one of the colours may even be zero; two balls are taken out at the same time, and one being looked at is white; what is the chance that the other is also white?

Now, it is argued:—" Since the urn may, previous to drawing, contain

m white and 0 black balls,

or $m-1$........ 1 ....,

or $m-2$........ 2 ....,

⋮ ⋮

or 0 .... m ....,

it is the same as if we had $m+1$ urns with these different combinations, and knew not from which urn the two balls are taken.

" Now each urn contains m balls and admits of $m(m-1)$ variations, two and two, and as there are $m+1$ urns, the total possible number of ways of drawing is $(m+1)m(m-1)$. The arrangements and not the combinations are here taken, for the question evidently requires that we pay attention to the order in which the two balls are drawn.

" There are as many drawings commencing with a black as with a white ball, hence $\frac{(m+1)m(m-1)}{2}$ expresses the number of cases in which a white ball may be drawn first.

" Let us now examine the number of cases favourable to the drawing of a second white ball.

" Two white balls may be drawn from the first urn which contains m white ones, in........ ..... $m(m-1)$ ways,

from the second, in ....$(m-1)(m-2)$,,

from the third, in..........$(m-2)(m-3)$,,

from the m^{th} and $(m+1)^{\text{th}}$, in............0 ways.

" Therefore the number of chances favourable to a second white ball is

$$m(m-1)+(m-1)(m-2)+\ldots.2.1 = \frac{(m+1)m(m-1)}{3};$$

hence the number of chances that the first is white and the second black, is

$$\frac{(m+1)m(m-1)}{2} - \frac{(m+1)m(m-1)}{3} = \frac{(m+1)m(m-1)}{6};$$

that is, just half the number of chances of drawing 2 white: hence the chances, whatever be the number m, are as 2 : 1 in favour of the second ball being also a white."

Now this extraordinary result which is true of $m+1$ urns arranged as in the preceding investigation, is not so for one urn with m balls as in the question; in fact, the error consists in having assimilated the two cases, by which it is assumed that there is as great a chance of any one of these combinations occurring as of any other.

But if we suppose the total number of white balls in the world to be A and of black balls B, then the probability that m balls taken indiscriminately are all white, is $\frac{A(A-1)\ldots\ldots(A-m+1)}{(A+B)(A+B-1)\ldots(A+B-m+1)}$; the probability of there being $m-1$ white ones and 1 black one, is $\frac{A(A-1)\ldots\ldots(A-m+2)B}{(A+B)(A+B-1)\ldots(A+B-m+1)}\cdot\frac{m}{1}$, a result which cannot be equal to the former for *all* values of m.

Let us consider the $A+B$ balls:—the probability of drawing two white ones from among them is $\frac{A(A-1)}{(A+B)(A+B-1)}$, and the probability of drawing firstly a white and then a black one is $\frac{A}{(A+B)}\cdot\frac{B}{(A+B-1)}$, and the ratio of the former probability to the latter is $\frac{A-1}{B}$, which, suffering no circumstance to have induced a preponderance of one colour over the other, (that is, taking $A=B$), becomes $\frac{A-1}{A}$, or less than 1.

Now the laying aside in an urn m of these balls taken indiscriminately, and then taking two balls from the urn, can noways alter the state of the probability, therefore the odds, instead of being 2 : 1 in favour of the second ball being white, are rather less than 1 : 1.

Thus, suppose $m=3$, and let A and B represent the total number of white and black balls respectively,

The chance that the 3 balls are all white is $\frac{A(A-1)(A-2)}{(A+B)(A+B-1)(A+B-2)}$,

The chance that 2 are white and 1 black is $\frac{A(A-1)B}{(A+B)(A+B-1)(A+B-2)}\cdot\frac{3}{1}$,

..............1............2 ...... $\frac{A.B.(B-1)}{(A+B)(A+B-1)(A+B-2)}\cdot\frac{3}{1}$,

..............0............3 ..... $\frac{B(B-1)(B-2)}{(A+B)(A+B-1)(A+B-2)}$.

Now the probability that 2 white balls are drawn in succession is, in the first case, $\frac{2}{3}\cdot\frac{2}{2}$, in the second, $\frac{2}{3}\cdot\frac{1}{2}$, in the third and fourth 0; these mul-

tiplied by the respective à priori probabilities and the results added, give

$$\frac{A(A-1)(A+B-2)}{(A+B)(A+B-1)(A+B-2)} = \frac{A(A-1)}{(A+B)(A+B-1)}.$$

The probability that the balls will be the first white, the second black, is, in the first case, 0, in the second, $\frac{2}{3}\cdot\frac{1}{2}$, in the third, $\frac{1}{3}\cdot\frac{2}{2}$, and in the fourth 0; these multiplied as before and the results added gives for the probability in favour of the second ball being black,

$$\frac{A.B(A+B-2)}{(A+B)(A+B-1)(A+B-2)} = \frac{A.B}{(A+B)(A+B-1)};$$

$$\therefore\ \frac{\text{probability in favour of white}}{\text{probability in favour of black}} = \frac{A-1}{B}, \text{ as before.}$$

St. Mark's Road, Jersey, Oct. 27th, 1847.

SOLUTIONS OF MATHEMATICAL EXERCISES.

CXVI. *By J. J.*

There are n candidates for a situation, to whom the voters attribute every degree of merit contained between the limits 0 and m. Shew how to find the mean value of all the candidates' merits.

[SOLUTION. *By J. J., the proposer.*]

A solution to this question may easily be obtained from the "Théorie Analitique des Probabilities," p. 273: the following answer differs in some respects from the process given in that work.

Let $t_1, t_2, t_3 \ldots t_r$ denote the respective merits of the n candidates, t_1 being the merit attributed to the candidate whom the voters deem most worthy, t_2 the merit ascribed to the next candidate in point of merit, and t_r the merit attributed to the candidate in the rank r.

First, suppose that t_1 may have any positive value comprised between the limits of 0 and m, and that the mean of all such positive values of t_1 is required. Let w be the whole number of the values and Σt_1 their sum; then the required mean M is equal to $\frac{\Sigma t_1}{w}$. Suppose that the values of t_1 increase by equal intervals Δ, and that they become $0, \Delta, 2\Delta, \ldots m$; then $w = \frac{\Sigma\Delta}{\Delta}$; in this case, $M = \frac{\Delta\Sigma t_1}{\Delta\Sigma}$; but because t_1 is supposed to receive all values between 0 and m the quantity Δ may be considered infinitely small, and may consequently be represented by the differential dt_1; hence the sums Σ may be changed into integrals, and

$$\therefore\quad M = \frac{\int_0^m t_1 dt_1}{\int_0^m dt_1} = \frac{\frac{1}{2}m^2}{m} = \tfrac{1}{2}m,$$

therefore the mean of all these values is equal to the mean of the two extreme values.

Next suppose t_1, t_2 to be the merits of two candidates according to the opinion of any voter, t_1 being the merit that he attributes to the most worthy candidate, t_1 therefore being greater than t_2. Here t_2 may have any values between 0 and t_1; and t_1 any values between $t_2 = 0$ and $t_2 = t_1$.

The aggregate of all the values of t_1 contained between $t_2 = 0$ and $t_2 = t_1$ is equal to $\dfrac{t_1\int_0^{t_1} dt_2}{dt_2}$, and the sum of all the values of t_1 is equal to $\dfrac{\int_0^m (t_1 dt_1 \int_0^{t_1} dt_2)}{dt_1 dt_2}$; the number of these values is $\dfrac{\int_0^m dt_1 \int_0^{t_1} dt_2}{dt_1 dt_2}$; therefore the mean value of t_1 is $\dfrac{\int_0^m\int_0^{t_1} t_1 dt_1 dt_2}{\int_0^m\int_0^{t_1} dt_1 dt_2}$.

Similarly, if there be n candidates and t_r be the merit similarly attributed to the a candidate in rank r, the mean value of t_1 will be

$$\frac{\int_0^m\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} t_r dt_1 dt_2 \ldots dt_n}{\int_0^m\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} dt_1 dt_2 \ldots dt_n}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}\cdots\frac{1}{n-r}\cdot\frac{1}{1-r+2}\cdot\frac{1}{1-r+3}\cdots\frac{1}{n+1}m^{n+1}}{\frac{1}{2}\cdot\frac{1}{3}\cdots\cdots\cdots\frac{1}{n}m^n} = \frac{n-r+1}{n+1}m.$$

Now if n be an odd number $t_{\frac{n+1}{2}}$ will be the middle term and its mean value $\frac{1}{2}m$. If n be an even number $t_{\frac{n}{2}}$, $t_{\frac{n+2}{2}}$ are the two middle terms; their mean values are $\frac{1}{2}m + \dfrac{\frac{1}{2}m}{n+1}$ and $\frac{1}{2}m - \dfrac{\frac{1}{2}m}{n+1}$, and the mean of these values is $\frac{1}{2}m$. The sum of n mean values is $\frac{1}{2}mn$, and therefore the mean of all the values is $\frac{1}{2}m$, half of the two extreme values, which is the answer required.

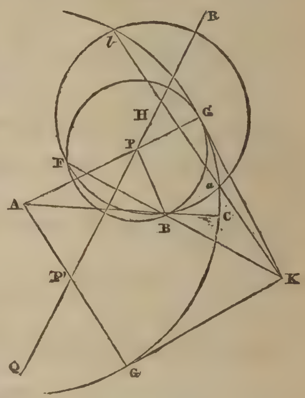
CXXIV. *Mr. Thomas Weddle, Wimbledon.*

A straight line is given in position, and the axes of an ellipse both in position and magnitude, to determine (graphically) the points of intersection of the straight line and ellipse, without describing the curve.

[SOLUTION. *By Amicus; and similarly by Mr. Ambler.*]

This problem differently enunciated, is solved in Simson's Geometry, 3rd edition, p. 223; the following solution is however more simple and direct than that referred to.

Let A and B be the foci of the ellipse (which are readily found from the data) and RQ the given straight line, and in AB produced take AC, the major axis. From centre A with radius AC describe a circle aGb; from H a point in QR as centre, with radius HB, describe a circle Bab cutting the former circle in a and b; join ba, and draw BF perpendicular to QR and meeting the circle Bab in F; produce FB to meet ba produced in K. From K draw KG and KG′ touching the circle aGb in G and G′, draw GA and G′A cutting QR in P and P′; P and P′ shall be the points of intersection required.



For by the circle aGb, $bK \cdot Ka = KG^2$; and by the circle Bab, $FK \cdot KB = bK \cdot Ka$; $\therefore FK \cdot KB = KG^2$; $\therefore$ KG touches the circle which passes through the points F, B and G; and because PP′ bisects FB at right angles, and PG is a perpendicular to KG, the centre of that circle is the point of intersection P of those two lines; hence $AP + PB = AG = AC$; $\therefore$ P is a point of intersection of QR with the ellipse, and similarly P′ is the other point of intersection.

Good solutions have also been received from Dr. Burns, Rochester; and Messrs. Hugh Godfray, Jersey; Joseph Paul, Shooter's Hill; A. Roy, Dundee Academy; Nicholas Smyth, Galway; and Thomas Weddle, the proposer (two solutions). The solutions of the proposer and Mr. Joseph Paul are very elegant and are omitted with much regret.

CXXV. *Mr. Thomas Weddle, Wimbledon.*

The sum of the squares of the reciprocals of the areas of the sections of an ellipsoid made by any three diametral planes at right angles to each other, is equal to the sum of the squares of the reciprocals of the areas of the sections made by the three principal planes.

[First Solution. *Mr. Thomas Weddle, the proposer; and similarly by Mr. A. Roy, Dundee.*]

Let the equation to the ellipsoid referred to any three diametral planes at right angles to each other be

$$Ax^2+By^2+Cz^2+2Dyz+2Exz+2Fxy=H \quad \ldots\ldots\ldots\ldots (1).$$

If $2a$, $2b$, $2c$ be the three principal diameters, and

$$r^2 = x^2+y^2+z^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

then will a, b, c be the values of r when (2) is a maximum or minimum. Equate the differentials of (1) and (2) to zero,

$$\therefore (Ax+Ez+Fy)dx+(By+Dz+Fx)dy+(Cz+Dy+Ex)dz=0 \quad \ldots (3),$$

$$xdx+ydy+zdz=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Multiply (4) by λ, add the resulting equation to (3), and equate the coefficients of dx, dy, and dz to zero,

$$\therefore \left.\begin{aligned} Ax+Ez+Fy+\lambda x&=0 \\ By+Dz+Fx+\lambda y&=0 \\ Cz+Dy+Ex+\lambda z&=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5).$$

Multiply these equations by x, y, z respectively, add, and note (1, 2),

$$\therefore H+\lambda r^2=0, \quad \text{or } \lambda = -\frac{H}{r^2}.$$

Hence the equations (5) become

$$\left.\begin{aligned} -\left(\frac{H}{r^2}-A\right)x+Fy+Ez=0 \\ Fx-\left(\frac{H}{r^2}-B\right)y+Dz=0 \\ Ex+Dy-\left(\frac{H}{r^2}-C\right)z=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(6).$$

Eliminate x, y, and z from these equations by cross multiplication,

$$\therefore \left(\frac{H}{r^2}-A\right)\left(\frac{H}{r^2}-B\right)\left(\frac{H}{r^2}-C\right)$$
$$-D^2\left(\frac{H}{r^2}-A\right)-E^2\left(\frac{H}{r^2}-B\right)-F^2\left(\frac{H}{r^2}-C\right)-2DEF=0;$$

Or, $$\frac{H^3}{r^6}-\left(A+B+C\right)\frac{H^2}{r^4}+\left(BC-D^2+AC-E^2+AB-F^2\right)\frac{H}{r^2}$$
$$-(ABC+2DEF-AD^2-BE^2-CF^2)=0\ldots\ldots\ldots(7).$$

This is a cubic in $\frac{1}{r^2}$, of which the three roots are $\frac{1}{a^2}$, $\frac{1}{b^2}$, and $\frac{1}{c^2}$; hence by the theory of equations

$$\frac{1}{b^2c^2}+\frac{1}{a^2c^2}+\frac{1}{a^2b^2}=\frac{BC-D^2}{H^2}+\frac{AC-E^2}{H^2}+\frac{AB-F^2}{H^2}\ldots.(8).$$

Let S_1, S_2, S_3 be the areas of the sections made by the planes of yz, xz and xy respectively, and s_1, s_2, s_3 those of the sections made by the principal diametral planes. Putting $x=0$ in (1), we have

$$By^2+Cz^2+2Dyz=H,$$

for the equation of the ellipse which is the intersection of the plane of yz with the ellipsoid (1), and the area of this ellipse will, by a process similar to the preceding*, be found to be $\dfrac{\pi.H}{(BC-D^2)^{\frac{1}{2}}}$; hence

$$S_1^2=\frac{\pi^2.H^2}{BC-D^2}.\quad \text{Similarly, } S_2^2=\frac{\pi^2H^2}{AC-E^2}, \text{ and } S_3^2=\frac{\pi^2H^2}{AB-F^2}.$$

Also it is evident that $s_1=\pi bc$, $s_2=\pi ac$, and $s_3=\pi ab$; hence, dividing (8) by π^2 we have

$$\frac{1}{s_1^2}+\frac{1}{s_2^2}+\frac{1}{s_3^2}=\frac{1}{S_1^2}+\frac{1}{S_2^2}+\frac{1}{S_3^2};$$

and thus the exercise is established.

Cor. 1. From (7), we also have,

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{A}{H}+\frac{B}{H}+\frac{C}{H},$$

which is easily seen to amount to the well known theorem :—The sum of the squares of the reciprocals of any three diameters of an ellipsoid at right angles to one another, is equal to the sum of the squares of the reciprocals of the three principal diameters.

* See Mathematician, Vol. II., p. 230 or p. 322; or Gregory's Examples, p. 123.

Cor. 2. Recurring once more to (7), we get,

$$\frac{1}{a^2b^2c^2} = \frac{\text{ABC} + 2\text{DEF} - \text{AD}^2 - \text{BE}^2 - \text{CF}^2}{\text{H}^3};$$

hence, volume of the ellipsoid $= \frac{4}{3}\pi abc$

$$= \frac{4\pi}{3} \cdot \frac{\text{H}^{\frac{3}{2}}}{(\text{ABC} + 2\text{DEF} - \text{AD}^2 - \text{BE}^2 - \text{CF}^2)^{\frac{1}{2}}}.$$

(See Gregory's Examples, p. 121.)

[SECOND SOLUTION. *Mr. J. W. Elliott, Greatham.*]

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then the areas of the elliptic sections made by the principal planes are πab, πbc, πca.

Again, let A, A_1, A_2 be the areas of the sections of (1) made by the diametral planes

$$\left.\begin{aligned} l\,x + m\,y + n\,z &= 0 \\ l_1x + m_1y + n_1z &= 0 \\ l_2x + m_2y + n_2z &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

l, m, n, *etc.* being the direction cosines of the planes; then as usual (see *Gregory's Solid Geom.*, p. 102.)

$$\frac{1}{\text{A}^2} = \frac{a^2l^2 + b^2m^2 + c^2n^2}{(\pi abc)^2}, \quad \frac{1}{\text{A}_1^2} = \frac{a^2l_1^2 + b^2m_1^2 + c^2n_1^2}{(\pi abc)^2},$$

$$\frac{1}{\text{A}_2^2} = \frac{a^2l_2^2 + b^2m_2^2 + c^2n_2^2}{(\pi abc)^2}.$$

Moreover, since the planes (2) are at right angles to each other,

$$l^2 + l_1^2 + l_2^2 = 1, \quad m^2 + m_1^2 + m_2^2 = 1, \quad n^2 + n_1^2 + n_2^2 = 1.$$

Hence,

$$\frac{1}{\text{A}^2} + \frac{1}{\text{A}_1^2} + \frac{1}{\text{A}_2^2} = \frac{1}{(\pi ab)^2} + \frac{1}{(\pi bc)^2} + \frac{1}{(\pi ca)^2};$$

which proves the truth of this beautiful theorem.

[THIRD SOLUTION. *Mr. Peter Elliott, Lamesley, Durham.*]

Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

be the equation to an ellipsoid, and

$$\left.\begin{aligned} l\,x + m\,y + n\,z &= 0 \\ l_1x + m_1y + n_1z &= 0 \\ l_2x + m_2y + n_2z &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the equations to three planes passing through the origin; the conditions that these should be mutually perpendicular are

$$\begin{aligned} l^2 + m^2 + n^2 &= 1, & l\,l_1 + m\,m_1 + n\,n_1 &= 0, \\ l_1^2 + m_1^2 + n_1^2 &= 1, & l\,l_2 + m\,m_2 + n\,n_2 &= 0, \\ l_2^2 + m_2^2 + n_2^2 &= 1, & l_1l_2 + m_1m_2 + n_1n_2 &= 0. \end{aligned}$$

But these are the same relations which exist among the direction-cosines of three straight lines mutually at right angles, we must therefore also have

$$\left.\begin{array}{l} l^2+l_1^2+l_2^2=1 \\ m^2+m_1^2+m_2^2=1 \\ n^2+n_1^2+n_2^2=1 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From the expression in *Gregory's Solid Geometry*, p. 103, we obtain the sum of the squares of the reciprocals of the areas of the sections made by the planes (2) with the ellipsoid (1),

$$=\frac{a^2(l^2+l_1^2+l_2^2)+b^2(m^2+m_1^2+m_2^2)+c^2(n^2+n_1^2+n_2^2)}{(\pi abc)^2}$$

$$=\frac{a^2+b^2+c^2}{(\pi abc)^2}\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

by means of conditions (3).

Also the sum of the squares of the reciprocals of the areas of the principal sections,

$$=\frac{1}{(\pi ab)^2}+\frac{1}{(\pi ac)^2}+\frac{1}{(\pi bc)^2}=\frac{a^2+b^2+c^2}{(\pi abc)^2},$$

the same as (4), which proves the theorem.

Mr. Nicholas Smyth favoured us with a good solution.

CXXVI. *Mr. George Fred. Lacey, M.A., Woolwich Common.*

Let an uniform beam resting horizontally on a spherical surface be drawn from its position by two unequal weights W and P, W being greater than P, and the surface being rough enough to prevent sliding. Shew that the angle θ which it makes with the horizon $=\frac{W-P}{W+P}\cdot\frac{l}{r}\cdot\frac{90^\circ}{\pi}$, where $l=$ length of beam, $r=$ radius of sphere; the weight of beam being neglected. And if the weights W and P are suddenly removed, and the weight of beam w is taken into consideration, the tendency to recover its horizontal position $=wr\pi\frac{\theta}{180^\circ}\cos\theta$.

[First Solution. *Mr. Thomas Weddle.*]

Let AB be the beam, D its middle point, which before the weights were attached coincided with E, the highest point of the sphere; also let F be the point of contact of the beam and sphere. Then O being the centre of the sphere it is evident that $\angle EOF=\theta$, $\therefore DF=EF=r\theta.\frac{\pi}{180}$; also the perpendiculars from F on the directions of the forces W, P and w are $BF.\cos\theta$, $AF.\cos\theta$, and $DF.\cos\theta$ respectively; hence taking the moments of the forces about F (so as to get rid of the normal pressure and the friction), we have

$$W.BF.\cos\theta=P.AF.\cos\theta+w.DF.\cos\theta;$$

$$\therefore W.(BD-DF) \text{ or } W.(AD-DF)=P.(AD+DF)+w.DF;$$

$$\therefore DF=\frac{(W-P)AD}{W+P+w};$$

but $DF=r\theta.\frac{\pi}{180}$, and $AD=\frac{1}{2}l$,

$$\therefore \theta=\frac{W-P}{W+P+w}\cdot\frac{l}{r}\cdot\frac{90}{\pi}.$$

This value of θ coincides with that given in the enunciation, when $w=0$, or when the weight of the beam is neglected.

Again, the components of w parallel and perpendicular to the beam are $w\sin\theta$ and $w\cos\theta$ respectively, the former of which is counteracted by the friction, and the latter tends to turn the beam round F; hence, when the weights W and P are removed, the tendency of the beam to recover its horizontal position will be measured by $w\cos\theta.\mathrm{DF}=w.r\pi\dfrac{\theta}{180}\cos\theta$.

[SECOND SOLUTION. *Messrs. Peter Elliott, Lamesley, and Hugh Godfray, Jersey.*]

Taking the moments about the point of contact F, the resulting equation will contain neither the resistance of the sphere nor friction, and calling x the distance of F from the middle point D of the beam, or which is the same thing, the arc FE subtending the angle θ, we have

$$\mathrm{W}(\tfrac{1}{2}l-x)=\mathrm{P}(\tfrac{1}{2}l+x);$$

whence, $$\frac{\mathrm{W}-\mathrm{P}}{\mathrm{W}+\mathrm{P}}=\frac{2x}{l},\text{ or }x=\frac{l}{2}\,\frac{\mathrm{W}-\mathrm{P}}{\mathrm{W}+\mathrm{P}}:$$

and $\theta=\dfrac{x}{r}.\dfrac{180^\circ}{\pi}=\dfrac{\mathrm{W}-\mathrm{P}}{\mathrm{W}+\mathrm{P}}.\dfrac{l}{r}.\dfrac{90^\circ}{\pi}$, which proves the first relation.

Now the tendency of the weight w of the beam applied at F to bring the beam back to its original position (W and P being cancelled) will be

$$w\,x\cos\theta=wr\pi\frac{\theta}{180^\circ}\cos\theta.$$

The solutions of Dr. Burns, Lieutenant A. R. Clarke, R. E., and Messrs. Joseph Paul, A. Roy, and Nicholas Smyth, reflect much credit on their authors, but being the same in principle as the preceding, they are reluctantly omitted. Solutions were also received from Messrs. J. W. Elliott and W. Mortimer.

CXXVIII. *By Calculus.*

On the Lancaster and Carlisle railway, the incline on Shap Fell is 8 miles in length, and rises 1 in 75. Now admitting locomotive resistance when the speed is 30 miles an hour to be $\frac{1}{100}$ part of the pressure on the rails, in what time will a train moving with the above speed on entering the incline, descend to the bottom, and what will be the final velocity, the steam being shut off at the commencement of the descent?

[SOLUTION. *By Calculus, the proposer.*]

Let $g=32{\cdot}2$ft. represent the force of gravity, i the angle which the plane makes with the horizon, and x any variable distance from the summit; then the component of gravity in the direction of the plane $=g\sin i$, and the accelerating force arising from the resistance $=\dfrac{g\cos i}{100}$; hence the whole force acting on the train $=g\left(\sin i-\dfrac{\cos i}{100}\right)=gw$, calling $\left(\sin i-\dfrac{\cos i}{100}\right)$, w.

By mechanics, $\dfrac{d^2x}{dt^2}=gw$; integrating, $v^2=2gwx+c$; but when $x=0$, $v=44$, $\therefore c=44^2=1936$, and hence $v=\sqrt{\{2gwx+1936\}}$.

Also $t = \int \frac{dx}{v} = \int \frac{dx}{\sqrt{\{2gwx + 1936\}}} = \frac{\sqrt{\{2gwx + 1936\}}}{gw} + C'$, but $t = 0$ when $x = 0$, $\therefore C' = -\frac{44}{gw}$, hence $t = \frac{\sqrt{\{2gwx + 1936\}} - 44}{gw}$.

Computation: $i = 45' \, 50''$, $w = \cdot 00333422$, $x = (8 \text{ miles}) \; 42240$,
$g = 32 \cdot 2$; hence $\sqrt{\{2gwx + 1936\}} = 105 \cdot 0375$ feet,
$t = 568 \cdot 52$ seconds, $= 9$ minutes $28\frac{1}{2}$ seconds, the time of descent.

The final velocity = 105 feet per second, or 71 miles an hour.

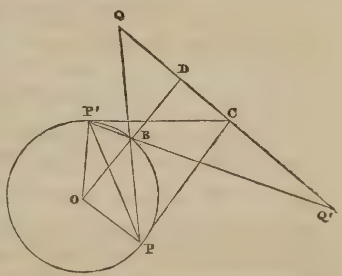
Mr. Nicholas Smyth also favoured us with a solution.

CXXIX. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*

Let CP and CP′ be tangents to a circle whose centre is O, and CD a perpendicular from C to a radius OB; then if PB, P′B be drawn to meet CD in Q,Q′ respectively, the four points P, P′, Q, Q′ lie in the circumference of a circle whose centre is C.

[FIRST SOLUTION. *Messrs. P. Elliott, Lamesley; S. Gaskill, at Mr. Hoole's Academy, Blackburn; John Stafford, St. Ann's School, Newcastle; and similarly by Dr. Burns, Rochester; Messrs. W. H. Levy, Shalbourn; S. Barry, City School, London; William Mortimer, Seedley; A. D. B., North Tawton, Crediton, Devonshire; and Lieutenant A. R. Clarke, R. E.*]

For draw the lines OP, OP′ and PP′; then since the angles OPB and BPC are together equal to a right angle, and the angles BQD and QBD are together also equal to a right angle; therefore the two angles OPB and BPC are equal to the two angles BQD and QBD; but angle OPB is equal to angle QBD or its opposite angle OBP; therefore the angle BPC is equal to the angle BQD, and the triangle CQP isosceles. In a similar manner, the triangle P′CQ′ is shewn to be isosceles; hence since CP is equal to CP′, the four points P, P′, Q, Q′ lie in the circumference of a circle whose centre is C.



[SECOND SOLUTION. *Mr. C. Atkinson Weddle, Ridsdale Iron Works; and similarly by Messrs. P. Elliott (Second Solution); and David Maver, Bon accord School, Aberdeen.*]

Draw OP, OP′ and PP′; then the angle QCP′ is equal to the angle DOP′, for each is the excess of a right angle above the angle formed by the intersection of the lines CP′ and OBD. Now twice the angle CP′Q′ is equal to the angle P′OB (Euc. iii. 32), and therefore equal to the angle P′CQ; but the angle P′CQ is equal to the two angles CP′Q′ and CQ′P′; hence twice the angle CP′Q′ is equal to the two angles CP′Q′ and CQ′P′; therefore the angle CP′Q′ is equal to the angle CQ′P′, and hence CQ′ is equal to CP′. In the same manner it may be shewn that CQ is equal to CP; but CP is equal to CP′; therefore the four lines CP′, CP, CQ′ and CQ are all equal, and the truth of the proposition is manifest.

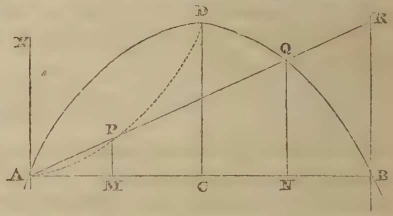
Neat demonstrations of this exercise have also been received from Mr. Stephen Watson, Castle Side, near Shotley Bridge, Durham; Mr. A. Roy; and the Proposer.

CXXX. *Dr. Burns, Rochester.*

AB is the double ordinate to the axis of a given parabola, BR a diameter, AR any line cutting the curve in Q; then if AP be taken in AR equal to QR, the locus of P is a parabola: required a demonstration.

[FIRST SOLUTION. *Dr. Burns, the proposer; and similarly by Messrs. S. Barry, City School, London; David Maver, Bon accord School, Aberdeen; and A. Roy.*]

Draw PM and QN parallel to RB; and let $AB = 2a$, $CD = b$, $AM = x$, $PM = y$, and the latus rectum $= m$; then by a property of the parabola,



$CB^2 : CN^2 :: CD : CD - NQ$,

or since $NB = AM = x$,

$a^2 : (a - x)^2 :: b : b - NQ$;

and by division,

$a^2 : 2ax - x^2 :: b : NQ \ldots\ldots (1)$.

Again, $MP : MA :: NQ : NA$, or

$y : x :: NQ : 2a - x \ldots (2)$.

Hence from (1) and (2) we get

$a^2y : x^2(2a - x) :: b : 2a - x$, or $a^2y : x^2 :: b : 1$; hence

$bx^2 = a^2y$; but since $a^2 = mb$, we have $bx^2 = mby$, or $x^2 = my$;

therefore the locus of P is a parabola, whose vertex is A, and principal parameter equal to that of the given parabola.

[SECOND SOLUTION. *Mr. C. Atkinson Weddle, Ridsdale Iron Works, Woodburn, Northumberland.*]

Let the abscissa $CD = b$, and $AB = 2a$, then the equation of the parabola gives the relation $a^2 = 4mb$, between a and b. Now if through A, the diameter AX be drawn, and if the equation of the parabola be transformed to the axes AB and AX, by putting $x + b$ for x, and $y - a$ for y, we shall have for the equation of the parabola

$$y^2 - 2ay - 4mx = 0 \ldots\ldots\ldots\ldots (1).$$

Let angle $XAQ = \theta$, then the equations of the straight lines AQ and BR are respectively

$$y = x \tan\theta \ldots\ldots (2), \quad \text{and } y = 2a \ldots\ldots\ldots (3).$$

Combining (1) and (2), we have for the coordinates of the point Q,

$$x = 2a \cot\theta + 4m \cot^2\theta, \quad y = 2a + 4m \cot\theta.$$

Combining (2) and (3), we have for the coordinates of the point R,

$$x = 2a \cot\theta, \quad \text{and } y = 2a.$$

But the coordinates of the point P are evidently equal to the differences of the coordinates Q and R; hence if α and β are the coordinates of P, we have

$$\alpha = 4m \cot^2\theta, \quad \beta = 4m \cot\theta.$$

Eliminating $\cot\theta$ from these, we get the relation

$$\beta^2 = 4m\alpha,$$

which shows that the locus of P is a parabola equal in all respects to the given parabola, and whose vertex is A.

[THIRD SOLUTION. *Mr. P. Elliott, Lamesley, Durham.*]

Let A be the origin of polar coordinates, and AX, from which θ is measured, a diameter to the parabola. Then the equation to the given parabola is (Waud's Algebraical Geometry, p. 127)

$$r = \frac{p \cos \theta - 2y' \sin \theta}{\sin^2 \theta} \ldots\ldots\ldots\ldots (1),$$

and since AB $= -2y'$, the equation to the diameter BR is

$$r = -\frac{2y'}{\sin \theta} \ldots\ldots\ldots\ldots (2).$$

The difference between the values of r and r in (1) and (2) gives the value of QR = AP, and

$$r = \frac{p \cos \theta}{\sin^2 \theta},$$

is the equation to the required curve, whose vertex is A and axis AX. On comparing this equation with (1), we observe that they both represent the same curve, but occupying positions already stated.

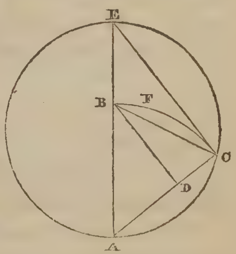
Excellent solutions were sent by Messrs. W. Mortimer and John Stafford, and Lieut. A.R. Clarke, R.E., which are omitted with much regret.

132. *By A. R.*

From the extremity A of a diameter of a given circle as centre, a circular arc is described cutting the diameter in the point B, and the circumference in the point C; to find the values of the angle ABC when (BD) the perpendicular upon AC, the chord BC, the arc BFC, and the segment BFCB are severally the greatest possible.

[SOLUTION. *Messrs. P. Elliott, Lamesley; C. Atkinson Weddle; and A. R., the proposer.*]

Let the diameter AE $= 2r$, and the length of an arc of a sector to radius 1, similar to the sector BAC $= \theta$, θ' being its corresponding angular measurement. Then we have AB = AC $= 2r \cos \theta$.



(1) Since BD $=$ AB $\sin \theta = 2r \sin \theta \cos \theta = r \sin 2\theta =$ a maximum, it is obvious that BD will be a maximum when $\sin 2\theta$ is a maximum, or when $2\theta' = 90°$;

$\therefore \theta' = 45°$, and angle ABC $= \frac{1}{2}(180° - 45°) = 67° \, 30'$.

(2) The chord BC $= \dfrac{\text{BD}}{\cos \frac{1}{2}\theta} = \dfrac{r \sin 2\theta}{\cos \frac{1}{2}\theta} = \dfrac{2r \sin \theta \cos \theta}{\cos \frac{1}{2}\theta}$

$$= \frac{4r \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta (1 - 2 \sin^2 \frac{1}{2}\theta)}{\cos \frac{1}{2}\theta}$$

$$= 4r (\sin \tfrac{1}{2}\theta - 2 \sin^3 \tfrac{1}{2}\theta) = \text{maximum};$$

hence if $u = \sin \frac{1}{2}\theta - 2 \sin^3 \frac{1}{2}\theta$, we have $\dfrac{du}{d\theta} = \frac{1}{2} \cos \frac{1}{2}\theta - 3 \sin^2 \frac{1}{2}\theta \cos \frac{1}{2}\theta = 0$;

therefore $6 \sin^2 \frac{1}{2}\theta = 1$, and $\sin \frac{1}{2}\theta = \frac{1}{6}\sqrt{6} = \cdot 4082483$; and hence $\frac{1}{2}\theta' = 24° \, 5' \, 41''$, $\theta' = 48° \, 11' \, 22''$, and angle ABC $= 65° \, 54' \, 19''$ nearly.

(3) The arc BFC $= \theta \times 2r\cos\theta = 2r\theta\cos\theta =$ maximum; hence if $u = \theta\cos\theta$, $\frac{du}{d\theta} = \cos\theta - \theta\sin\theta = 0$; $\therefore \theta = \cot\theta$, and the approximate value of θ' is $49^\circ\ 17'\ 36''$, therefore angle ABC $= 65^\circ\ 21'\ 12''$ nearly.

(4) The segment BFCB $=$ sector ABFC $-$ triangle ABC

$$= 2r^2\theta\cos^2\theta - 2r^2\sin\theta\cos^2\theta$$
$$= 2r^2\cos^2\theta(\theta - \sin\theta);$$

hence if $u = \cos^2\theta(\theta - \sin\theta)$; then we have

$$\frac{du}{d\theta} = -2\cos\theta\sin\theta(\theta - \sin\theta) + \cos^2\theta(1 - \cos\theta) = 0;$$

$$\therefore\ 2\theta\sin\theta = 2\sin^2\theta + \cos\theta - \cos^2\theta = 3\sin^2\theta + \cos\theta - 1;$$

$$\therefore\ 2\theta = 3\sin\theta + \cot\theta - \operatorname{cosec}\theta,$$

from which we obtain an approximate value of $\theta' = 56^\circ\ 2'\ 16'$; and angle ABC $= 61^\circ\ 58'\ 52''$ nearly.

Mr. William Mortimer favoured us with a good solution. Answers were also sent by A. D. B., North Tawton, near Crediton, Devon; and Lieutenant A. R. Clarke, R. E.

CXXXI. *Mr. Hugh Godfray, Jersey.*

Two points being given, it is required to find ten other points which with these two will form the angular points of a regular dodecagon, using the compasses only.

[SOLUTION. *Messrs. W. Mortimer, Seedley; David Maver, Aberdeen; W. Maver, Glasgow; and A. Roy, Dundee.*]

On AB (A and B are the two given points) describe a circle (centre O) to contain the regular hexagon ABCDEF (Euc. iv. 15). This can be done by the compasses alone, as AO $=$ AB. Then with centres F and C, and radius FD or CE, describe arcs to intersect in G. Again, with centre F and radius OG describe an arc cutting the circle ABC in H; then H is the centre and HA the radius of the circle which will contain the required dodecagon. For since FG$=$CG, and FO$=$OC (FO and OC are evidently in the same straight line) the angle GOF is a right angle, and hence $FO^2 + FH^2 = FO^2 + OG^2 = FG^2 = FD^2 = 3FO^2$, or $FH^2 = 2FO^2$; consequently H is the middle of the arc ED, and therefore HA$=$HB. It hence follows, since the angle AHB$=\frac{1}{2}$AOB$=\frac{1}{12}$ of four right angles, that A and B are two of the angular points of a regular dodecagon inscribed in the circle whose centre is H and radius HA or HB.

Good solutions were received from Dr. Burns, Rochester; A. D. B., North Tawton; and Messrs. S. Gaskell, Blackburn; W. H. Levy, Shalbourne; and C. A. Weddle, Woodburn.

CXXXIV. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*

Find the value of $\phi(x,y) = \frac{(x+y)^n}{x^n + y^n}$, when $x = 0, y = 0$.

[SOLUTION. *Mr. Thomas Weddle.*]

Assume $y = vx$, $\quad\therefore\ \phi(x, y) = \frac{(1+v)^n}{1+v^n}$ ....(1).

Now since x and y are entirely independent of each other, $\frac{y}{x}$ or v is

indeterminate when $x=0$ and $y=0$; hence the right hand member of (1), and therefore $\phi(x, y)$ is *indeterminate* (within certain limits) when $x=0$ and $y=0$.

Or view the question in this way:

Let $$z=\frac{(x+y)^n}{x^n+y^n} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
be the equation of a surface; now the right hand member of (2) is evidently a function of $\frac{y}{x}$; hence denoting it by $\Psi\left(\frac{y}{x}\right)$, the equation to the surface becomes

$$z=\Psi\left(\frac{y}{x}\right).$$

In this form we recognise (*Gregory's Solid Geometry*, p. 166) the *conoidal* surface, which has the axis of z for the fixed axis or directrix through which the generating straight line always passes; hence a portion of the axis of z being situated in the surface, z is indeterminate when $x=0$ and $y=0$.

Solutions were received from the proposer, and Messrs. W. Mortimer and A. Roy.

CXXXV. *Mr. Thomas Weddle, at Mr. Stoton's Academy, Wimbledon.*

Let E, E_1, E_2, E_3 denote the spherical excesses of any system of associated spherical triangles, and E', E'_1, E'_2, E'_3 those of the polar associated system: prove that

$$\frac{2}{n}=\cot\tfrac{1}{2}E+\cot\tfrac{1}{2}E_1+\cot\tfrac{1}{2}E_2+\cot\tfrac{1}{2}E_3$$

and $$\frac{2}{N}=\cot\tfrac{1}{2}E'+\cot\tfrac{1}{2}E'_1+\cot\tfrac{1}{2}E'_2+\cot\tfrac{1}{2}E'_3,$$

where, as usual, $n^2=\sin s\sin(s-a)\sin(s-b)\sin(s-c)$,

and $N^2=-\cos S\cos(S-A)\cos(S-B)\cos(S-C)$.

[FIRST SOLUTION. *Mr. Weddle, the proposer.*]

By Davies's Hutton, vol. ii. p. 40, we have $\sin B\sin C=\frac{2N^2}{n\sin A}$;

hence $$\sin^2\tfrac{1}{2}a=\frac{-\cos S.\cos(S-A)}{\sin B.\sin C}=\frac{n}{2}.\frac{-\cos S.\cos(S-A)\sin A}{N^2}$$

$$=\frac{n}{2}.\frac{\sin\{S-B+S-C\}}{\cos(S-B).\cos(S-C)}=\frac{n}{2}\Big\{\tan(S-B)+\tan(S-C)\Big\}(1)$$

Also, $$\cos^2\tfrac{1}{2}a=\frac{\cos(S-B).\cos(S-C)}{\sin B.\sin C}=\frac{n}{2}.\frac{\cos(S-B).\cos(S-C).\sin A}{N^2}$$

$$=\frac{n}{2}.\frac{\sin\{S-(S-A)\}}{-\cos S.\cos(S-A)}=\frac{n}{2}\Big\{\tan(S-A)-\tan S.\Big\}..(2).$$

Add (1, 2), and divide by $\frac{n}{2}$;

$$\therefore\quad \frac{2}{n}=-\tan S+\tan(S-A)+\tan(S-B)+\tan(S-C)\ldots.(3).$$

By similarly transforming the analogous expressions for $\cos^2\frac{1}{2}A$ and $\sin^2\frac{1}{2}A$, we get

$$\frac{2}{N} = -\cot s + \cot(s-a) + \cot(s-b) + \cot(s-c) \ldots\ldots (4).$$

Now, Davies's Hutton, p. 50, $S = \frac{1}{2}(\pi + E)$, $S - A = \frac{1}{2}(\pi - E_1)$, $S - B = \frac{1}{2}(\pi - E_2)$, and $S - C = \frac{1}{2}(\pi - E_3)$; also (recollecting that the angles of the polar triangle are $\pi - a$, $\pi - b$, and $\pi - c$) it may easily be shown that $s = \pi - \frac{1}{2}E'$, $s - a = \frac{1}{2}E_1'$, $s - b = \frac{1}{2}E_2'$, and $s - c = \frac{1}{2}E_3'$; hence (3) and (4) become

$$\frac{2}{n} = \cot\tfrac{1}{2}E + \cot\tfrac{1}{2}E_1 + \cot\tfrac{1}{2}E_2 + \cot\tfrac{1}{2}E_3 \ldots\ldots (5),$$

and $$\frac{2}{N} = \cot\tfrac{1}{2}E' + \cot\tfrac{1}{2}E_1' + \cot\tfrac{1}{2}E_2' + \cot\tfrac{1}{2}E_3' \ldots\ldots (6).$$

The expressions (5, 6) are remarkably neat, and neither these nor (3, 4) from which they are derived, have, as far as I am aware, been given before. By means of (3), n may be computed when the angles of the spherical triangle are known, and by (4), N may in like manner be computed, when the sides are given, and these seem to be good working formulas, especially if the reciprocals only of n and N be required.

Other interesting expressions might also be given, only two or three of which however I shall put down here. Since $\cos a = \cos^2\frac{1}{2}a - \sin^2\frac{1}{2}a$, we have (1, 2, 3),

$$\cos a = \frac{-\tan S + \tan(S-A) - \tan(S-B) - \tan(S-C)}{-\tan S + \tan(S-A) + \tan(S-B) + \tan(S-C)};$$

also, $$\tan^2\tfrac{1}{2}a = (1, 2), \ \frac{\tan(S-B) + \tan(S-C)}{\tan(S-A) - \tan S}.$$

Similarly, $$\cos A = \frac{\cot s - \cot(s-a) + \cot(s-b) + \cot(s-c)}{-\cot s + \cot(s-a) + \cot(s-b) + \cot(s-c)},$$

and, $$\cot^2\tfrac{1}{2}A = \frac{\cot(s-b) + \cot(s-c)}{\cot(s-a) - \cot s}.$$

[SECOND SOLUTION. *Mr. W. Mortimer, Seedley.*]

Here $\cot\frac{1}{2}E = \dfrac{\cot\frac{1}{2}a.\cot\frac{1}{2}b + \cos C}{\sin C}$ (Hind's Trig., p. 211,) and making the proper substitutions of $\pi - B$, $\pi - C$, $\pi - b$, $\pi - c$, *etc.* for B, C, b, c, *etc.*

$$\cot\tfrac{1}{2}E_1 = \frac{\cot\frac{1}{2}a.\tan\frac{1}{2}b - \cos C}{\sin C}, \quad \cot\tfrac{1}{2}E_2 = \frac{\tan\frac{1}{2}a.\cot\frac{1}{2}b - \cos C}{\sin C},$$

and $$\cot\tfrac{1}{2}E_3 = \frac{\tan\frac{1}{2}a.\tan\frac{1}{2}b + \cos C}{\sin C}; \ \therefore \cot\tfrac{1}{2}E + \cot\tfrac{1}{2}E_1 + \cot\tfrac{1}{2}E_2 +$$

$$\cot\tfrac{1}{2}E_3 = \frac{(\cot\frac{1}{2}a + \tan\frac{1}{2}a)(\cot\frac{1}{2}b + \tan\frac{1}{2}b)}{\sin C} = \frac{4\operatorname{cosec} a.\operatorname{cosec} b}{2\operatorname{cosec} a.\operatorname{cosec} b.n} = \frac{2}{n}.$$

Similarly, $$\cot\tfrac{1}{2}E' + \cot\tfrac{1}{2}E_1' + \cot\tfrac{1}{2}E_2' + \cot\tfrac{1}{2}E_3'$$

$$= \frac{(\cot\frac{1}{2}A + \tan\frac{1}{2}A)(\cot\frac{1}{2}B + \tan\frac{1}{2}B)}{\sin c} = \frac{2}{N}.$$

[THIRD SOLUTION. *Mr. Peter Elliott, Lamesley.*]

By a well known formula

$$\cot\frac{E}{2} = \frac{\cot\frac{1}{2}a\cot\frac{1}{2}b + \cos C}{\sin C},$$

$$\cot\frac{E_1}{2} = \frac{\cot\frac{1}{2}a\cot\frac{1}{2}(\pi - b) + \cos(\pi - C)}{\sin(\pi - C)},$$

$$\cot\frac{E_2}{2} = \frac{\cot\frac{1}{2}(\pi - a)\cot\frac{1}{2}b + \cos(\pi - C)}{\sin(\pi - C)},$$

$$\cot\frac{E_3}{2} = \frac{\cot\frac{1}{2}(\pi - a)\cot\frac{1}{2}(\pi - b) + \cos C}{\sin C}.$$

Summing and giving reduced forms to the functions where necessary, we obtain

$$\cot\frac{E}{2} + \cot\frac{E_1}{2} + \cot\frac{E_2}{2} + \cot\frac{E_3}{2}$$

$$= \frac{1}{\sin C}(\tan\tfrac{1}{2}a + \cot\tfrac{1}{2}a)(\tan\tfrac{1}{2}b + \cot\tfrac{1}{2}b) = \frac{\operatorname{cosec}^2\frac{1}{2}a\operatorname{cosec}^2\frac{1}{2}b}{\cot\frac{1}{2}a\cot\frac{1}{2}b.\sin C}$$

$$= \frac{1}{\sin\frac{a}{2}\cos\frac{a}{2}\sin\frac{b}{2}\cos\frac{b}{2}\sin C} = \frac{4}{\sin a\sin b\sin C} = \frac{2}{n}.$$

(Spher. Trig. Soc. Diff. U. K., page 11.)

For the polar triangles we have only to substitute in the value of n, the approximate trigonometrical functions arising from the consideration that s must become

$$\frac{3\pi}{2} - S;\ s - a,\ \frac{\pi}{2} - (S - A);\ s - b,\ \frac{\pi}{2} - (S - B);\ s - c,\ \frac{\pi}{2} - (S - C),$$

and we obtain

$$\cot\frac{E'}{2} + \cot\frac{E_1'}{2} + \cot\frac{E_2'}{2} + \cot\frac{E_3'}{2} = \frac{2}{N}.$$

[FOURTH SOLUTION. *Mr. J. W. Elliott, Greatham.*]

Let a, b, c be the sides of a spherical triangle ABC: then by *De Gua's* formula for the spherical excess,

$$2n\cot\tfrac{1}{2}E = 1 + \cos a + \cos b + \cos c \text{ .... } (1).$$

Taking ABC as the fundamental triangle of the associated system, the sides of the other three, or supplemental triangles, are severally $a, \pi - b, \pi - c$; $\pi - a, b, \pi - c$; $\pi - a, \pi - b, c$. Inserting these in (1), we have,

$$\left.\begin{aligned} 2n\cot\tfrac{1}{2}E_1 &= 1 + \cos a - \cos b - \cos c \\ 2n\cot\tfrac{1}{2}E_2 &= 1 - \cos a + \cos b - \cos c \\ 2n\cot\tfrac{1}{2}E_3 &= 1 - \cos a - \cos b + \cos c \end{aligned}\right\} \text{ .... } (2).$$

Adding (1) and (2): then

$$\cot\tfrac{1}{2}E + \cot\tfrac{1}{2}E_1 + \cot\tfrac{1}{2}E_2 + \cot\tfrac{1}{2}E_3 = \frac{2}{n};$$

which proves the first property.

Proceeding in a similar manner, in respect to the polar associated system, we obtain

$$2N\cot\tfrac{1}{2}E' = 1 - \cos A - \cos B - \cos C,$$
$$2N\cot\tfrac{1}{2}E_1' = 1 - \cos A + \cos B + \cos C,$$
$$2N\cot\tfrac{1}{2}E_2' = 1 + \cos A - \cos B + \cos C,$$
$$2N\cot\tfrac{1}{2}E_3' = 1 + \cos A + \cos B - \cos C;$$

and by addition of these,

$$\cot \tfrac{1}{2} E' + \cot \tfrac{1}{2} E_1' + \cot \tfrac{1}{2} E_2' + \cot \tfrac{1}{2} E_3' = \frac{2}{N},$$

which proves the second property.

Mr. Elliott sent another solution to this exercise.
The solutions of Lieut. R. A. Clarke; and Messrs. S. Gaskell, Mr. Hoole's Academy, Blackburn, Lancashire; A. Roy, Dundee; and C. A. Weddle, Woodburn, Northumberland, are very elegant, and are omitted with considerable regret.

Letter from G. W. Hearn, Esq., to the Editors relative to an article in the Mechanic's Magazine.

TO THE EDITORS OF THE MATHEMATICIAN.

Gentlemen,

An accusation having appeared against me in the Mechanic's Magazine of the 12th inst., of having acted unfairly towards your very able contributor Thomas Weddle, Esq., by the publication of a mode of obtaining Logarithms and Anti-logarithms to ten places of figures, by very concise tables, I embrace this opportunity of freeing myself from the said accusation.

My paper on Logarithms and Anti-logarithms was written (according to the best of my recollection) in June, 1845, and was seen by my friend Mr. Godward in the September of that year. The " Mathematician" of November appeared, and to the great surprise of both Mr. Godward and myself contained a paper by Mr. Weddle in many parts closely resembling my own. In that same month of November, I handed personally my paper to Mr. Rutherford in the presence of Mr. Godward, giving at the same time a full explanation as to its originality, as far as I myself was concerned, which was fully corroborated by Mr. Godward. We also, naturally, had some conversation upon the singularity of the coincidence that Mr. Weddle and myself should, quite independently of each other, have about the same time attended to the same subject, and treated it in a very similar manner, and yet that manner entirely different from all pre-existing methods.

My paper did not appear in the " Mathematician" until March, 1847.

I will merely add, that the motive which induced me to write the paper was to satisfy myself of the possibility of calculating Logarithms and Anti-logarithms, *mentally*, to ten figures. I considered that the very concise tables I had formed might easily be committed to memory, and that the easy calculation required might be done mentally by a person skilled in mental arithmetic. I had heard of a person who had astonished all Cambridge by this intellectual feat of calculating Logarithms *mentally*, and I endeavoured to invent a method by which it might be done.

By your inserting the above letter you will greatly oblige,

Gentlemen, your most obedient,

R. M. Coll. Sandhurst, 19*th Feb*. 1848. GEO. W. HEARN.

Gentlemen, London, 21st Feb., 1848.

I hereby confirm the whole of the declaration which Mr. Hearn has made with respect to myself; and as a corroborative fact, I add the following extract from a letter which I had from him, dated Nov. 24, 1845, shortly after the number of the " Mathematician" was published, containing Mr. Weddle's paper:—" How singular it is that Mr. Weddle should have published something on Logarithms so like what I had just done, and which you saw." I am, Gentlemen, your's obediently,

To the Editors of the " Mathematician." WM. GODWARD.

MATHEMATICAL EXERCISES—(*continued.*)

139. *Mr. A. Roy, Dundee Academy.*

Let ABC and A′B′C′ be two triangles having the angles A, B, C equal to the angles A′, B′, C′, each to each: show by the first book of Euclid that if the sides about equal angles be taken, the rectangle contained by a side in the one triangle and a side not corresponding to it in the other is equal to the rectangle contained by the remaining sides. For example, the rectangle contained by AC and A′B′ is equal to the rectangle contained by A′C′ and AB.

140. *Lieut. A. R. Clarke, R. E.*

If lines drawn from one of the centres of the escribed circles of a plane triangle, or from the centre of the inscribed circle, to the other three, represent in magnitude and direction three forces, then the line joining this centre and that of the circumscribing circle will represent in magnitude and direction one fourth of the resultant.

141. *By the same Gentleman.*

Let ABC be any plane triangle, and let straight lines be drawn through the angular points, making each an angle a with the sides, and forming by their intersection a triangle DEF. Let O_1, O_2, O_3 be the centres; and R_1, R_2, R_3 the radii of the circles circumscribed about the triangles BCF, DCA, and ABE: then if R be the radius of the circle circumscribed about the triangle ABC, and Δ, Δ_1 the areas of the triangles ABC and $O_1O_2O_3$, we shall have

$$(1)\quad 16\,\Delta\,\Delta_1 = a^2b^2 + b^2c^2 + c^2a^2.$$

$$(2)\quad R^3 = R_1R_2R_3.$$

142. *By Samuel Square.*

From a given point in the circumference of a given circle, to draw a chord intersecting two sides of a given inscribed triangle, so that the rectangle of its extreme segments may be equal to a given square.

143. *Lieut. St. Andrew St. John, R. E.*

The centres of two spheres, radii r_1, r_2, are at the extremities of a line $2a$, on which as a diameter a circle is described: find the point in the circumference of this circle from which the greatest portion of spherical surface is visible.

144. *Mr. John Riddle.*

If AC be the diagonal of a parallelogram ABCD, and AX a line through A in any direction in space, then show by *plane trigonometry* that

$$AC \cos CAX = AB \cos BAX + AD \cos DAX.$$

145. *Dr. August Weigand.*

AB is a given straight line and C a given point in it: on AB as hypothenuse describe an isosceles right angled triangle ADB, and let a parabola be described about the triangle ADB having its vertex in the angular point D; through C draw CH perpendicular to AB meeting the parabola in H, then CH will be an harmonic mean between AC and CB.

146. *Mr. Weddle.*

If the squares of the sides of a triangle be in arithmetical progression, the lines drawn from the angles to a point within the triangle, so as to make equal angles with each other, are in arithmetical progression.

147. *Mr. S. Barry, City School, London.*

Find the locus of the middle points of equal chords in an ellipse.

148. *Mr. John Walker, Trin. Coll., Dublin.*

If a and b be the extreme, and c the mean, angles of a harmonic pencil of four lines, prove that

(1) $\cos(a+c)\cos(b+c)=\cos(a+b)$;

(2) $\tan D \tan c - \tan a \tan b = 2\tan a \tan b \tan c \tan D$,

where $D = a + b + c$.

149. *By the same Gentleman.*

Sum the series

$$x.1^y + \frac{x(x-1)}{1.2}.2^y + \frac{x(x-1)(x-2)}{1.2.3}3^y + \ldots + x(x-1)^y + x^y,$$

by the method of the Calculus of Finite Differences, and as an example prove that

$$x.1^5 + \frac{x(x-1)}{1.2}2^5 + \ldots + x^5 = 2^{x-5}\left\{x(x+5)^2 - 10(x+1)\right\}x^2.$$

150. *Dr. Burns, Rochester.*

AB, BC are two equal straight beams whose length is L, they are united at B and form an acute angle θ; AB rests on the top of an upright wall and is perpendicular to its plane, BC is in the same vertical plane with AB and C touches the face of the wall; determine the power which applied horizontally at A will sustain a weight w placed on AB at the distance l from the wall, the beams being supposed inflexible and without weight.

151. *By Simon Lhuilier.*

Let a sphere and plane which cut each other (not in a great circle) be given, and likewise any number of points in space: it is required to inscribe in the circle of the sphere made by the plane, a spherical polygon (composed of arcs of great circles) so that the planes of these great circles shall pass through the given points.

152. *Mr. Weddle, Wimbledon.*

Let $\phi(x_1, x_2 \ldots x_n)$ be a homogeneous function of n variables, and of the m^{th} degree, then will

$$m(m-1)\ldots(m-p+1).\phi(x_1, x_2 \ldots x_n) =$$
$$\left\{a_1\frac{d}{dx_1} + a_2\frac{d}{dx_2} \ldots + a_n\frac{d}{dx_n}\right\}^p.\phi(x_1, x_2 \ldots x_n);$$

in the right hand member of which $x_1, x_2 \ldots x_n$ are to be written for $a_1, a_2 \ldots a_n$, after performing the operation

$$\left\{a_1\frac{d}{dx_1} + a_2\frac{d}{dx_2} \ldots + a_n\frac{d}{dx_n}\right\}^p \text{ on } \phi(x_1, x_2 \ldots x_n).$$

153. *Mr. Fenwick.*

Assuming that the diameter of a curve is the "locus of the middle points of a series of parallel chords," it is required to prove that all diameters of the Witch of Agnesi (the principal axis excepted) are curves of the third order. The equation of the Witch to rectangular coordinates is

$$xy^2 = 4a^2(2a - x).$$

154. *By Disïota.*

In a given circle to inscribe a quadrilateral whose sides shall pass through four given points, all of which are situated in the same straight line.

THE MATHEMATICIAN.

Vol. III. JULY, 1848. No. 3.

APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

[*Mr. Finlay, Professor of Mathematics and Natural Philosophy, Manchester New College.*]

(Continued from page 32.)

PRELIMINARY PROPERTIES OF POLES AND POLAR PLANES.

XV.

If a straight line be drawn from a given point P, meeting a given surface of the second degree in Q and Q′, and if P and P′ be conjugate harmonic points with regard to Q and Q′, it is required to find the locus of P′.

Let x_1, y_1, z_1 be the coordinates of the given point P,

$$x-x_1=a(z-z_1), \quad y-y_1=\beta(z-z_1)$$

the equations of the straight line PP′, and

$$\mathrm{A}x^2+\mathrm{A}'y^2+\mathrm{A}''z^2+2\mathrm{B}yz+2\mathrm{B}'zx+2\mathrm{B}''xy+2\mathrm{C}x+2\mathrm{C}'y+2\mathrm{C}''z+\mathrm{E}=0 \,..(1)$$

the equation of the given surface. Put $\mathrm{PQ}=r$, $\mathrm{PP}'=\rho$, and let x', y', z' be the coordinates of P′; then

$$r=(z-z_1)\sqrt{(1+a^2+\beta^2+2a\beta\cos xy+2\beta\cos yz+2a\cos zx)}=\frac{z-z_1}{\gamma}, \text{ suppose };$$

$$\therefore \quad z-z_1=r\gamma, \quad y-y_1=r\beta\gamma, \quad x-x_1=ra\gamma \,........(a),$$
$$z'-z_1=\rho\gamma, \quad y'-y_1=\rho\beta\gamma, \quad x'-x_1=\rho a\gamma \,........(b);$$

hence, if we assume, for the sake of brevity,

$$s=\mathrm{A}a^2+\mathrm{A}'\beta^2+\mathrm{A}''+2\mathrm{B}\beta+2\mathrm{B}'a+2\mathrm{B}''a\beta,$$
$$t=\mathrm{A}ax_1+\mathrm{A}'\beta y_1+\mathrm{A}''z_1+\mathrm{B}(y_1+\beta z_1)+\mathrm{B}'(x_1+az_1)+\mathrm{B}''(\beta x_1+ay_1)$$
$$+\mathrm{C}a+\mathrm{C}'\beta+\mathrm{C}'',$$
$$u\mathrm{A}x={}_1^2+\mathrm{A}'y_1^2+\mathrm{A}''z_1^2+2\mathrm{B}y_1z_1+2\mathrm{B}'z_1x_1+2\mathrm{B}''x_1y_1$$
$$+2\mathrm{C}x_1+2\mathrm{C}'y_1+2\mathrm{C}''z_1+\mathrm{E};$$

by eliminating x, y, z from equations (a) and (1) we shall have

$$s\gamma^2r^2+2t\gamma r+u=0 \,....................(c).$$

Now since PQ and PQ′ are the values of r in this equation, we obtain by quadratics

$$\frac{1}{\mathrm{PQ}}+\frac{1}{\mathrm{PQ}'}=\frac{-2t\gamma}{u};$$

and since PP′ is a harmonic mean between PQ and PQ′, we have

$$\frac{1}{PQ}+\frac{1}{PQ'}=\frac{2}{PP'}=\frac{2}{\rho},$$

$$\therefore\quad u+t\gamma\rho=0.$$

By restoring the value of t, and substituting for $\gamma\rho$, $\alpha\gamma\rho$, $\beta\gamma\rho$ their values given by equations (b), this becomes

$$u+Ax_1(x'-x_1)+A'y_1(y'-y_1)+A''z_1(z'-z_1)+C(x'-x_1)$$
$$+C'(y'-y_1)+C''(z'-z_1)$$
$$+B\{y_1(z'-z_1)+z_1(y'-y_1)\}+B'\{z_1(x'-x_1)+x_1(z'-z_1)\}$$
$$+B''\{x_1(y'-y_1)+y_1(x'-x_1)\}=0;$$

and by substituting for u its value, and making slight reductions, we get

$$(Ax_1+B''y_1+B'z_1+C)x'+(B''x_1+A'y_1+Bz_1+C')y'$$
$$+(B'x_1+By_1+A''z_1+C'')z'+Cx_1+C'y_1+C''z_1+E=0,$$

from which it is evident that the point P′ always lies on the plane

$$(Ax_1+B''y_1+B'z_1+C)x+(B''x_1+A'y_1+Bz_1+C')y$$
$$+(B'x_1+By_1+A''z_1+C'')z+Cx_1+C'y_1+C''z_1+E=0\ldots.(2).$$

The plane represented by this equation is called the *polar plane* of the point $P(x_1y_1z_1)$ with regard to the surface (1); and the point P is called the *pole* of the plane (2) with respect to that surface.

When the surface (1) is a cone having its vertex at the origin, we have $C=0$, $C'=0$, $C''=0$, $E=0$, and equation (2) becomes

$$(Ax_1+B''y_1+B'z_1)x+(B''x_1+A'y_1+Bz_1)y+(B'x_1+By_1+A''z_1)z=0\,..(2');$$

hence *the polar plane of any point taken with respect to a cone passes through the vertex or centre of the cone.*

If the pole be at the origin, we have $x_1=0$, $y_1=0$, $z_1=0$, and therefore the equation to the polar plane of the origin is

$$Cx+C'y+C''z+E=0\ldots\ldots\ldots\ldots\ldots\ldots.(2'').$$

XVI.

The *centre* of a surface of the second degree is a point which bisects every straight line drawn through it and terminated both ways by the surface. When the pole P coincides with the centre we have PQ=PQ′, hence, in virtue of the harmonic relation, P′Q=P′Q′, and therefore the point P′ is at infinity. Hence we see that *the polar plane of the centre of the given surface is at infinity.* In this case the coefficients of x, y, z in equation (2) must vanish; hence if t, u, v denote the centre of the surface (1), we shall have

$$\left.\begin{aligned}At+B''u+B'v+C&=0\\ B''t+A'u+Bv+C'&=0\\ B't+Bu+A''v+C''&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

from which the coordinates of the centre might be found by elimination.

By subtracting the first members of these equations from the coefficients of x, y, z in equation (2) we get

$$\{A(x_1-t)+B''(y_1-u)+B'(z_1-v)\}x+\{B''(x_1-t)+A'(y_1-u)+B(z_1-v)\}y$$
$$+\{B'(x_1-t)+B(y_1-u)+A''(z_1-v)\}z+C(x_1-t)+C'(y_1-u)+C''(z_1-v)$$
$$+Ct+C'u+C''v+E=0.$$

Let m and n be the indices of position of the straight line passing through the points tuv and $x_1y_1z_1$, so that

$$x_1-t=m(z_1-v),\quad y_1-u=n(z_1-v);$$

then by dividing the preceding equation by z_1-v, we obtain

$$(\mathrm{A}m+\mathrm{B}''n+\mathrm{B}')x+(\mathrm{B}''m+\mathrm{A}'n+\mathrm{B})y+(\mathrm{B}'m+\mathrm{B}n+\mathrm{A}'')z$$
$$+\mathrm{C}m+\mathrm{C}'n+\mathrm{C}''+\frac{\mathrm{C}t+\mathrm{C}'u+\mathrm{C}''v+\mathrm{E}}{z_1-v}=0\,\ldots\ldots\ldots\,(4),$$

which is another form of the equation to the polar plane of the point $x_1y_1z_1$, m and n being the indices of position of the diameter of the surface (1) passing through the pole.

By assigning different values to z_1 in this equation we shall obtain the equations to the polar planes of a system of points which range on the same diameter, and it is evident from the form of the equation that the polar planes of all such points are parallel.

When the given surface is a cone having its vertex at the origin, equation (4) becomes

$$(\mathrm{A}m+\mathrm{B}''n+\mathrm{B}')x+(\mathrm{B}''m+\mathrm{A}'n+\mathrm{B})y+(\mathrm{B}'m+\mathrm{B}n+\mathrm{A}'')z=0\ldots(4');$$

and since this is independent of the coordinates x_1, y_1, z_1 of the pole, it follows that equation (4′) denotes the polar plane of *any* point which lies in the straight line

$$x=mz,\quad y=nz\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

passing through the vertex of the cone; or that the pole of the plane (4′) may be any point in the straight line (5). Hence the straight line (5) is called the *polar line* of the plane (4′) with respect to the cone

$$\mathrm{A}x^2+\mathrm{A}'y^2+\mathrm{A}''z^2+2\mathrm{B}yz+2\mathrm{B}'zx+2\mathrm{B}''xy=0\ldots\ldots\ldots(1').$$

It is evident that the polar line of the plane (4′) is the diameter of the cone (1′) which is conjugate to that plane.

XVII.

When the pole P is at infinity, the transversal PP′ remains always parallel to the same straight line; and since, in virtue of the harmonic relation, we have P′Q = P′Q′, the point P′ bisects the part of the transversal intercepted within the surface. Consequently *when the pole is at infinity its polar plane is the diametral plane which bisects chords parallel to the straight line drawn from the centre of the surface in the direction of the pole.* Now by taking $z_1=\infty$ in equation (4) we obtain

$$(\mathrm{A}m+\mathrm{B}''n+\mathrm{B}')x+(\mathrm{B}''m+\mathrm{A}'n+\mathrm{B})y+(\mathrm{B}'m+\mathrm{B}n+\mathrm{A}'')z+\mathrm{C}m+\mathrm{C}'n+\mathrm{C}''=0\ldots(6),$$

which is therefore the equation of the diametral plane that bisects chords parallel to the diameter

$$x-t=m(z-v),\quad y-u=n(z-v)\ldots\ldots\ldots(5').$$

Since the planes represented by equations (4) and (6) are parallel when m and n are the same in both, it follows that *the polar plane* (4) *of any point in a diameter* (5′) *of a surface of the second degree, is parallel to the diametral plane* (6) *which is conjugate to that diameter.*

When the surface is a sphere, the diametral plane conjugate to any diameter is perpendicular to it; hence in this case the polar plane of any point is perpendicular to the diameter passing through the pole.

XVIII.

When the pole P is without the given surface (1), if the transversal PP′ be drawn so as to touch the surface, the points Q, Q′, P′ will evidently coincide with the point of contact, and therefore that point belongs to the polar plane of P. Hence *if a cone be described so as to envelop the surface* (1), *every point in the curve of contact lies in the polar plane of the vertex.*

If the pole move up to the surface along any diameter, its polar plane will move in the opposite direction, always remaining parallel to the diametral plane which is conjugate to that diameter (XVII) ; and when the pole comes to the surface, the enveloping cone having its vertex at the pole, expands into a plane touching the surface at the pole. Hence when the pole P is on the surface (1), or when the condition

$$Ax_1^2+A'y_1^2+A''z_1^2+2By_1z_1+2B'z_1x_1+2B''x_1y_1 \\ +2Cx_1+2C'y_1+2C''z_1+E=0$$

is satisfied, equation (2) denotes the tangent plane applied to the surface (1) at the point $x_1y_1z_1$.

It is evident from equation (2) that the preceding equation also expresses the condition that the polar plane (2) may pass through its pole $x_1y_1z_1$; hence when the polar plane passes through its pole, the polar plane touches the surface, and the point of contact is the pole.

XIX.

If x', y', z' be the coordinates of any point R, the condition that the plane (2) may pass through R will be

$$Ax_1x'+A'y_1y'+A''z_1z'+C(x_1+x')+C'(y_1+y')+C''(z_1+z') \\ +B(y_1z'+z_1y')+B'(z_1x'+x_1z')+B''(x_1y'+y_1x')+E=0 \,..\, (7);$$

but (XV) the equation of the polar plane of R is

$$(Ax'+B''y'+B'z'+C)x+(B''x'+A'y'+Bz'+C')y+(B'x'+By'+A''z'+C'')z \\ +Cx'+C'y'+C''z'+E=0,$$

from which it is evident that equation (7) also expresses the condition that the polar plane of R may pass through P; hence we have the following theorem :—

If the polar plane of any point pass through a second point, then reciprocally the polar plane of the second point will pass through the first point.

When two points are such that the polar plane of each, taken with respect to a given surface of the second degree, passes through the other, they are called *polar reciprocal points* with respect to the given surface. Hence P and R are polar reciprocal points with regard to the surface (1).

Since the polar plane of any point R in the plane (2) passes through the pole of that plane, it is evident that if any three points be taken in the plane (2) the common point of intersection of the polar planes of these points will be the pole of the plane (2). Hence *the pole of any plane is the point of intersection of the polar planes of any three points in that plane.*

XX.

When the given surface is a cone, the polar planes of P and R pass through the vertex O of the cone (XV) ; and the theorem obtained in the last number may be stated as follows :—

If two straight lines OP *and* OR, *drawn through the vertex* O *of a cone, be such that the polar plane of* OP *passes through* OR, *then reciprocally the polar plane of* OR *will pass through* OP. On account of this remarkable relation, the straight lines OP and OR are called *polar reciprocal straight lines* with respect to the cone.

Let OPR and OP′R be any two planes passing through the vertex O of the cone, and cutting each other in the line OR, and let OQ and OQ′ be the polar lines (XVI) of the planes OPR and OP′R respectively; then since OR lies in the plane OPR, its polar plane passes through OQ, and since OR is in the plane OP′R its polar plane passes through OQ′, and therefore OQQ′ is the polar plane of OR. Hence it is evident that *the polar line of any plane passing through the vertex of a cone, is the intersection of the polar planes of any two lines in that plane which pass through the vertex of the cone.*

XXI.

Take any two points P′ and R′ in the line of intersection of the polar planes of P and R (XIX); then since P′ is a point in the polar plane of P, its polar plane passes through P (XIX), and similarly the polar plane of R′ passes through P; hence also the intersection of the polar planes of P′ and R′ passes through P, and in like manner it may be shown that it passes through R. Hence we have the following theorem:—

If two straight lines PR *and* P′R′ *be so related that the polar planes of any two points* P *and* R *on the former, taken with regard to a surface of the second degree, pass through the latter; then reciprocally the polar planes of any two points* P′ *and* R′ *on the latter will pass through the former.*

On account of this important relation PR and P′R′ are called *polar reciprocal straight lines.*

When the equations to one of the lines are given, the equations to its reciprocal polar may be found as follows. Let $x'y'z'$ denote the pole of the plane

$$ax + by + cz = 1 \ldots\ldots\ldots\ldots (a),$$

then by identifying this with the equation to the polar plane of the point $x'y'z'$ (XIX) we obtain

$$\begin{aligned}
&A x' + B''y' + B'z' + C + a(Cx' + C'y' + C''z' + E) = 0,\\
&B''x' + A'y' + B z' + C' + b(Cx' + C'y' + C''z' + E) = 0,\\
&B'x' + B y' + A''z' + C'' + c(Cx' + C'y' + C''z' + E) = 0.
\end{aligned}$$

Let the equations to the given line be

$$x = mz + p, \quad y = nz + q \ldots\ldots\ldots\ldots (8),$$

then the conditions that the plane (a) may pass through this line are

$$ma + nb + c = 0, \quad pa + qb - 1 = 0;$$

and by substituting in these the values of a, b, c, given by the above equations, we shall have

$$\left.\begin{aligned}
&m(Ax' + B''y' + B'z' + C) + n(B''x' + A'y' + Bz' + C')\\
&\qquad + B'x' + By' + A''z' + C'' = 0\\
&p(Ax' + B''y' + B'z' + C) + q(B''x' + A'y' + Bz' + C')\\
&\qquad + Cx' + C'y' + C''z' + E = 0
\end{aligned}\right\} \ldots\ldots (9).$$

The straight line represented by the last two equations is evidently the locus of the poles of all planes passing through the straight line (8), and therefore the straight lines (8) and (9) are reciprocal polars.

If a system of straight lines be drawn parallel to the diameter (5′) of the surface (1), it is evident from the first of equations (9) that the polar reciprocals of these lines will lie in the diametral plane (6) which bisects chords parallel to the diameter (5′).

XXII.

To find the coordinates of the pole of any given plane.

Let the equation of the given plane be

$$ax+by+cz=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and let x', y', z' be the coordinates of its pole; then, as in No. XXI, we obtain

$$\left.\begin{array}{l} Ax'+B''y'+B'z'+C+a(Cx'+C'y'+C''z'+E)=0, \\ B''x'+A'y'+Bz'+C'+b(Cx'+C'y'+C''z'+E)=0, \\ B'x'+By'+A''z'+C''+c(Cx'+C'y'+C''z'+E)=0 \end{array}\right\}\ldots\ldots(a).$$

If, for brevity, we assume

$$\begin{aligned} N=&\{E(B^2-A'A'')+C'(A''C'-BC'')+C''(A'C''-BC')\}a \\ &+\{E(A''B''-B'B)+C(BC''-C'A'')+C''(B'C'-B''C'')\}b \\ &+\{E(A'B'-B''B)+C(BC'-C''A')+C'(B''C''-B'C')\}c \\ &+C(B^2-A'A'')+C'(A''B''-B'B)+C''(A'B'-B''B), \ldots\ldots(b), \end{aligned}$$

$$\begin{aligned} N'=&\{E(A''B''-B'B)+C'(B'C''-CA'')+C''(BC-B''C'')\}a \\ &+\{E(B'^2-A''A)+C''(C''A-B'C)+C(CA''-B'C'')\}b \\ &+\{E(AB-B'B'')+C(B''C''-BC)+C'(B'C-C''A)\}c \\ &+C(A''B''-B'B)+C'(B'^2-A''A)+C''(AB-B'B''), \ldots\ldots(c), \end{aligned}$$

$$\begin{aligned} N''=&\{E(A'B'-B''B)+C'(BC-B'C')+C''(B''C'-CA')\}a \\ &+\{E(AB-B'B'')+C''(B''C-AC')+C(B'C'-BC)\}b \\ &+\{E(B''^2-AA')+C(CA'-B''C')+C'(C'A-B''C)\}c \\ &+C(A'B'-B''B)+C'(AB-B'B'')+C''(B''^2-AA'), \ldots\ldots(d), \end{aligned}$$

$$\begin{aligned} D=&\{C(A'A''-B^2)+C'(BB'-B''A'')+C''(BB''-B'A')\}a \\ &+\{C'(A''A-B'^2)+C''(B'B''-BA)+C(BB'-B''A'')\}b \\ &+\{C''(AA'-B''^2)+C(B''B-B'A')+C'(B'B''-BA)\}c \\ &+A(A'A''-B^2)+B'(B''B-B'A')+B''(BB'-B''C) \ldots\ldots(e); \end{aligned}$$

we obtain from equations (a) by any of the ordinary methods of elimination

$$x'=\frac{N}{D},\quad y'=\frac{N'}{D},\quad z'=\frac{N''}{D} \ldots\ldots\ldots\ldots\ldots (11),$$

which determine completely the coordinates of the pole of the given plane (10).

To simplify the expressions for N, N′, N″, D, let us assume

$$\left.\begin{array}{l} A_1=E(B^2-A'A'')+C'(A''C'-BC'')+C''(A'C''-BC') \\ A_2=E(B'^2-A''A)+C''(C''A-B'C)+C(CA''-B'C'') \\ A_3=E(B''^2-A'A)+C(CA'-B''C')+C'(C'A-B''C) \end{array}\right\}\ldots\ldots(12),$$

$$\left.\begin{aligned}B_1&=E(AB-B'B'')+C''(B''C-AC')+C\,(B'C'-B\,C)\\B_2&=E(A'B'-B''B)+C\,(BC'-A'C'')+C'(B''C''-B'C')\\B_3&=E(A''B''-BB')+C'(B'C''-CA'')+C''(BC-B''C'')\end{aligned}\right\}\ldots\ldots(13),$$

$$\left.\begin{aligned}C_1&=C\,(B'^2-A'A'')+C'(A''B''-BB')+C''(A'B'-B''B)\\C_2&=C'(B''^2-A''A)+C''(AB-B'B'')+C(A''B''-B'B)\\C_3&=C''(B''^2-AA')+C\,(A'B'-B''B)+C'(AB-B'B'')\end{aligned}\right\}\ldots\ldots(14),$$

$$E_1=AB^2+A'B'^2+A''B''^2-AA'A''-2BB'B''\ldots\ldots\ldots\ldots(15);$$

$$\therefore\ N=A_1a+B_3b+B_2c+C_1\ldots\ldots\ldots\ldots(b'),$$

$$N'=B_3a+A_2b+B_1c+C_2\ldots\ldots\ldots\ldots(c'),$$

$$N''=B_2a+B_1b+A_3c+C_3\ldots\ldots\ldots\ldots(d'),$$

$$D=-C_1a-C_2b-C_3c-E_1\ldots\ldots\ldots\ldots(e').$$

By supposing the plane (a) to coincide successively with each of the coordinate planes, we obtain from (11) the coordinates of the poles of those planes. Thus, when the plane (a) coincides with the plane of xy, we have $a=0$, $b=0$, $c=\infty$; hence if h, h', h'' be the coordinates of the plane of xy, we shall have

$$h=-\frac{B_2}{C_3},\quad h'=-\frac{B_1}{C_3},\quad h''=-\frac{A_3}{C_3}\ldots\ldots\ldots\ldots(16).$$

Similarly, if $gg'g''$ denote the pole of the plane of zx, we have

$$g=-\frac{B_3}{C_2},\quad g'=-\frac{A_2}{C_2},\quad g''=-\frac{B_1}{C_2}\ldots\ldots\ldots\ldots(17),$$

and if $ff'f''$ denote the pole of the plane of yz, we have

$$f=-\frac{A_1}{C_1},\quad f'=-\frac{B_3}{C_1},\quad f''=-\frac{B_2}{C_1}\ldots\ldots\ldots\ldots(18);$$

hence by eliminating A_1, A_2, A_3, B_1, B_2, B_3 from equations (b'), (c'), (d'), (e') by means of these equations, we obtain

$$N=-C_1(fa+f'b+f''c-1)\ldots\ldots\ldots\ldots(b''),$$

$$N'=-C_2(ga+g'b+g''c-1)\ldots\ldots\ldots\ldots(c''),$$

$$N''=-C_3(ha+h'b+h''c-1)\ldots\ldots\ldots\ldots(d'').$$

If k, k', k'' denote the coordinates of the centre of the given surface, we obtain from equations (3), No. XVI,

$$k=-\frac{C_1}{E_1},\quad k'=-\frac{C_2}{E_1},\quad k''=-\frac{C_3}{E_1}\ldots\ldots\ldots\ldots(19),$$

$$\therefore\ D=+E_1(ka+k'b+k''c-1)\ldots\ldots\ldots\ldots(e'');$$

and by substituting these values of N, N′, N″, D in equations (11), we get

$$\left.\begin{aligned}x'&=\frac{k(fa+f'b+f''c-1)}{ka+k'b+k''c-1}\\y'&=\frac{k'(ga+g'b+g''c-1)}{ka+k'b+k''c-1}\\z'&=\frac{k''(ha+h'b+h''c-1)}{ka+k'b+k''c-1}\end{aligned}\right\}\ldots\ldots\ldots\ldots(11'),$$

which is a very simple form of the expressions for the coordinates of the pole of the given plane (10).

XXIII.

To find the polar line of a given plane, or the locus of the poles of that plane, with regard to a given cone of the second degree.

Let the required line be denoted by

$$\frac{x}{a} = \frac{y}{\beta} = \frac{z}{\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

then if $x'y'z'$ be any point on this line, the equation of its polar plane will be, XV,

$$(\mathrm{A}x'+\mathrm{B}''y'+\mathrm{B}'z')x+(\mathrm{B}''x'+\mathrm{A}'y'+\mathrm{B}z')y+(\mathrm{B}'x'+\mathrm{B}y'+\mathrm{A}''z')z=0;$$

which may also be written in the form

$$(\mathrm{A}a+\mathrm{B}''\beta+\mathrm{B}'\gamma)x+(\mathrm{B}''a+\mathrm{A}'\beta+\mathrm{B}\gamma)y+\mathrm{B}'a+\mathrm{B}\beta+\mathrm{A}''\gamma)z=0.$$

Since this plane coincides with the given one, if the equation of the latter be

$$mx+ny-pz=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the conditions for coincidence will be

$$\frac{\mathrm{A}a+\mathrm{B}''\beta+\mathrm{B}'\gamma}{\mathrm{B}'a+\mathrm{B}\beta+\mathrm{A}''\gamma}+\frac{m}{p}=0, \quad \frac{\mathrm{B}''a+\mathrm{A}'\beta+\mathrm{B}\gamma}{\mathrm{B}'a+\mathrm{B}\beta+\mathrm{A}''\gamma}+\frac{n}{p}=0;$$

hence by elimination we obtain

$$\left.\begin{aligned} \frac{a}{\gamma} &= \frac{m(\mathrm{A}'\mathrm{A}''-\mathrm{B}^2)+n(\mathrm{B}\mathrm{B}'-\mathrm{B}''\mathrm{A}'')-p(\mathrm{B}''\mathrm{B}-\mathrm{B}'\mathrm{A}')}{m(\mathrm{B}''\mathrm{B}-\mathrm{B}'\mathrm{A}')+n(\mathrm{B}'\mathrm{B}''-\mathrm{B}\mathrm{A})-p(\mathrm{A}\mathrm{A}'-\mathrm{B}''^2)} \\ \frac{\beta}{\gamma} &= \frac{m(\mathrm{B}\mathrm{B}'-\mathrm{B}''\mathrm{A}'')+n(\mathrm{A}\mathrm{A}''-\mathrm{B}'^2)-p(\mathrm{B}'\mathrm{B}''-\mathrm{A}\mathrm{B})}{m(\mathrm{B}''\mathrm{B}-\mathrm{B}'\mathrm{A}')+n(\mathrm{B}'\mathrm{B}''-\mathrm{B}\mathrm{A})-p(\mathrm{A}\mathrm{A}'-\mathrm{B}''^2)} \end{aligned}\right\} \ldots (22),$$

from which the position of the straight line (20) becomes known.

Let $x=a_3z$, $y=\beta_3z$ be the equations to the polar line of the plane of xy, then by taking $m=0$ and $n=0$ in (22), we obtain

$$a_3=\frac{\mathrm{B}''\mathrm{B}-\mathrm{B}'\mathrm{A}'}{\mathrm{A}\mathrm{A}'-\mathrm{B}''^2}, \quad \beta_3=\frac{\mathrm{B}'\mathrm{B}''-\mathrm{A}\mathrm{B}}{\mathrm{A}\mathrm{A}'-\mathrm{B}''^2} \quad \ldots\ldots\ldots\ldots (23).$$

Similarly, if $x=a_2y$, $z=\gamma_2y$ denote the polar line of the plane of xz, we have

$$a_2=\frac{\mathrm{B}\mathrm{B}'-\mathrm{B}''\mathrm{A}''}{\mathrm{A}\mathrm{A}''-\mathrm{B}'^2}, \quad \gamma_2=\frac{\mathrm{B}'\mathrm{B}''-\mathrm{B}\mathrm{A}}{\mathrm{A}\mathrm{A}''-\mathrm{B}'^2} \quad \ldots\ldots\ldots\ldots (24),$$

and if $y=\beta_1x$, $z=\gamma_1x$, be the polar line of the plane of yz, we have

$$\beta_1=\frac{\mathrm{B}\mathrm{B}'-\mathrm{B}''\mathrm{A}''}{\mathrm{A}'\mathrm{A}''-\mathrm{B}^2}, \quad \gamma_1=\frac{\mathrm{B}''\mathrm{B}-\mathrm{B}'\mathrm{A}'}{\mathrm{A}'\mathrm{A}''-\mathrm{B}^2} \quad \ldots\ldots\ldots\ldots (25);$$

and in virtue of these the general equations become

$$\left.\begin{aligned} \frac{a}{\beta} &= \frac{(\mathrm{B}^2-\mathrm{A}'\mathrm{A}'')(m+n\beta_1-p\gamma_1)}{(\mathrm{B}'^2-\mathrm{A}\mathrm{A}'')(ma_2+n-p\gamma_2)} \\ \frac{\beta}{\gamma} &= \frac{(\mathrm{B}'^2-\mathrm{A}\mathrm{A}'')(ma_2+n-p\gamma_2)}{(\mathrm{B}''^2-\mathrm{A}\mathrm{A}')(ma_3+n\beta_3-p)} \\ \frac{\gamma}{a} &= \frac{(\mathrm{B}''^2-\mathrm{A}\mathrm{A}')(ma_3+n\beta_3-p)}{(\mathrm{B}^2-\mathrm{A}'\mathrm{A}'')(m+n\beta_1-p\gamma_1)} \end{aligned}\right\} \ldots\ldots\ldots\ldots (22')$$

which is a simpler form of the equations for determining the polar line of the plane (21) with respect to the cone (1').

GENERAL THEORY OF POLAR RECIPROCAL SURFACES.

XXIV.

If two surfaces (A) and (B) be so related that the points P, P', P'', *etc.* of the former are the poles of tangent planes applied to the latter at the points Q, Q', Q'', *etc.* with regard to a given surface (C)* of the second degree; then reciprocally the points Q, Q', Q'', *etc.* are the poles of tangent planes applied to the surface (A) at the points P, P', P'', *etc.*

If the tangent planes applied to (B) at the points Q, Q', Q'' intersect one another in the point R, it is evident (XIX) that R is the pole of the plane $PP'P''$. Let the points P and Q remain fixed, while Q' and Q'' move up to Q and ultimately coincide with it. It is evident that R will then coincide with Q; and since the points P, P', P'' ultimately coalesce, the plane $PP'P''$ will become a tangent plane at P. But since R continues to be the pole of the plane $PP'P''$ in every position during the motion, it follows that its ultimate position Q is the pole of the plane $PP'P''$ in its ultimate position when it touches the surface (A) at P; and in like manner it may be demonstrated that Q' is the pole of the tangent plane at P', and so on.

When two surfaces are such that each of them may be considered either as the locus of the poles of tangent planes applied to the other, or the envelope of the polar planes of the points of the other, with regard to any surface of the second degree; they are called *polar reciprocal surfaces* in relation to the given surface, which is called the *director*. Thus the surfaces (A) and (B) are polar reciprocals with respect to the director (C).

Two points so related that each is the pole of the tangent plane applied at the other, are called *corresponding points* of the reciprocal surfaces (A) and (B).

XXV.

We have seen (XVII) that the polar plane of any point in relation to a surface of the second degree is parallel to the diametral plane which is conjugate to the diameter passing through that point. Hence, since the polar plane of any point P on the surface (A) touches the surface (B) at the corresponding point Q, we have the following theorem:—

If a tangent plane be applied at any point to the surface (B), *and if a diameter of the director be drawn through the corresponding point of* (A), *the diametral plane of the director* (C) *which is conjugate to that diameter is parallel to the tangent plane.*

When the director is a central surface, if the centre of that surface be taken as the origin of coordinates, the theorem may be expressed more simply as follows:—

Any tangent plane applied to the surface (B) *is parallel to the diametral plane of* (C) *which is conjugate to the corresponding radius vector of* (A).

Since every diametral plane of a sphere is perpendicular to its conjugate diameter, it follows from the above theorem, that *when the director is a sphere, a tangent plane applied to the surface* (B) *at any point is cut perpendicularly by the radius vector drawn to the corresponding point of the reciprocal surface* (A), and the converse.

* The surface (C) may be an ellipsoid, a hyperboloid, or a paraboloid. The case in which it is a cone is reserved for separate investigation.

XXVI.

Let the director be a sphere represented by the equation

$$x^2 + y^2 + z^2 = R^2;$$

then if $x_1y_1z_1$ denote any point P on the surface (A) the equation of its polar plane will be, (xv),

$$x_1x + y_1y + z_1z = R^2,$$

and since this plane passes through the corresponding point $x'y'z'$ of the surface (B), we shall have

$$x_1x' + y_1y' + z_1z' = R^2 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let r be the radius vector of the point P; α, β, γ the angles which it makes with the axes of coordinates, and p the part of it intercepted between the centre of the director and the perpendicular tangent plane applied at the corresponding point Q $(x'y'z')$ of the surface (B); and let r', α', β', γ', p' be the corresponding quantities for the point Q; then equation (1) becomes

$$rr'(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma') = R^2;$$

which may also be written in either of the following forms,

$$rp' = R^2, \quad r'p = R^2 \ldots\ldots\ldots\ldots\ldots\ldots(2);$$

because $\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$ is equal to the cosine of the angle made by the lines r and r'. These equations show that *the perpendicular to any tangent plane of the surface* (B) *is reciprocally proportional to the radius vector of the reciprocal surface* (A) *which coincides with it in direction*, and the converse. We shall see hereafter that this elegant property may be made the basis of the theory of sphero-polar reciprocal surfaces, and enables us to dispense altogether with the aid of the director.

XXVII.

Given the equation of any surface, to find the equation of its reciprocal surface, in relation to an ellipsoid, a hyperboloid, or a paraboloid taken as director.

Let $x_1y_1z_1$ denote any point P in the surface (A), $x'y'z'$ the corresponding point Q in the reciprocal surface (B), and let the equation of the director be

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy + 2Cx + 2C'y + 2C''z + E = 0;$$

then if $u = 0$ be the equation of the given surface (B), the equation of the tangent plane applied to it at the point $x'y'z'$ will be

$$(x - x')\frac{du'}{dx'} + (y - y')\frac{du'}{dy'} + (z - z')\frac{du'}{dz'} = 0:$$

but (xv) the equation of the polar plane of the point $x_1y_1z_1$ is

$$(Ax_1 + B''y_1 + B'z_1 + C)x + (B''x_1 + A'y_1 + Bz_1 + C')y$$
$$+ (B'x_1 + By_1 + A''z_1 + C'')z + Cx_1 + C'y_1 + C''z_1 + E = 0;$$

and since these two planes are identical, (xxiv), we shall have

$$\left.\begin{aligned}
&\frac{du'}{dx'}(Cx_1 + C'y_1 + C''z_1 + E) + \left(x'\frac{du'}{dx'} + y'\frac{du'}{dy'} + z'\frac{du'}{dz'}\right)(Ax_1 + B''y_1 + B'z_1 + C) = 0 \\
&\frac{du'}{dy'}(Cx_1 + C'y_1 + C''z_1 + E) + \left(x'\frac{du'}{dx'} + y'\frac{du'}{dy'} + z'\frac{du'}{dz'}\right)(B''x_1 + A'y_1 + Bz_1 + C') = 0 \\
&\frac{du'}{dz'}(Cx_1 + C'y_1 + C''z_1 + E) + \left(x'\frac{du'}{dx'} + y'\frac{du'}{dy'} + z'\frac{du'}{dz'}\right)(B'x_1 + By_1 + A''z_1 + C'') = 0
\end{aligned}\right\} \ldots\ldots (3).$$

The equation of the surface (A) may be obtained by eliminating x', y', z' from these equations by means of the equation $u'=0$.

When u is of the form $\phi-\theta$, where ϕ is a homogeneous function of x, y, z of n dimensions, and θ is constant, the above equations admit of an important simplification. For, by a well known property of homogeneous functions, we have

$$x\frac{d\phi}{dx}+y\frac{d\phi}{dy}+z\frac{d\phi}{dz}=n\phi;$$

$$\therefore\quad x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}=n\phi'=n\theta,$$

and equations (3) become

$$\left.\begin{aligned}\frac{du'}{dx'}(Cx_1+C'y_1+C''z_1+E)+n\theta(Ax_1+B''y_1+B'z_1+C)=0\\ \frac{du'}{dy'}(Cx_1+C'y_1+C''z_1+E)+n\theta(B''x_1+A'y_1+Bz_1+C')=0\\ \frac{du'}{dz'}(Cx_1+C'y_1+C''z_1+E)+n\theta(B'x_1+By_1+A''z_1+C'')=0\end{aligned}\right\}\ldots\ldots(3').$$

XXVIII.

To illustrate the method of applying these formulas, let it be proposed to find the reciprocal polar to the surface whose equation is

$$u=\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}+\left(\frac{z}{c}\right)^{2n}-1=0\ldots\ldots\ldots(4).$$

By differentiation we obtain

$$\frac{du'}{dx'}=\frac{2nx'^{2n-1}}{a^{2n}},\quad \frac{du'}{dy'}=\frac{2ny'^{2n-1}}{b^{2n}},\quad \frac{du'}{dz'}=\frac{2nz'^{2n-1}}{c^{2n}};$$

hence equations (3′) give

$$\frac{x'^{2n-1}}{a^{2n}}(Cx_1+C'y_1+C''z_1+E)=-(Ax_1+B''y_1+B'z_1+C),$$

$$\frac{y'^{2n-1}}{b^{2n}}(Cx_1+C'y_1+C''z_1+E)=-(B''x_1+A'y_1+Bz_1+C'),$$

$$\frac{z'^{2n-1}}{c^{2n}}(Cx_1+C'y_1+C''z_1+E)=-(B'x_1+By_1+A''z_1+C'');$$

and by multiplying the first of these by a, the second by b, the third by c, raising each to the power $\frac{2n}{2n-1}$, adding the results, and putting $2m$ for $\frac{2n}{2n-1}$, we obtain the required equation

$$a^{2m}(Ax_1+B''y_1+B'z_1+C)^{2m}+b^{2m}(B''x_1+A'y_1+Bz_1+C')^{2m}+$$
$$c^{2m}(B'x_1+By_1+A''z_1+C'')^{2m}=(Cx_1+C'y_1+C''z_1+E)^{2m}\ldots.(5).$$

When $n=1$ the given surface is the ellipsoid

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1\ldots\ldots\ldots\ldots\ldots\ldots(4'),$$

and its reciprocal polar is the surface of the second degree

$$a^2(Ax+B''y+B'z+C)^2+b^2(B''x+A'y+Bz+C')^2+c^2(B'x+By+A''z+C'')^2$$
$$=(Cx+C'y+C''z+E)^2\ldots\ldots(5').$$

When c is imaginary, or c^2 negative, equation (4′) denotes a hyperboloid of one sheet, and it is evident from (5′) that the reciprocal surface will, in this case, pass through four of the intersections of the four planes represented by the following equations;—

$$a(\mathrm{A}x+\mathrm{B}''y+\mathrm{B}'z+\mathrm{C})+c(\mathrm{B}'x+\mathrm{B}y+\mathrm{A}''z+\mathrm{C}'')=0,$$
$$a(\mathrm{A}x+\mathrm{B}''y+\mathrm{B}'z+\mathrm{C})-c(\mathrm{B}'x+\mathrm{B}y+\mathrm{A}''z+\mathrm{C}'')=0,$$
$$b(\mathrm{B}''x+\mathrm{A}'y+\mathrm{B}z+\mathrm{C}')+(\mathrm{C}x+\mathrm{C}'y+\mathrm{C}''z+\mathrm{E})=0,$$
$$b(\mathrm{B}''x+\mathrm{A}'y+\mathrm{B}z+\mathrm{C}')-(\mathrm{C}x+\mathrm{C}'y+\mathrm{C}''z+\mathrm{E})=0.$$

XXIX.

As another example let it be required to find the surface reciprocal to

$$u=\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n+n\left(\frac{z}{c}\right)^{n-1}=0 \ldots\ldots\ldots\ldots (6).$$

By differentiation we get

$$\frac{du'}{dx'}=\frac{nx'^{n-1}}{a^n},\quad \frac{du'}{dy'}=\frac{ny'^{n-1}}{b^n},\quad \frac{du'}{dz'}=\frac{n(n-1)z'^{n-2}}{c^{n-1}};$$

$$\therefore x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}=n\left(\frac{x'^n}{a^n}+\frac{y'^n}{b^n}+(n-1)\frac{z'^{n-1}}{c^{n-1}}\right)=-n\left(\frac{z'}{c}\right)^{n-1}:$$

hence equations (3) may be written in the form

$$\left(\frac{x}{a}\right)^{n-1}=a\mathrm{X}.\left(\frac{z}{c}\right)^{n-1},\quad \left(\frac{y}{b}\right)^{n-1}=b\mathrm{Y}.\left(\frac{z}{c}\right)^{n-1},$$

$$(n-1)\frac{z^{n-2}}{c^{n-1}}=\mathrm{Z}\left(\frac{z}{c}\right)^{n-1} \text{ or } z\mathrm{Z}=n-1;$$

from which we readily obtain

$$\left\{(a\mathrm{X})^{\frac{n}{n-1}}+(b\mathrm{Y})^{\frac{n}{n-1}}\right\}\left(\frac{z}{c}\right)^n=\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=-n\left(\frac{z}{c}\right)^{n-1},$$

$$\text{or } (a\mathrm{X})^{\frac{n}{n-1}}+(b\mathrm{Y})^{\frac{n}{n-1}}+\frac{n}{n-1}c\mathrm{Z}=0.$$

Putting m for $\dfrac{n}{n-1}$, and restoring the values of X, Y, Z, we obtain the equation of the reciprocal surface

$$a^m(\mathrm{A}x_1+\mathrm{B}''y_1+\mathrm{B}'z_1+\mathrm{C})^m+b^m(\mathrm{B}''x_1+\mathrm{A}'y_1+\mathrm{B}z_1+\mathrm{C}')^m$$
$$+mc(\mathrm{B}'x_1+\mathrm{B}y_1+\mathrm{A}''z_1+\mathrm{C}'')(\mathrm{C}x_1+\mathrm{C}'y_1+\mathrm{C}''z_1+\mathrm{E})^{m-1}=0 \ldots . (7).$$

Taking $n=1$ and c negative, the given surface (6) becomes the elliptic paraboloid

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-2\left(\frac{z}{c}\right)=0,$$

and the reciprocal polar (7) is the surface of the second degree

$$a^2(\mathrm{A}x+\mathrm{B}''y+\mathrm{B}'z+\mathrm{C})^2+b^2(\mathrm{B}''x+\mathrm{A}'y+\mathrm{B}z+\mathrm{C}')^2$$
$$=2c(\mathrm{B}'x+\mathrm{B}y+\mathrm{A}''z+\mathrm{C}'')(\mathrm{C}x+\mathrm{C}'y+\mathrm{C}''z+\mathrm{E}).$$

When b is imaginary, or b^2 negative, the paraboloid becomes hyperbolic, and the reciprocal surface passes through four of the intersections of the four planes represented by the equations

$$a(\mathrm{A}x+\mathrm{B}''y+\mathrm{B}'z+\mathrm{C})+b(\mathrm{B}''x+\mathrm{A}'y+\mathrm{B}z+\mathrm{C}')=0,$$
$$a(\mathrm{A}x+\mathrm{B}''y+\mathrm{B}'z+\mathrm{C})-b(\mathrm{B}''x+\mathrm{A}'y+\mathrm{B}z+\mathrm{C}')=0,$$
$$\mathrm{B}'x+\mathrm{B}y+\mathrm{A}''z+\mathrm{C}''=0, \text{ and } \mathrm{C}x+\mathrm{C}'y+\mathrm{C}''z+\mathrm{E}=0.$$

XXX.

Hitherto the equation of the director has been left in its most general form, and the axes of coordinates have been selected so as to simplify as much as possible the equation of the given surface (B.) In this manner the eliminations, in the preceding examples, were rendered extremely easy. The axes may however be so assumed that the equation of the director shall take its simplest form, and the equations in No. XXVII. may thus be considerably modified. The following appear to be the most important cases.

First let the director be a paraboloid represented by the equation

$$Ax^2+By^2-2ABz=0;$$

then the equation of the tangent plane applied to the surface (B) at the point $x'y'z'$ being

$$z-z'=\frac{dz'}{dx'}(x-x')+\frac{dz'}{dy'}(y-y'),$$

and the equation of the polar plane at the point $x_1y_1z_1$ being

$$Ax_1x+By_1y-AB(z_1+z)=0;$$

we obtain by identifying these equations

$$\frac{dz'}{dx'}=\frac{x_1}{B},\quad \frac{dz'}{dy'}=\frac{y_1}{A},\quad z'-x'\frac{dz'}{dx'}-y'\frac{dz'}{dy'}=-z_1 \ldots(8);$$

and the equation of the reciprocal surface (A) will be obtained by eliminating x',y',z' from these by means of the equation of the given surface (B).

Secondly, let the director be a central surface represented by the equation

$$Ax^2+By^2+Cz^2=1,$$

the origin being at the centre of the director, and the axes coinciding with a system of conjugate diameters. In this case equations (3) of No. XVII. become

$$\left.\begin{aligned}\frac{du'}{dx'}&=Ax_1\left(x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}\right)\\ \frac{du'}{dy'}&=By_1\left(x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}\right)\\ \frac{du'}{dz'}&=Cz_1\left(x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}\right)\end{aligned}\right\}\ldots\ldots\ldots\ldots(9),$$

and the equation of the reciprocal surface may be found as in the preceding examples.

If the director be a sphere we may assume $A=B=C=1$, and the last three equations become

$$\left.\begin{aligned}\frac{du'}{dx'}&=x_1\left(x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}\right)\\ \frac{du'}{dy'}&=y_1\left(x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}\right)\\ \frac{du'}{dz'}&=z_1\left(x'\frac{du'}{dx'}+y'\frac{du'}{dy'}+z'\frac{du'}{dz'}\right)\end{aligned}\right\}\ldots\ldots\ldots\ldots(10).$$

Example. To find the surface reciprocal to a given ellipsoid, the director being a concentric sphere.

Let the equation of the ellipsoid be

$$u = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 - 1 = 0 \ldots\ldots\ldots(11);$$

$$\therefore \quad \frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = \frac{2y}{b^2}, \quad \frac{du}{dz} = \frac{2z}{c^2};$$

$$x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} = 2;$$

hence equations (10) become

$$x' = a^2 x_1, \quad y' = b^2 y_1, \quad z' = c^2 z_1,$$

from which we easily obtain

$$a^2 x_1^2 + b^2 y_1^2 + c^2 z_1^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

the equation of the reciprocal surface, which is therefore an ellipsoid having its axes reciprocally proportional to those of the given one. The reciprocal ellipsoids (11) and (12) are of importance in the wave theory of light.

XXXI.

When the surface (B) is considered as the envelope of a series of planes which succeed one another according to any given law, the equation of the polar reciprocal surface (A) may be obtained, in many cases, with great facility. Thus, let the equation of the moveable plane be

$$lx + my + nz = v \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

v being the perpendicular drawn to it from the origin; l, m, n the direction cosines of v; and let the parameters l, m, n, v be connected by an equation of the form

$$f(l, m, n, v) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

(where f denotes any function), in addition to the well known condition

$$l^2 + m^2 + n^2 = 1;$$

thus leaving two of the parameters independent. If the sphere

$$x^2 + y^2 + z^2 = 1$$

be taken as director, the polar plane of the point $x_1 y_1 z_1$, corresponding to the point $x'y'z'$ in which the plane (13) touches the surface (B), is

$$x_1 x + y_1 y + z_1 z = 1;$$

and since (XXIV) this is identical to the plane (13), we shall have

$$l = vx_1, \quad m = vy_1, \quad n = vz;$$

$$\left.\begin{aligned} \therefore \quad 1 &= v^2(x_1^2 + y_1^2 + z_1^2) = v^2 r_1^2 \\ l = \frac{x_1}{r_1}&, \quad m = \frac{y_1}{r_1}, \quad n = \frac{z_1}{r_1} \end{aligned}\right\} \ldots\ldots\ldots\ldots(15),$$

and the equation of the surface (A) will be obtained by substituting these values of l, m, n, v in equation (14).

The reciprocal surface, in relation to any surface of the second degree taken as director, may be determined with almost equal facility.

It is evident from equations (15), that equation (14), which defines the law of motion of the variable plane (13), may be considered as the polar equation of the reciprocal surface (A), v being the reciprocal of the radius vector, and l, m, n its direction cosines. The same result might be obtained from the investigation in No. XXVI.

XXXII.

To illustrate the method of applying these principles, let it be required to find the reciprocal polar of the surface enveloped by the plane

$$lx + my + nz = v,$$

where l, m, n, v are connected by the relation

$$l^2(v^2-b^2)(v^2-c^2)+m^2(v^2-c^2)(v^2-a^2)+n^2(v^2-a^2)(v^2-b^2)=0.$$

Eliminating l, m, n, v by means of equations (15), we obtain

$$\frac{x_1^2}{1-a^2r_1^2}+\frac{y_1^2}{1-b^2r_1^2}+\frac{z_1^2}{1-c^2r_1^2}=0 \ldots\ldots\ldots\ldots(16),$$

which is the required equation. The equation of the envelope being

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0 \ldots\ldots\ldots\ldots\ldots(17),*$$

if we assume

$$aa_1 = bb_1 = cc_1 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

equation (16) may be written in the form

$$\frac{a_1^2x_1^2}{a_1^2-r_1^2}+\frac{b_1^2y_1^2}{b_1^2-r_1^2}+\frac{c_1^2z_1^2}{c_1^2-r_1^2}=0 \ldots\ldots\ldots\ldots(16')$$

which is exactly similar to the equation of the reciprocal surface (17).

As another example, let the given surface be the envelope of the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

where $abc = k^3$, k being a given constant. In this case the equation of the envelope is

$$27\,xyz = k^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19);$$

and since $ax_1 = 1$, $by_1 = 1$, $cz_1 = 0$, the equation of the reciprocal surface is

$$x_1y_1z_1 = k^{-3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

This result, like the last, is remarkable on account of the reciprocal surfaces being of the same kind. The most obvious property of these surfaces is, that the volume of the tetrahedron intercepted between any tangent plane and the coordinate planes is constant for the same surface.

XXXIII.

To find the locus of the poles of the tangent planes applied to a given developable surface.

Let the given surface by the envelope of the plane

$$z = a + xf(a) + y\phi(a) \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

in which a is a variable parameter, and f and ϕ denote given functions. If the director be a sphere, represented by the equation

$$x^2+y^2+z^2=1,$$

the polar plane of the point $x_1y_1z_1$ will be

$$x_1x + y_1y + z_1z = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22),$$

and by identifying this with equation (21) we get

$$az_1 - 1 = 0,\quad z_1f(a) + x_1 = 0,\quad z_1\phi(a) + y_1 = 0;$$

* This is the equation to the surface of a wave of light propagated through a biaxal crystal, and is therefore called the *wave surface*. Equation (16) shows that *the polar reciprocal of the wave surface is also a wave surface.* The latter has been called by M. Cauchy the *surface of refraction*, and Sir William R. Hamilton calls it the surface of *wave slowness*.

hence by eliminating a we obtain

$$\frac{x_1}{z_1}+f\left(\frac{1}{z_1}\right)=0,\ \frac{y_1}{z_1}+\phi\left(\frac{1}{z_1}\right)=0\ \ldots\ldots\ldots\ldots(23),$$

the equations of the required locus, which is therefore in general a *curve of double curvature.*

When the given surface is a conical one, all the tangent planes pass through the vertex of the cone, and therefore their poles lie in the polar plane of that vertex (XIX); hence *the locus of the poles of the tangent planes applied to a conical surface is a plane curve situated in the polar plane of the vertex of the cone.*

If the given surface be cylindrical, the tangent plane is always parallel to the axis of the cylinder, and therefore (XXI) the poles of the tangent planes lie in the diametral plane of the director which bisects chords parallel to the axis of the cylinder; hence *the locus of the poles of the tangent planes applied to a cylindrical surface is also a plane curve.*

For example, let us take the equation

$$\frac{x}{a\cos a}+\frac{y}{a\sin a}+\frac{z}{c}=1,$$

which denotes a series of planes passing through a fixed point in the axis of z, and having their traces on the plane of xy all of the same length a. By identifying this with equation (22) we get

$$ax_1\cos a=1,\quad ay_1\sin a=1,\quad cz_1=1;$$

hence by eliminating a we have

$$x_1^2+y_1^2=a^2x_1^2y_1^2,\quad cz_1=1;$$

and therefore the locus of the poles is a plane curve of the fourth degree. By passing to polar coordinates we have

$$ar\sin\theta\cos\theta=1;$$

hence the perpendicular drawn from the foot of any ordinate to the corresponding radius vector is constant for all points of the curve.

The envelope of the proposed system of planes is evidently a cone having its vertex at the given point on the axis of z.

The curve which is the locus of the poles of the tangent planes applied to a developable surface, is called the *polar curve* of that surface.

XXXIV.

To find the polar curve of a cone of the second degree in relation to a sphere.

The principal axes of the cone being taken as axes of coordinates, its equation is of the form

$$Ax^2+By^2+Cz^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

and the equation of the director is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\delta^2,$$

$\alpha\beta\gamma$ being its centre and δ its radius. Now if $x'y'z'$ be any point on the surface of the cone, and $x_1y_1z_1$ the corresponding point on the polar curve, the equation of the tangent plane at the former point is

$$Ax'x+By'y+Cz'z=0,$$

and the equation of the polar plane of the latter point is

$$(a)\ldots(x_1-\alpha)x+(y_1-\beta)y+(z_1-\gamma)z=\alpha(x_1-\alpha)+\beta(y_1-\beta)+\gamma(z_1-\gamma)+\delta^2;$$

and since these two planes are identical, we have

$$\alpha(x_1-\alpha)+\beta(y_1-\beta)+\gamma(z_1-\gamma)+\delta^2=0 \ldots\ldots \ldots (25),$$
$$Ax'(z_1-\gamma)-Cz'(x_1-\alpha)=0,\ By'(z_1-\gamma)+Cz'(y_1-\beta)=0 \ldots\ldots (b);$$

but since the polar plane (a) passes through the point $x'y'z'$ we obtain in virtue of (25)

$$(x_1-\alpha)x'+(y_1-\beta)y'+(z_1-\gamma)z'=0;$$

hence, eliminating x', y', z' by means of equations (b), we obtain

$$\frac{(x_1-\alpha)^2}{A}+\frac{(y_1-\beta)^2}{B}+\frac{(z_1-\gamma)^2}{C}=0 \ldots\ldots\ldots (26).$$

The required curve being the intersection of the cone (26) and the plane (25), it follows that the polar curve of a cone of the second degree is a conic section, such that the cone (26) having this curve for its base, and the centre of the sphere for its vertex, has its focal lines perpendicular to the circular sections of the proposed cone, and its circular sections perpendicular to the focal lines of the proposed cone.

XXXV.

If a developable surface be so related to a curve of double curvature, that the points of the curve are the poles of the tangent planes applied to the surface, with regard to any surface of the second degree taken as director; then, reciprocally, the lines of contact of the tangent planes to the developable surface will be polar reciprocals to the tangents of the curve.

Let PQ and P′Q′ be the lines of contact of two tangent planes applied to the developable surface, and T, T′ their poles; then if SS′ be the line of intersection of these tangent planes, the straight lines SS′ and TT′ will be polar reciprocals (XXI). Let the tangent plane SPQ, and consequently its pole T, remain fixed, while the tangent plane SP′Q′ moves up to, and ultimately coincides with it. The line SS′ will ultimately coincide with PQ; and TT′ will become a tangent at T to the curve of the poles; consequently the tangent at T is the polar reciprocal of the line of contact at PQ.

In consequence of this property the developable surface and curve of double curvature are called *polar reciprocals* in relation to the given surface of the second degree. The former is called the *polar surface* of the curve, and the latter the *polar curve* of the surface.

XXXVI.

To find the equation of the polar surface of a given curve in relation to a sphere.

Let $x'y'z'$ denote any point on the given curve, and $x_1y_1z_1$ any point on the line of contact which is the polar reciprocal of the tangent at $x'y'z'$; and let the equation of the director be

$$x^2+y^2+z^2=1.$$

The equations of the tangent at the point $x'y'z'$ being

$$x-x'=\frac{dx'}{dz'}(z-z'),\quad y-y'=\frac{dy'}{dz'}(z-z') \ldots\ldots (21),$$

the equations of its polar line will be

$$\left.\begin{aligned} x_1dx'+y_1dy'+z_1dz'&=0 \\ x_1(x'Az'-z'dx')+y_1(y'dz'-z'dy')-1&=0\end{aligned}\right\} \ldots\ldots\ldots (27);$$

and the equation of the required surface will be obtained by eliminating x', y', z' from these equations by means of the equation of the given curve.

By multiplying the former of these equations by z', and adding the latter to the result, we obtain

$$x_1x'+y_1y'+z_1z'=1\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

which may sometimes be advantageously substituted for the last equation.

As an example, let it be required to find the polar surface of a given conic section with regard to a sphere. Let the equation of the director be

$$x^2+y^2+z^2=1,$$

and those of the given curve

$$Ax'^2+By'^2+2Cx'+2Dy'+E=0,\ z'=c\ldots\ldots\ldots\ldots(29).$$

By differentiating the latter equations we get

$$(Ax'+C)dx'+(By'+D)dy'=0,\ dz'=0,$$

and by the first of equations (27), $x_1dx'+y_1dy'=0$;

$$\therefore -\frac{dy'}{dx'}=\frac{Ax'+C}{By'+D}=\frac{x_1}{y_1},\ \text{or}\left(x'+\frac{C}{A}\right)\div\left(y'+\frac{D}{B}\right)=\frac{Bx_1}{Ay_1}.$$

But the first of equations (29) may evidently be written in the form

$$A\left(x'+\frac{C}{A}\right)^2+B\left(y'+\frac{D}{B}\right)^2=\frac{C^2}{A}+\frac{D^2}{B}-E=E_1^2 \text{ suppose};$$

$$\therefore \left(\frac{B^2x_1^2}{Ay_1^2}+B\right)\left(y'+\frac{D}{B}\right)^2=E_1^2,$$

$$\therefore y'+\frac{D}{B}=\left(\frac{E_1y_1}{B}\right)\div\left(\frac{x_1^2}{A}+\frac{y_1^2}{B}\right)^{\frac{1}{2}},\ x'+\frac{C}{A}=\left(\frac{E_1x_1}{A}\right)\div\left(\frac{x_1^2}{A}+\frac{y_1^2}{B}\right)^{\frac{1}{2}}.$$

Multiplying the former of these by y_1, the latter by x_1, and adding the results, we get

$$x_1x'+y_1y'+\frac{Cx_1}{A}+\frac{Dy_1}{B}=E_1\left(\frac{x_1^2}{A}+\frac{y_1^2}{B}\right)^{\frac{1}{2}};$$

but by equations (28) and (29), $x_1x'+y_1y'+z_1z'=1$; hence by subtraction

$$E_1\left\{\left(\frac{x_1^2}{A}+\frac{y_1^2}{B}\right)\right\}^{\frac{1}{2}}=1-cz_1+\frac{Cx_1}{A}+\frac{Dy_1}{B},\ \text{or}$$

$$\left(\frac{C^2}{A}+\frac{D^2}{B}-E\right)\left(\frac{x_1^2}{A}+\frac{y_1^2}{B}\right)=\left(1-cz_1+\frac{Cx_1}{A}+\frac{Dy_1}{B}\right)^2\ldots\ldots(30),$$

which is the equation of the reciprocal cone.

By taking $z_1=c$ in this equation, we obtain

$$\left(\frac{C^2}{A}+\frac{D^2}{B}-E\right)\left(\frac{x_1^2}{A}+\frac{y_1^2}{B}\right)=\left(1-c^2+\frac{Cx_1}{A}+\frac{Dy_1}{B}\right)^2\ldots\ldots(30'),$$

which is the equation of the curve in which the cone (30) is cut by the plane of the given conic section (29); and the equation of the curve in which the director is cut by the same plane is

$$x^2+y^2=1-c^2.\ \ldots\ldots\ldots\ldots\ldots\ldots(a):$$

now since the curve (29) and (30′) are polar reciprocals with respect to the circle (a) (VIII), we have the following theorem;—

The polar surface of a conic section with regard to a sphere is a cone of the second degree, whose vertex is the pole of the plane of the conic section, and whose base on that plane is the polar reciprocal curve of the given conic section, taken with regard to the circle in which the plane of the given curve cuts the director.

When $B = A = 1$ the given conic section is a circle, and when z_1 is constant, equation (30) denotes a conic section having the origin for a focus and the straight line

$$1 - cz_1 + \frac{Cx_1}{A} + \frac{Dy_1}{B} = 0$$

for the corresponding directrix: hence we see that *the polar surface of a circle with regard to a sphere is a cone of the second degree, having the straight line drawn from the centre of the sphere perpendicular to the plane of the circle, for one of its focal lines.*

XXXVII.

If two cones (A) and (B) be so related to a cone (C) of the second degree, that the polar planes of the sides of the former touch the latter; then the polar planes of the sides of the latter will be tangent planes to the former.

Let O be the common vertex of the three cones. Let any two tangent planes be applied to (B), touching it along the lines OP′ and OR′, and cutting each other in the line OT; and let OP and OR be the polar lines of those planes, which, by hypothesis, lie on the surface of the cone (A); then will OT be the polar line of the plane OPR (XX). Let OP and OP′ remain fixed while OR moves up to OP along the surface of the cone (A); then when OR coincides with OP, the plane OPR touches the cone (A) along the line OP, and its polar line OT coincides with OP′. Hence the side OP′ of the cone (B) is the polar line of the plane which touches the cone (A) along the line OP; and in like manner it may be shown that OR′ is the polar line of the plane which touches the cone (B) along the line OR, *etc.*

When two conical surfaces (A) and (B) are such that the sides of each are the polar lines of tangent planes applied to the other, with respect to a given cone (C) of the second degree, they are called *polar reciprocal cones* in relation to (C) which is called the director.

XXXVIII.

Let the three cones be cut by any plane PP′S, meeting OP in P, the tangent plane to the surface (B) along the line OP′, in P′S, and the cones (A), (B), (C), in the curves (A′), B′), (C′) respectively; and let the straight line PP′ meet the curve (C′) in the points Q and Q′. Because Q and Q′ are points on the surface of the cone (C), P and P′ are conjugate harmonic points with respect to Q and Q′ (XV); from which it is evident (I) that the straight line P′S is the polar of the point P with respect to the curve (C′). But SP′ evidently touches the curve (B′) at the point P′; hence (III) the curves (A′) and (B′) are reciprocal polars with respect to the conic section (C′), and we have the following theorem:—

If a plane be drawn in any direction, cutting two conical surfaces which are polar reciprocals in relation to a given cone of the second degree, the curves in which it cuts the two reciprocal surfaces will be polar reciprocals with respect to the curve in which it cuts the director.

Hence it is evident that *if any two curves in the same plane be polar reciprocals with respect to a conic section situated in their plane, their central projections on any other plane will be polar reciprocals in relation to the projection of the conic section on that plane.*

XXXIX.

To find the equation of the conical surface which is reciprocal to a given conical surface with respect to a given cone of the second degree.

Let $x'y'z'$ be any point on the given conical surface, and $x_1y_1z_1$ the corresponding point on the other, the equation of the director being

$$Ax^2+A'y^2+A''z^2+2Byz+2B'zx+2B''xy=0.$$

The equation of the tangent plane applied at the former point is

$$z=\frac{dz'}{dx'}.x+\frac{dz'}{dy'}.y,$$

and the polar plane of the latter is

$$(Ax_1+B''y_1+B'z_1)x+(B''x_1+A'y_1+Bz_1)y+(B'x_1+By_1+A''z_1)z=0;$$

and since these equations belong to the same plane (XXVII), we obtain

$$\frac{dz'}{dx'}+\frac{Ax_1+B''y_1+B'z_1}{B'x_1+By_1+A''z_1}=0,\ \frac{dz'}{dy'}+\frac{B''x_1+A'y_1+Bz_1}{B'x_1+By_1+A''z_1}=0\ldots(31).$$

The equation of the reciprocal cone will be obtained by eliminating $\frac{x'}{z'}$, and $\frac{y'}{z'}$, from the equations of the given cone, by means of these equations.

Ex. To find the reciprocal polar of any cone of the second degree, in relation to a given cone of the second degree taken as director.

Let the equation of the director referred to any system of conjugate diameters be

$$ax^2+by^2+cz^2=0,$$

and the equation of the given surface referred to the same axes

$$Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy=0\ldots\ldots\ldots\ldots(32).$$

If $x'y'z'$ be any point on the latter, the equation of the tangent plane applied at that point will be

$$(Ax'+Fy'+Ez')x+(Fx'+By'+Dz')y+(Ex'+Dy'+Cz')z=0,$$

and if $x_1y_1z_1$ be the corresponding point of the required surface, the equation of its polar plane is

$$ax_1x+by_1y+cz_1z=0;$$

and as this is identical with the tangent plane, we shall have

$$cz_1(Ax'+Fy'+Ez')=ax_1(Ex'+Dy'+Cz'),$$
$$cz_1(Fx'+By'+Dz')=by_1(Ex'+Dy'+Cz').$$

Comparing these with the formulas in No. XXIII, we deduce

$$\frac{x'}{z'}=\frac{(D^2-BC)ax_1+(FC-DE)by_1+(BE-DF)cz_1}{(BE-DF)ax_1+(AD-EF)by_1+(F^2-AB)cz_1},$$
$$\frac{y'}{z'}=\frac{(CF-DE)ax_1+(E^2-AC)by_1+(AD-EF)cz_1}{(BE-DF)ax_1+(AD-EF)by_1+(F^2-AB)cz_1};$$

but since the polar plane of $x_1y_1z_1$ passes through the point $x'y'z'$, we have

$$ax_1x'+by_1y'+cz_1z'=0,$$

hence by eliminating x', y', z', we obtain

$$(D^2-BC)a^2x_1^2+(E^2-AC)b^2y_1^2+(F^2-AB)c^2z_1^2$$
$$+2(AD-EF)bcy_1z_1+2(BE-FD)caz_1x_1+2(CF-DE)abx_1y_1=0\ldots(33),$$

which is the equation of the reciprocal cone.

The equation $D^2-BC=0$ expresses the condition that any section of the

given cone parallel to the plane of yz may be a parabola, or that the plane of yz may touch the given cone; and the same equation expresses the condition that the axis of x may lie on the surface of the reciprocal cone; hence we have the following theorem :—

If any tangent plane be drawn to the given cone, the diameter of the director which is conjugate to it will lie on the surface of the reciprocal cone.

If the equations of the polar lines of the planes of yz, zx, xy with respect to the given cone (32) be respectively

$$y=\beta_1 x,\ z=\gamma_1 x;\ z=\gamma_2 y,\ x=\alpha_2 y;\ x=\alpha_3 z,\ y=\beta_3 z;$$

it is evident from the formulas in No. XXIII that the equation of the reciprocal cone may be written in the form

$$\begin{aligned}&(D^2-BC).ax.(ax+b\beta_1 y+c\gamma_1 z)\\ &+(E^2-AC).by.(a\alpha_2 x+by+c\gamma_2 z)\\ &+(F^2-AB).cz.(a\alpha_3 x+b\beta_3 y+cz)=0 \dots\dots\dots(33').\end{aligned}$$

When $\gamma=0$ we have $\alpha_3=0$, and the term involving xz disappears from this equation; hence when the polar line of the plane of yz lies in the plane of xy, or the polar line of the plane of xy in the plane of yz, the axes of z and x are conjugate diameters of the reciprocal cone.

If $\beta_1=0$ and $\gamma_1=0$, we have $\alpha_2=0$ and $\alpha_3=0$, and the terms containing xy and zx vanish from equation (33′); hence when the polar line of the plane yz in relation to the given cone (32) coincides with the axis of x, that axis is the diameter of the reciprocal cone which is conjugate to the plane of yz.

When the reciprocal cone passes through the axis of x the first term of equation (33′) vanishes, and therefore the reciprocal cone passes through four of the intersections of the four planes $y=0$, $z=0$,

$$a\alpha_2 x+by+c\gamma_2 z=0,\ a\alpha_3 x+b\beta_3 y+cz=0,$$

which are evidently the polar planes of the axes of y and z and of the straight lines $z=\gamma_2 y$, $x=\alpha_2 y$, and $x=\alpha_3 z$, $y=\beta_3 z$, with respect to the director.

XL.

It will be observed that, in order to simplify the formulas, the director has been supposed, in many cases, to be a sphere, with radius unity, and having its centre at the origin of coordinates. It is evident, however, that this does not diminish the generality of the formulas; for, by comparing equations (3) and (10), we see that the equation of the polar reciprocal surface, referred to any axes, and taken with respect to any surface of the second degree as director, will be obtained by substituting

$$\frac{Ax_1+B''y_1+B'z_1+C}{Cx_1+C'y_1+C''z_1+E},\ \frac{B''x_1+A'y_1+Bz_1+C'}{Cx_1+C'y_1+C''z_1+E},\ \frac{B'x_1+By_1+A''z_1+C''}{Cx_1+C'y_1+C''z_1+E},$$

for $-x_1$, $-y_1$. $-z_1$, respectively, in the equation of the sphero-polar reciprocal surface.

It may also be observed that if a sphere be described having its centre at the common vertex of two reciprocal cones, the curves in which it cuts those cones will be polar reciprocals in relation to the spherical conic in which it cuts the director: so that the whole theory of polar reciprocal curves in a plane may be extended to curves traced on the surface of a sphere. As this remark appears to be altogether new, I shall probably investigate the subject at some length in a subsequent part of this paper.

EQUATION OF A CONIC SECTION THROUGH FIVE GIVEN POINTS.

[*The Rev. Thomas Gaskin, M.A.**]

In your edition of Hutton's Course you state that you have not seen any method by which the constants are actually determined in the equation to a conic section passing through five points any how situated. I enclose for your inspection the method I have adopted; and I shall be glad if you find it satisfactory, and that you deem it to fulfil the conditions you have in view.

Let x_1y_1, x_2y_2, *etc.*, be the coordinates of the five given points, P_1, P_2, *etc.*, referred to oblique coordinates which are inclined at an angle ω; and let the angles P_1AX, P_2AX, *etc.*, be denoted by θ_1, θ_2, *etc.*, and AP_1, AP_2, *etc.*, by r_1, r_2, *etc.* Then,

$$x_1 = \frac{r_1 \sin(\omega - \theta_1)}{\sin \omega}; \quad y_1 = \frac{r_1 \sin \theta_1}{\sin \omega};$$

and similarly for x_2, y_2, *etc.* Whence

$$y_2x_1 - y_1x_2 = \frac{r_1r_2\{\sin\theta_2 \sin(\omega-\theta_1) - \sin\theta_1 \sin(\omega-\theta_2)\}}{\sin^2\omega}$$

$$= \frac{r_1r_2 \sin\omega \sin(\theta_2-\theta_1)}{\sin^2\omega} = \frac{2\Delta AP_1P_2}{\sin\omega}.$$

Similar relations hold good for the other pairs of coordinates taken in couples.

Now the equation to the line P_1P_2 is

$$u_{1,2} = (x_2-x_1)y - (y_2-y_1)x + (y_2x_1 - x_2y_1) = 0 \ldots\ldots\ldots\ldots (a),$$

and if $u_{1,2,5}$ be the value of $u_{1,2}$ when x_5, y_5 (the coordinates of P_5) are substituted for x and y, we have

$$u_{1,2,5} = (y_5x_2 - x_5y_2) - (y_5x_1 - x_5y_1) + (y_2x_1 - x_2y_1) \ldots\ldots\ldots (b)$$

$$= \frac{2}{\sin\omega}\left\{\Delta AP_2P_5 - \Delta AP_1P_5 + \Delta AP_1P_2\right\}$$

$$= \frac{2}{\sin\omega}.\ \Delta P_1P_2P_5.$$

These expressions are true for all positions of P_1, P_2, P_5; in whatever curve they may be situated; and, this premised, the problem is solved in the following manner.

Let $u_{1,2} = 0$, $u_{2,3} = 0$, $u_{3,4} = 0$, $u_{4,1} = 0$ be the equations to P_1P_2, P_2P_3, P_3P_4, P_4P_1, each of which with suitable changes is of the form denoted by (a). Then the equation to the conic section is

$$u_{1,2}\, u_{3,4} + \lambda u_{2,3}\, u_{4,1} = 0;$$

where λ is constant.

Since the conic section passes through the point x_5y_5, we have

$$u_{1,2,5}\, u_{3,4,5} + \lambda u_{2,3,5}\, u_{4,1,5} = 0;$$

where $u_{1,2,5}\, u_{3,4,5}$, *etc.*, are the values of $u_{1,2}$, $u_{3,4}$, *etc.*, when x_5y_5 are substituted for x,y. Hence,

$$u_{2,3,5}\, u_{4,1,5}\, u_{1,2}\, u_{3,4} - u_{1,2,5}\, u_{3,4,5}\, u_{2,3}\, u_{4,1} = 0 \ldots\ldots\ldots\ldots (1)$$

is the equation required; since by (b) $u_{2,3,5}$, $u_{4,1,5}$ .. may be expressed in terms of the coordinates of the given points.

* Extracted from a letter to Mr. Davies.

Cor. 1. When the distances between the given points and the equations to the straight lines joining the points P_1P_2, P_2P_3, P_3P_4, P_4P_1 are known, the coefficients of the final equation may be readily obtained; for

$$u_{2,3,5} = \frac{2}{\sin\omega}\,\Delta P_2P_3P_5, \textit{etc.},\text{ and hence the equation becomes}$$

$$\Delta P_2P_3P_5 \cdot \Delta P_1P_4P_5 \cdot u_{1,2} \cdot u_{3,4} - \Delta P_1P_2P_5 \cdot \Delta P_3P_4P_5 \cdot u_{2,3} \cdot u_{4,1} = 0 \ldots\ldots (2).$$

Wherefore, since the areas of the triangle are expressible in terms of the distances between the angular points, the equation of the conic section is easly written down at length.

Cor. 2. If three of the points be in one line, let them be considered as corresponding with P_1P_2 and P_5, then $\Delta P_1P_2P_5 = 0$, and equation (2) becomes reduced to a single term. Whence in this case the expression is reduced to an equation which represents the two straight lines joining the points P_1P_2, and P_3P_4 respectively.

Cor. 3. If the conic section be required to pass through the five points $(1, -1;\ 2, 1;\ -2, 3;\ 3, 2;\ -1, -3)$ then the curve is readily found by the preceding process to be

$$61y^2 - 17xy - 65x^2 + 36y + 174x - 151 = 0;$$

which is a solution of one of the last Senate House problems.

Cor. 4. Since the areas of the four triangles $P_2P_3P_5$ are constant, and alike independent of the origin of coordinates and of the angle of ordination, the constant multiplier λ will be the same, however the axes be assumed; provided the equations of the lines $u_{1,2} = 0$, $u_{3,4} = 0$ be put under the proper forms

$$u_{1,2} = (x_2 - x_1)y - (y_2 - y_1)x + y_2x_1 - x_2y_1 = 0, \textit{etc.}$$

Cor. 5. If P_6 be any other point in the conic section, it is manifest that

$$\frac{P_1P_2P_5 \cdot P_3P_4P_5}{P_2P_3P_5 \cdot P_4P_1P_5} = \frac{P_1P_2P_6 \cdot P_3P_4P_6}{P_2P_3P_6 \cdot P_4P_1P_6}.$$

Cambridge, 18*th March*, 1848.

ON A CERTAIN INTEGRAL TRANSFORMATION.

[*The Rev. Brice Bronwin.*]

The transformation here given is one that has been given by Mr. Boole, in No. 17 of the Cambridge Mathematical Journal, first series. But I have obtained four different results instead of the one which he has found. Moreover my method is more simple and easy than his, and the function $\phi(x)$ is more general. In the theorem

$$x^r\phi(x) = \phi(E)x^r \ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$E = 1 + \Delta$ operates on r thus: $\Delta r = 1$, $Er = r + 1$, $E^m r = r + m$, and therefore $E^m x^r = x^{r+m}$, where m may be integer or fractional, positive or negative. Give to m any values and any number of them; multiply the several results by any constants, and take their sum, and we have the theorem (a).

Operating with $\left(\frac{d}{dx}\right)^i$ on each member of (a), we have

$$\left(\frac{d}{dx}\right)^i x^r\phi(x)=\phi(\text{E})\left(\frac{d}{dx}\right)^i x^r=\phi(\text{E})r(r-1)\ldots.(r-i+1)x^{r-i}$$

$$=\phi(\text{E})\frac{\Gamma(r+1)}{\Gamma(r-i+1)}x^{r-i} \text{ and } \int x^{p-1}\,dx(1-x)^{q-1}\left(\frac{d}{dx}\right)^i x^r\phi(x)$$

$$=\phi(\text{E})\frac{\Gamma(r+1)}{\Gamma(r-i+1)}\int x^{r+p-i-1}dx(1-x)^{q-1}$$

$$=\Gamma(q)\phi(\text{E})\frac{\Gamma(r+1)\Gamma(r+p-i)}{\Gamma(r-i+1)\Gamma(r+p+q-i)}\ldots\ldots\ldots\ldots\ldots\ldots(b),$$

the integrals being taken from $x=0$ to $x=1$, which are the limits throughout this paper.

Make $q=i-p+1$, and the second member of (b) becomes

$$\Gamma(i-p+1)\phi(\text{E})\frac{\Gamma(r+p-i)}{\Gamma(r-i+1)}=\frac{\Gamma(i-p+1)}{\Gamma(i-p)}\phi(\text{E})\frac{\Gamma(r+p-i)\Gamma(1-p)}{\Gamma(r-i+1)}$$

$$=\frac{\Gamma(i-p+1)}{\Gamma(1-p)}\phi(\text{E})\int x^{r+p-i-1}\,dx(1-x)^{-p}$$

$$=\frac{\Gamma(i-p+1)}{\Gamma(1-p)}\int x^{r+p-i-1}dx(1-x)^{-p}\phi(x),$$

the last step being made, as will be easily seen, by (a). Therefore

$$\int x^{r+p-i-1}dx(1-x)^{-p}\phi(x)=\frac{\Gamma(1-p)}{\Gamma(i-p+1)}\int x^{p-1}\,dx(1-x)^{i-p}\left(\frac{d}{dx}\right)^i x^r\phi(x).$$

In this change p into $1-p$, and afterwards r into $r+p+i-1$; and we have

$$\int x^{r-1}\,dx(1-x)^{p-1}\phi(x)=\frac{\Gamma(p)}{\Gamma(i+p)}\int x^{-p}dx(1-x)^{i+p-1}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x)$$
$$\ldots\ldots\ldots(1).$$

Make $p=1$, and the second member of (b) becomes

$$\Gamma(q)\phi(\text{E})\frac{\Gamma(r+1)}{\Gamma(r+q-i+1)}=\frac{\Gamma(q)}{\Gamma(q-i)}\phi(\text{E})\frac{\Gamma(r+1)\Gamma(q-i)}{\Gamma(r+q-i+1)}=$$

$$\frac{\Gamma(q)}{\Gamma(q-i)}\phi(\text{E})\int x^r dx(1-x)^{q-i-1}=\frac{\Gamma(q)}{\Gamma(q-i)}\int x^r dx(1-x)^{q-i-1}\phi(x) \text{ by } (a).$$

$$\text{Therefore}\int x^r dx(1-x)^{q-i-1}\phi(x)=\frac{\Gamma(q-i)}{\Gamma(q)}\int dx(1-x)^{q-1}\left(\frac{d}{dx}\right)^i x^r\phi(x).$$

In this change r into $r-1$, and q into $p+1$; and there results

$$\int x^{r-1}dx(1-x)^{p-1}\phi(x)=\frac{\Gamma(p)}{\Gamma(p+i)}\int dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{r-1}\phi(x)\ldots(2).$$

It is obvious that x must not go out of (b) by the operation $\left(\frac{d}{dx}\right)^i$ performed on it, and that the elements of the integrals must not become infinite, except of a power less than unity.

Now suppose $\phi(x)=ax^m+a_1x^{m_1}+\ldots m$ being the least exponent. Put x^m for $\phi(x)$ in (1) and (2), and there result

$$\int x^{m+r-1}dx(1-x)^{p-1}=\frac{\Gamma(p)\Gamma(m+r+p+i)}{\Gamma(p+i)\Gamma(m+r+p)}\int x^{m+r-1}dx(1-x)^{p+i-1}$$

$$\int x^{m+r-1}dx(1-x)^{p-1}=\frac{\Gamma(p)\Gamma(m+r)}{\Gamma(p+i)\Gamma(m+r-i)}\int x^{m+r-i-1}dx(1-x)^{p+i-1}.$$

These follow from the common formulæ of reduction, and are therefore a verification of the results obtained in (1) and (2). Now we know that we must have $p>0$, $m+r>0$, and $m>-r$.

In (2) change $\phi(x)$ into $x^k\left(\frac{d}{dx}\right)^n x^m\phi(x)$, and we have

$$\int x^{r+k-1}d(1-x)^{p-1}\left(\frac{d}{dx}\right)^n x^m\phi(x)$$

$$=\frac{\Gamma(p)}{\Gamma(p+i)}\int dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{r+k-1}\left(\frac{d}{dx}\right)^n x^m\phi(x).$$

In this change p into $p+i$, k into $1-r-p$, n into i, i into n, and m into $r+p+i-1$; and it becomes

$$\int x^{-p}dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x)$$

$$=\frac{\Gamma(p+i)}{\Gamma(p+i+n)}\int dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{-p}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x).$$

The first member of the last is the second member of (1), and therefore by substitution, there results

$$\int x^{r-1}dx(1-x)^{p-1}\phi(x)$$

$$=\frac{\Gamma(p)}{\Gamma(p+i+n)}\int dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{-p}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x)\,..\,(3),$$

which is the theorem before alluded to, given by Mr. Boole, or equivalent to it; and it may doubtless be verified in the same way in which we have verified (1) and (2). By making $\phi(x)=x^m$, we see that x must be greater than $-r$, and m is supposed to be the least exponent contained in the series which represents the value of $\phi(x)$. It will not, I presume, be necessary to repeat this observation in what follows.

In (1) change $\phi(x)$ into $x^k\left(\frac{d}{dx}\right)^n x^m\phi(x)$, and it gives

$$\int x^{r+k-1}dx(1-x)^{p-1}\left(\frac{d}{dx}\right)^n x^m\phi(x)$$

$$=\frac{\Gamma(p)}{\Gamma(p+i)}\int x^{-p}dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{r+p+k+i-1}\left(\frac{d}{dx}\right)^n x^m\phi(x).$$

In this if we change k into $1-r$, p into $p+i$, n into i, i into n, and m into $r-1$; it becomes

$$\int dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{r-1}\phi x$$

$$=\frac{\Gamma(p+i)}{\Gamma(p+i+n)}\int x^{-p-i}dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{p+i+n}\left(\frac{d}{dx}\right)^i x^{r-1}\phi(x).$$

The first member of this is the second member of (2), and therefore by substitution

$$\int x^{r-1}dx(1-x)^{p-1}\phi(x)$$

$$=\frac{\Gamma(p)}{\Gamma(p+i+n)}\int x^{-p-i}dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{p+i+n}\left(\frac{d}{dx}\right)^i x^{r-1}\phi(n) \ldots (4).$$

We may derive another from the same from which the last was derived thus; change p into $p+i$, k into $1-r-p$, n into i, i into n, and m into $r+p+i-1$, and we find

$$\int x^{-p}dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x)$$

$$=\frac{\Gamma(p+i)}{\Gamma(p+i+n)}\int x^{-p-i}dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{i+n}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x).$$

Here the first member is the second of (1), and therefore by equality

$$\int x^{r-1}dx(1-x)^{p-1}\phi(x)=$$

$$\frac{\Gamma(p)}{\Gamma(p+i+n)}\int x^{-p-i}dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{i+n}\left(\frac{d}{dx}\right)^i x^{r+p+i-1}\phi(x) \ldots (5).$$

We may observe that (3), (4), and (5), might have been found directly, or in the same manner as (1) and (2) were found; and in this way I have found in addition to the preceding

$$\int x^{r-1}dx(1-x)^{p-1}\phi(x)$$

$$=\frac{\Gamma(p)}{\Gamma(p+i+n)}\int x^{-i}dx(1-x)^{p+i+n-1}\left(\frac{d}{dx}\right)^n x^{i+n}\left(\frac{d}{dx}\right)^i x^{r-n-1}\phi(x) \ldots (6)$$

We may now particularize some of these theorems a little.

In (2) make $p=1$, then $r=1$, and also $p=1$, $r=1$; and we have the following remarkable formulæ.

$$\int x^{r-1}dx\,\phi(x)=\frac{1}{\Gamma(i+1)}\int dx(1-x)^i\left(\frac{d}{dx}\right)^i x^{r-1}\phi(x);$$

$$\int dx(1-x)^{p-1}\phi(x)=\frac{\Gamma(p)}{\Gamma(p+i)}\int dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i\phi(x);$$

$$\int dx\phi(x)=\frac{1}{\Gamma(i+1)}\int dx(1-x)^i\left(\frac{d}{dx}\right)^i\phi(x).$$

From (1) by the same steps, we find

$$\int x^{r-1}dx\phi(x)=\frac{1}{\Gamma(i+1)}\int\frac{dx}{x}(1-x)^i\left(\frac{d}{dx}\right)^i x^{r+i}\phi(x);$$

$$\int dx(1-x)^{p-1}\phi(x)=\frac{\Gamma(p)}{\Gamma(p+i)}\int x^{-p}dx(1-x)^{p+i-1}\left(\frac{d}{dx}\right)^i x^{p+i}\phi(x);$$

$$\int dx\,\phi(x)=\frac{1}{\Gamma(i+1)}\int\frac{dx}{x}(1-x)^i\left(\frac{d}{dx}\right)^i x^{i+1}\phi(x).$$

The four following, of which the second and fourth are very singular, are derived from (3) and (4) by making $p=1$, and then $n=-i$.

$$\int x^{r-1}dx\,\phi(x)=\frac{1}{\Gamma(i+n+1)}\int dx(1-x)^{i+n}\left(\frac{d}{dx}\right)^n x^{-1}\left(\frac{d}{dx}\right)^i x^{r+i}\phi(x);$$

$$\int x^{r-1}\,dx\phi(x)=\int dx\left(\frac{d}{dx}\right)^{-i}x^{-1}\left(\frac{d}{dx}\right)^i x^{r+i}\phi(x)\,;$$

$$\int x^{r-1}dx\phi(x)=\frac{1}{\Gamma(i+n+1)}\int x^{-i-1}dx(1-x)^{i+n}\left(\frac{d}{dx}\right)^n x^{i+n+1}\left(\frac{d}{dx}\right)^i x^{r-1}\phi(x);$$

$$\int x^{r-1}dx\phi(x)=\int x^{-i-1}dx\left(\frac{d}{dx}\right)^{-i}x\left(\frac{d}{dx}\right)^i x^{r-1}\phi(x).$$

In the second and fourth of these make $r=1$. Then

$$\int dx\,\phi(x)=\int dx\left(\frac{d}{dx}\right)^{-i}x^{-1}\left(\frac{d}{dx}\right)^i x^{i+1}\phi(x),$$

$$\int dx\,\phi(x)=\int x^{-i-1}dx\left(\frac{d}{dx}\right)^{-i}x\left(\frac{d}{dx}\right)^i\phi(x).$$

I am unwilling to multiply these particular formulæ, but there are two others from (5) and (6) which should be exhibited with the preceding; they are

$$\int dx\phi(x)=\frac{1}{\Gamma(i+n+1)}\int x^{-i-1}dx(1-x)^{i+n}\left(\frac{d}{dx}\right)^n x^{i+n}\left(\frac{d}{dx}\right)^i x^{i+1}\phi(x),$$

$$\int dx\phi(x)=\frac{1}{\Gamma(i+n+1)}\int x^{-i}dx(1-x)^{i+n}\left(\frac{d}{dx}\right)^n x^{i+n}\left(\frac{d}{dx}\right)^i x^{-n}\phi(x).$$

We might transform these theorems by making $x=\cos v$, and $x=\cos^2 v$, and $x=\sin v$, $x=\sin^2 v$; but if the differential symbols $\left(\frac{d}{dx}\right)^n$, $\left(\frac{d}{dx}\right)^i$ were transformed into functions of $\frac{d}{dv}$, their second members would contain a series of terms, even when n and i are integers. In many cases, however, they might be reduced to finite, simple, and useful forms.

We have remarked that (3), (4), and (5) might be found directly in the the same manner as (1) and (2) were found.

Thus, taking $\left(\frac{d}{dx}\right)^i x^r\phi(x)=\phi(\mathrm{E})\frac{\Gamma(r+1)}{\Gamma(r-i+1)}x^{r-i}$, which was found in the investigation of (6); we derive from it

$$\left(\frac{d}{dx}\right)^n x^k\left(\frac{d}{dx}\right)^i x^r\phi(x)=\phi(\mathrm{E})\frac{\Gamma(r+1)}{\Gamma(r-i+1)}\left(\frac{d}{dx}\right)^n x^{r+k-i}$$

$$=\phi(\mathrm{E})\frac{\Gamma(r+1)\Gamma(r+k-i+1)}{\Gamma(r-i+1)\Gamma(r+k-i-n+1)}x^{r+k-i-n}.$$ And this will give

$$\int x^m dx(1-x)^p\left(\frac{d}{dx}\right)^n x^k\left(\frac{d}{dx}\right)^i x^r\phi(x)=\phi(\mathrm{E})f\int x^{r+k+m-i-n}dx(1-x)^p$$

$$=\Gamma(d+1)\phi(\mathrm{E})\frac{\Gamma(r+1)\Gamma(r+k-i+1)\Gamma(r+k+m-i-n+1)}{\Gamma(r-i+1)\Gamma(r+k-i-n+1)\Gamma(r+k+m+p-i-n+2)},$$

where f has been put for the suppressed factorials to abridge. We must here find k, m, p so as to reduce the six factorials to three, and so that those three may express a definite integral of the form of the first member of (1), (2), *etc.* But we must take care not to make k, m, *etc.* functions of r, because it is operated upon by E. As many ways as we can do this, so many theorems we shall find.

We might extend the subject to integrals containing $\left(\frac{d}{dx}\right)^m$, $\left(\frac{d}{dx}\right)^n$, $\left(\frac{d}{dx}\right)^i$, or even more of these operating factors. But I shall conclude this paper by observing, that I have elsewhere applied the formula (a) and other kindred ones to the transformation of definite integrals in various ways with great facility, and think these formulæ very worthy of attention.

Note. In my last paper, No. 1, vol. 3, page 12, of the Mathematician, the values of T, T_1, *etc.*, are not sufficiently complete to show their form. They should have one or two terms containing $\phi^{(2)}$, $\phi^{(3)}$, *etc.*, or the second and third differential coefficients of ϕ. But these terms are easily found by the theorem (a).

Gunthwaite Hall, Dec. 18*th*, 1847.

HISTORICAL NOTICES RESPECTING AN ANCIENT PROBLEM.

[*Mr. Davies.*]

(Continued from page 87.)

One or two particulars were overlooked in the former paper which properly belong to the section of these notes, already printed: but the oversight will be here repaired.

In speaking of the trigonometrical solutions, I did not consider it necessary to quote the elementary books into which any of those solutions might have been copied; and on that account, did not refer to Meyer Hirsch's Geometry,—a mere collection of exercises, effected, for the most part, either by algebra or trigonometry. At the close of the volume, he gives Lagrange's solution of the triangle, and Carnot's of the polygon, very briefly. I mention this work more, however, for the purpose of shewing how easily historical errors are perpetuated, and thereby injustice often done.

Hirsch's work bears the date " Berlin, Jan. 15, 1805," (I quote from the English Translation). Lhuilier's paper had appeared in the *Berlin Mémoires* nine years before; yet he makes no reference to that paper, though he does to those of Castillon and Lagrange, which anteceded it by twenty years. To say the least, this is indicative of extraordinary negligence; and it would always be better to omit such historical notices where there exists too much indolence (for confined means of information cannot be pleaded by a geometer resident in *Berlin* of the *Berlin Mémoires*) to permit a moderately careful investigation of the facts.

Another instance of Hirsch's negligence may be quoted; viz. that he speaks of *three* solutions in the fourth volume of the Memoirs of the Italian Society, respectively given by *Al*tajano, Malfatti, and *Giordano.* Besides giving the first name wrong (in which he is not solitary though probably original), there is no third paper on the subject in the volume; and Giordano is one of the christian names of Ottajano! It is the fortune of the young

Neapolitan to be "pulled to pieces." He also gives a reference to a work by Romano on the application of algebra to geometry, printed at Venice in 1793 *(Metodo nuovo di applicare alla sintesi la soluzione analitica di qualunque problema geometrico)*, in which he says that a geometrical construction of Lagrange's formula is given.

Strange to say, a reference to this work will shew that Romano does not even profess to construct Lagrange's formulæ—nor yet any other *trigonometrical* equations whatever. The work is of some ability : but its object is very special and limited ; viz. to shew how to construct expressions of the first and second degrees involving one unknown, with complicated coefficients, the elements of which are the representatives of linear magnitudes, and only *integer powers* of these involved. His transformations are of the most familar kind (in the elementary algebraical works of our time at least, and not particularly *nuovo*, even then), and his constructions consist wholly of the most inartificial, but most laborious kind. Hirsch, indeed, speaks in the highest terms of the whole work : but if his familiarity with it were of the kind that could lead him to say that Romano had constructed, or even attempted to construct, Lagrange's formulæ, he must have allowed himself to somewhat overstep the duties of his mission as a public teacher.

Romano first copies Lagrange's solution verbatim. He then proceeds to shew that lines actually drawn in his own figure (agreeing with Lagrange's to the same extent) being represented by x, y, z, contain relations amongst them that may be expressed by equations, no one term in which is a trigonometrical function *in form*.* By a succession of transformations, he eliminates two of the unknowns, obtaining a final equation of the second degree for finding the remaining unknown. The linear factors of each term, however, rise to the *eighth* degree. By particular substitutions he reduces its coefficients to a form analogous to that of Lagrange : but in which neither the data nor quesitæ involved are the same as those in the solution by the trigonometrical method. This he calls the "analysis of the problem."

He then proceeds to the "synthesis," by which he means the same thing that we call the "construction of the problem." No one of his steps involves the least difficulty, or displays the most common-place degree of ingenuity ; and yet taken as whole, it has never been my lot to wade through a series of such unquestionable truisms, and yet finish with so indistinct an impression of the argument. It is very like the impression made upon the mind, by a long series of algebraic reductions, or of mere arithmetical computations. At the same time it may be said that a very moderate familiarity with geometrical construction would have enabled Romano to form his (so called) "synthesis" with fewer than half the steps he has taken in effecting it.† The reasoning in the celebrated 20th prop. of Huygens's Dioptrics is positive "child's play" in comparison with this synthesis of Romano.

It is due to Romano, however, to say that he does not propound this solution as superior to the geometrical ones given by Castillon, Ottajano, and Malfatti : but rather as a difficult problem which *could be resolved* by his own method—that method which it appears to be the main object of his work to develop. It is certainly the most difficult and intricate problem discussed in the entire volume. He says in language the import of which it is almost

* Romano considers it so *in reality :* the difference however is formal or real according to the theory of mathematical philosophy we may adopt. *Formal*, as I view it.

† The concluding part of Romano's remarks seems to imply his entertaining a similar suspicion, pp. 193—198.

impossible to mistake for a pun: "Non puossi però negare, che l'algebra non sia ammirabile, posciachè può dirsi che penetri l'impenetrabile."

Romano seems to have no distinct knowledge of the papers by Euler, Fuss, and Lexell; as he does not mention the two latter, and Euler's name (I think) only occurs in a quotation (preface, p. 11,) from the preliminary note of Ottajano.

Again, in the Journale de l'Ecole Polytechnique, there appeared a paper by Brianchon, which I shall more particularly analyse hereafter, as being in some respects remarkable. It is referred to here in connection with its historical negligence. The date of the 10th cahier of the work, on the title page, is November, 1810. He speaks of the three solutions of Ottajano, Malfatti, and *Giordano**, differing from Hirsch, only by being correct in the first letter of the sirname. He mentions Romano, too, with the same date as Hirsch; but in such a form that I was led to examine the *Petersburgh Mémoires* in search of it—in vain, of course. Brianchon, the geometer, is not however to be measured by Brianchon, the historian of geometry. His contributions to the progress of the *géométrie superiéure* of the modern French school are not few or of small value: but that very circumstance renders it more important to guard against his negligence on points like these.

However Brianchon has a more grave mistake. He says: "*Pappus* rapporte encore que les Grecs avaient inutilement essayé de résoudre ce problème plus général:"—the "more general problem" being that resolved by Castillon and Lagrange. That Pappus gives any authority for such an assertion has, I believe been discovered by Brianchon only. I shall be glad, at all events, if Pappus has said so, to be informed where.

One passage, however, strikingly illustrative of the state of geometry on the Continent, at, and antecedent to, the period at which Brianchon wrote, is worthy of being preserved here. Speaking of Castillon's solution, he says: "celui-ci même parvint à une construction d'autant plus remarquable, qu'elle est obtenue par *l'analyse géométrique* des anciens qui est *entièrement oubliée* depuis que toutes les branches des mathématiques sont soumises à *l'analyse algébrique*." p. 5.

It has been urged upon me by friends in whose judgment I have great confidence, that I ought to give some further account of Castillon's solution than a mere reference to an English work which "contains a solution very similar to it." I fear that it would, for want of space, be impossible to fully comply with those requests; and therefore shall "compromise the matter," by giving Castillon's analysis of the general case, and likewise a notice of the particular cases which he discusses. The geometrical reader will so readily infer the composition and demonstration from the

* A lady (a near relative) to whom I am greatly indebted for valuable assistance in this series of notices, suggests the following explanation.

"The name given to the young Neapolitan, at the head of his paper, is *Sig. D. Annibale Giordano di Ottajano:* and he is always mentioned by Romano by the name of *Giordano* only, without the *di Ottajano* affixed. It is, therefore, very likely that "Giordano" was the family name, and "Ottajano" the territorial appellation derived from his paternal *roccare* or villa. This accords, at least with Italian usage; and it is similarly of perpetual occurrence in the French also."

This is very probable: but it is no apology for those who speak of Giordano and Ottajano as different persons, and as having written separate papers on the same subject in the same volume of the Memoirs of the Italian Society. That those who have spoken of this difference of person, have been misled by Romano, is, I think, quite clear; but the fault is not that of the Venetian Engineer.

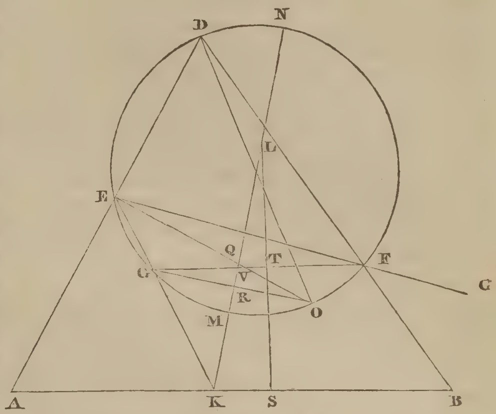
analysis, that I hope the middle course I have adopted will be satisfactory to my friends.

CASTILLON, *Berlin Mémoires*, 1776.

He begins with three lemmas which are familiarly known to English geometers, and which are involved in the steps of his process.

PROBLEM. *Three points*, A, B, C, *being given, to inscribe a triangle* DEF *in the given circle, so that its sides shall pass through the given points* A, B, C.

Analysis. Suppose it done, and let E, F be the points where AD, BD meet the circle; and join AB.



Draw FG parallel to AB; and join EG meeting AB in K.

Then, since FG is parallel to AB, we have DFG = DBA; and since the angles DFG and DEG are equal to two right angles, the angles DEK and DBK are equal to two right angles; and the points D,E,K,B, are in one circle, and hence also

BA . AK = DA . AE.

But DA.AE and AB are given, and hence the point K is given.

Through the centre L of the circle, draw KL cutting the circumference in M and N. Then since the point K and the circle are given, the segments NK, KM and the ratio NK : KM are given.

From G draw GO perpendicular to KL meeting it in R and the circle in O; and join EO intersecting KN in Q. Then

NQ : QM :: NK : KM (Pappus, vii. 156.)

But we have proved that NK : KM is given, and hence NQ : QM is also given; and hence again the point Q is given. Wherefore one of the legs EO, of the angle FEO, passes through a given point, Q; and the other, FE, through the given point, C.

From the centre L of the circle, draw LS perpendicular to AB: this is given in position. Also the line LK is given in position; and hence the angle KLS is given.

Let the line LS meet GF (which is parallel to AB) in T; then each of the angles LTG, LTF is a right angle.

Let V be the point in which FG meets LK. Then the triangles GRV, LTV have the angles GRV, LTV right angles; and the angles GVR, LVT equal; and hence VGR = VLT. Wherefore, we have,

QEC=QEF=OEF=OGF=RGV=VLT=KLS, a given angle.

Also, the points C and Q are given, and the angle QEC; and hence the segment on CQ to contain that angle, is given; and the intersections of this circle with the given one being given, the point E is given.

The construction given is that which immediately flows from this analysis; and the demonstration consists only in a reversal of the analysis. As is *always* the case (either immediately or virtually) respecting loci meeting in a point, Castillon is obliged to resort to the *ex absurdo* method. There is

also a slight oversight, but not very material, in one step of the demonstration.

The rest of the paper is devoted to a consideration of the different cases; of which the following is the substance.

(1). The different relative positions of the points A, B, C with respect to the circle and to each other, give rise to several diversities of case, which will require some modifications of the analysis and composition.

(2). When one of the points C, Q is within the circle and the other without, the construction is always possible, and two different triangles will fulfil the conditions.

(3). The same construction may be made for B, or for C, that was made with respect to A. The constructions for those cases are given; and it is *proved* that the resulting triangles are the same.

(4). A property of a point, k, employed in the demonstrations before referred to in (3); which might, possibly, have suggested one of Euler's theorems which will be noticed presently. The point k is in AB, and so situated that $Sk=SK$.

(5). The point K cannot coincide with S when AB cuts the circle, and S is between A and B.

(6). Case of K and S being coincident.

(7). Let LS cut the circle in c and d; then if $AS.SB=dS.Sc$, we shall have BA.AS equal to the square of the tangent drawn from A to the circle.

(8). *Euc.* iii. 36, deduced as a case of this: with some other properties which are now well known in connection with a quadrilateral inscribed in a circle.

(9). When K and S coincide, the *line* CQ cuts the circle in the required points E and F. Also the triangles are the same for C undergoing any change of position *in the line* defined by its first position.

(10). The case when AB passes through the centre of the given circle; with a variation of the construction adapted to a special case with respect to C.

(11). Notice of the "determinations" which are necessary under given circumstances. With respect to this, however, he says "Mais la détermination n'est bonne que lorsqu'elle épargne une partie considérable de la construction; et j'avoue que je n'en ai pas pu trouver de telle pour ce problème."

Lexell's attempt to construct Lagrange's formula was printed in the second part of the Petersburgh Acts for 1780: but in the first part there had appeared two *mémoires* by Euler and Fuss, of which Euler's stands foremost in the volume, and was therefore probably the first presented to the Academy. Both Fuss and Lexell were the disciples and ardent admirers of Euler; and hence all idea of rivalry is out of the question.

Though the geometry of Euler is so evidently that of a novice in the study, the paper bears testimony to the wonderful power of invention (even in a subject to which he had scarcely given common-place attention) which characterises all his writings. For instance, he considers that *Euc.* ii. 12, is a new property, and gives a demonstration of it; whilst on the other hand, he deduces one or two properties which are even yet, I believe, unremarked by any other writer, notwithstanding the subject to which they relate has passed through the hands of the ablest geometers.

He gives as a lemma, the property which constitutes the second of Stewart's General Theorems; and it is, indeed, for this circumstance that

M. Chasles* considers the paper to be principally remarkable. It was, doubtless, the first continental notice of the theorem. Euler employs it in demonstrating a property of the circle, which I shall give here, but with a different and more direct demonstration. All Euler's terms involve *three* factors of segments of a line, or else fractional forms. Neither of which is admissible in the strictness of the ancient geometry.

Let DEF *be a given circle, whose centre is* O, *and* A, B *be given points. Then,*

(1). *If any line* ADP *be drawn to cut the circle in* P, D *: then (however the point* P *be taken) if a circle* PDB *be described to cut* AB *in* K, *the point* K *is always the same. Similarly the circle* PEA (P, E, B *being in one line) will always cut* AB *in the same point* H.

This property is well known, and constantly used in the Apollonian problem of Tangencies.

(2). *From the centre* O *draw the perpendicular* OL *to* AB, *and take* LC *equal to the tangent drawn from* L *to the circle: then if from any point* A *whatever in* AB *a tangent be drawn to the circle, it will be equal to* AC.

This, also, is a familiar property.

(3). *The segment* KH *is bisected in* L.

For draw the tangents AF, BG, and join AC, BC. Then,

$BA.AK = AF^2 = AC^2 = AL^2 + LC^2$; and

$AB.BH = BG^2 = BC^2 = BL^2 + LC^2$; hence

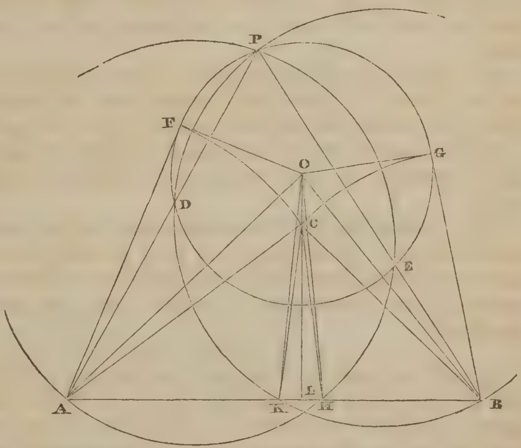
$AB(AK - BH) = AL^2 - LB^2 = AB(AL - LB)$; or

$AK - BH = AL - LB$, and

$AL - AK = BL - BH$;

that is, $KL = LH$;

which is one of Euler's properties.



(4). *The rectangle* KA.HB *is equal to the square of* KC *or* CH.

For, $AC^2 - CK^2 = AK^2 + 2AK.KL = AK^2 + AK.KH = AK.AH$.

But $AC^2 = AK.AB$; and hence

$$CK^2 = AK.AB - AK.AH = AK.BH.$$

This is the second of Euler's theorems.

(5). *The triangles* ACK, HCB *are equiangular.*

For, since $KC = CH$, the angles CKH, CHK are equal; and hence the angles CKA, CHB are equal. Also, since $AK.BH = CK^2 = CK.CH$, we have

$$AK : KC :: CH : HB.$$

* Aperçu Historique sur l'Origine et le Développement des Méthodes en Géométrie, p. 328.

Whence the sides about the equal angles at K and H being proportional, the triangles are equiangular; and therefore CAK = HCB, and ACK = CBH.

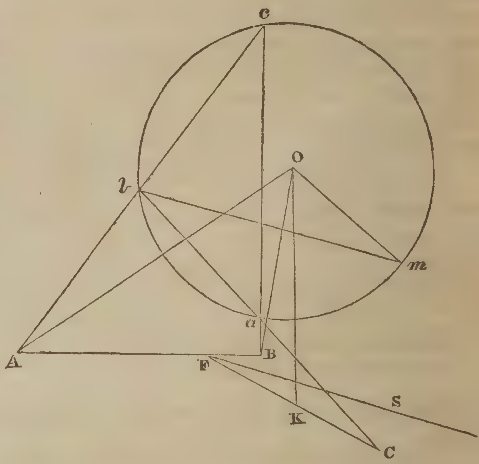
(6). *Join* AO, BO, KO, HO; *then we shall have*

$$AB.AK = AF^2 = AO^2 - OF^2, \textit{ etc.}; \text{ or, } AK = \frac{AO^2 - r^2}{AB};$$

in which form both Euler and Fuss use the property. It was the customary mode of the time, of writing geometrical equalities on the continent.

A considerable number of elegant properties of the circle may be deduced from these two theorems of Euler's; and some of the investigations respecting the pole and polar in respect to the circle may be advantageously modified. I have examined, with very little expenditure of time, several of them: but this is not the appropriate place to pursue the subject further.

The following is Euler's construction:—



Let O be the centre of the given circle, and A, B, C the three given points. Draw AO, BO, CO, which denote by a, b, c respectively; and put the radius of the circle equal to 1.

In BA take $BF = \frac{b^2 - 1}{AB}$:

then $FO^2 - 1 = \frac{BF(a^2 - 1)}{AB}$.

Join FC, upon which take

$$FK = \frac{FO^2 - 1}{FC} = \frac{BF(a^2 - 1)}{AB.FC};$$

and we shall have

$$KO^2 - 1 = \frac{FK(c^2 - 1)}{FC}.$$

Next, from the centre O draw the radius Om, so that $\cos KOm = \frac{\cos BFC}{KO}$; bisect the angle BFC by the line FS; and draw mb parallel to FS meeting the circle again in b. Then b is one of the angular points of the required triangle, and the rest follows.

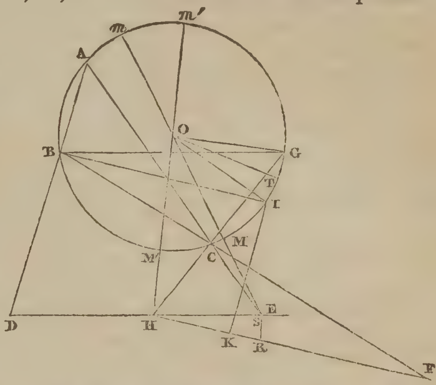
No one will call this " a geometrical construction of the problem;" and what is more, Euler offers no demonstration of it—obviously considering it to follow, as a matter of course, from the two theorems already quoted. As I have given my own demonstrations of those, it may be expected that I should do the same with respect to this. A strictly geometrical demonstration of a construction which lies beyond the recognised principles of geometry, is, of course, not to be attempted: but, as a proof is not difficult, and may form a useful " Exercise," I shall leave it for the consideration of the correspondents of that department of the Mathematician.

Fuss, 1780.

The great step made by Euler towards the *general* solution, consists in *joining* FC *and repeating the process by finding* K *similarly to the mode in which Castillon determined* F. In fact, there cannot be a mo-

ment's doubt that the solution of Euler suggested that of Fuss (in the same volume) in which the geometrical character of the construction is unquestionably pure. It bears traces also of Castillon's solution having been carefully studied. One remarkable circumstance, however, may be noticed: that what Fuss gives as "a demonstration of the construction" is in reality, except one single step, "the analysis of the problem." It is not a mere mistake of writing, but results from a total misconception of the essential character of the *ancient geometrical analysis*. I have, therefore, thought it better to give his processes their right names, and place them in their appropriate order, in this sketch: but I have made no other material alteration, except the omission of his construction for *Pappus*, vii. 117; which differs in no respect from that usually given in our elementary works. The subject will, however, be again referred to.

Analysis. Let ABC be the required triangle whose sides produced pass through the three given points D, E, F; from B draw the line BG parallel to DE meeting the circle in G; and from G draw GC to meet DE in H. Join HF, and draw BI parallel to it meeting the circle in I; and draw GI to meet HF in K.



Now we have by the circle and by parallels,

$$EHC=BGC=BAC:$$

and hence the triangles ECH, EDA are similar; and

$$EC : EH :: ED : EA, \text{ or}$$
$$EC.EA = ED.EH.$$

Draw EO (O being the centre of the circle) to meet the circle in M, m. Then,

$$EC.EA=EM.Em = (EO+OM)(EO - OM) = OE^2—OM^2.$$

Again, we have in a similar manner the angles HFC, IBC, IGC equal, and the triangles HFC, HGK similar. Whence

$$HC : HF :: HK : HG, \text{ or } HF.HK=HC.HG.$$

Wherefore, drawing the diameter $HM'm'$ through H, we obtain

$$HC.HG=HM'.Hm'=(HO-OM)(HO+OM)=HO^2—OM^2.$$

It follows from this, that the points H and K are given; since ED, EO, OM are given; and thence HF, HO, OM are also given. Moreover, the lines BG, BI being parallel to the given lines EH, HF, the angle GBI is given in magnitude; and therefore also the chord GI is given in magnitude. Whence it will be a tangent to a given circle concentric with the circle ABC; and it passes through a given point K. It is hence given both in magnitude and position; and the points I and G are given, and hence also the point B is given.

Construction. Join DE, and make $EH=\dfrac{EO^2—OM^2}{DE}$; then join HF, and make $HK=\dfrac{HO^2—OM^2}{HF}$. On HF take HR=MO, and draw RS perpendicular to DE. With radius HS describe a circle from centre O; and

draw the tangent KT to it from K meeting the given circle in I and G. From G draw GB parallel to DE, or from I draw IB parallel to HF: then B will be one angular point of the triangle; and BD, BF being drawn, we obtain the other two A and C.

A formal demonstration would be superfluous, as it is constituted by a mere reversal of the analysis; except, perhaps, as regards one simple step.

Let T be the point of contact of IG with the interior circle; and join IO, TO, GO. Then it is known that IOG is bisected by OT, and hence the angle IOT = IBG = SHR; and since HR = OI, we have HS=OT. We thus have the reason for this part of the construction.

Fuss then gives two corollaries respecting the cases where one, or where two of the points are *within* the circle; then two scholia respecting the cases dependent on which of the tangents to the interior circle drawn from K is used; and finally on the total number of cases that can exist.

Lexell, whose attempt to construct Lagrange's equations has already been mentioned, gave also what he considers a variation of Castillon's construction. It is very complex in its operations, and the demonstration is given partly by geometry and partly by his preceding trigonometrical reasoning. The problem, in short, did not progress in his hands; the recapitulation of his method would therefore be altogether uninteresting, and is hence omitted. He, however, gave a neat property of the inscribed quadrilateral, to which he affixed a proof partly geometrical and partly trigonometrical. Ottajano has given another, which had the merit of being strictly geometrical.*

The great obstacle to the extension of the method of construction given by Fuss turned upon the slight difference in the final part, which exists between the cases of the polygons being of odd and even numbers of sides. When the polygon is odd-sided the line which determines the principal point of the construction touches a determinable circle; and when the polygon is even-sided, the corresponding line is parallel to a determinable line. This difference is of perpetual occurrence in inquiries relating to polygons; and it is not less so in algebraic and trigonometrical functions, for odd and even values of the number denoting the order of the functions or the order of the terms. Even in some cases such circumstances depend on more complex relations than merely odd and even. It was, however, a complete obstacle to the extension of the method till Ottajano took up the subject.

Ottajano appears to have been a student in some college at Naples, and from his prefatory remarks, the only mathematics to which his attention as a student was directed, was "analysis;" whilst his own attachment was to the geometrical method of the ancients. At this period he was only sixteen years of age; and whether he lived to publish more than the one paper on which his celebrity rests, I am not sufficiently acquainted with Italian science to be able to say. This unexpected solution excited great attention; and Malfatti, Professor of Mathematics at Ferrara, was also induced to attempt the general solution before that of Ottajano was made public by the Italian Society, to which it had been sent. Malfatti's solution was sent to the Chevalier Lorgna, President of the Society, "to do with it as he judged best." The two solutions were decreed to be printed, and they appeared in the fourth volume of the *Memorie della Societa Italiana*, at the same time. There is not a question that the only suggestion made to Malfatti was, that it was possible to solve the general problem, and that such solution had been effected: but that he had either seen Ottajano's paper, or

* Memorie della Societa Italiana, tom iv.

been in the slightest degree informed of his method, there is not a shadow of a reason to believe. Yet the two investigations are so strictly identical in principle and method, that one could almost think Malfatti's was merely an arrangement of Ottajano's, in a form such as an experienced geometer would give to it (viewed as an inexperienced but clever boy's manuscript,) in preparing it for publication. The method, however, is little more than an extension of the process of Fuss, the cases of odd and even-sided polygons creating the greatest source of difficulty.

Very little notice has been taken of these solutions by English writers, and no one actually professes to give any idea of the method employed. Mr. Bonnycastle briefly alluded to them, in his geometry (p. 348 note); as also has Mr. Potts, in his Appendix to Euclid, (p. 97), correcting one or two mistakes (evidently typographical) in Bonnycastle's history. Doctor Trail, in his Life of Simson (p. 95), gives a somewhat more detailed description of the labours of Ottajano and Malfatti, for which he expresses himself to be indebted to Professor Playfair. Even this account is not strictly accurate in all its particulars; and he gives, besides, no description of the actual methods by which a reader could form the slightest notion of the processes pursued. It is, however, stated very truly by Playfair that Ottajano "appears to have had but a very imperfect notion of the ancient geometrical analysis:"—but he might further have said that the youthful Italian had so confused a notion of the ancient geometry that he absolutely mistakes a synthetic process for an analytical one. The state of geometrical knowledge on the continent at that time, may, however, be pleaded as a sufficient apology for so young a geometer committing this mistake.

In giving the researches of Ottajano and Malfatti, I have deemed it the better way to translate their enunciations, with a few occasional remarks upon them; and in order to make plain their processes, to give the "general solution" of the former and the solution of the latter for the polygons of four and of five sides. My own conviction is, that when the original language can be appropriately given, a better idea is formed by the reader of the character of any research than any paraphrase could afford, however carefully it should be made. Many cases may arise when neither the one nor the other is necessary—as in that of Lexell and Romano in this series—and when moreover, if given, they could not be of the least possible utility.

It will be remarked, that both these writers content themselves with giving merely the *analysis of the problem:* evidently considering that in the ancient geometry as well as in the algebraic, a problem was solved, when the analysis had been effected. It appears also quite to coincide with the general opinion on the continent at that time; and it was as little the custom to construct a problem so analysed, as it was to construct the equations which resulted from an algebraic investigation of the same problem.

OTTAJANO, 1784.

I. *In a given circle* NG *to inscribe a triangle* DEF, *whose sides shall pass through three given points* A, B, C, *situate in a straight line.*

Except as to form, the construction is identical with that of *Pappus* (*lib. vii.* 117.)

II. *In a given circle* NG *to inscribe a triangle* DEF *whose sides shall pass through three given points* A, B, C, *anyhow situated.*

The analysis differs in no essential point from that of Fuss: but the construction is not detailed.

III. *In a given circle to inscribe a triangle two of whose sides pass through two given points, and the remaining side makes a given angle with a line drawn from its extremity to a third given point.*

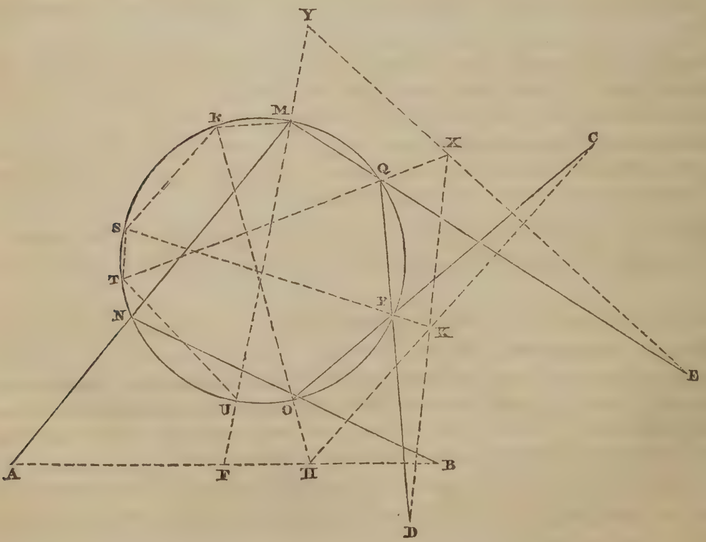
IV. *In a given circle to inscribe a triangle of which one side passes through a given point, and the other two make given angles with lines drawn from their extremities to two other given points.*

V. *In a given circle to inscribe a triangle, such that the lines drawn from two of the angular points to three given points, shall make with the sides given angles.*

These three problems (3, 4, 5) were, I think, proposed for the first time in the paper under consideration. The constructions given by Ottajano bespeak great sagacity and geometrical power; but they are very incomplete as to enumeration of the cases. A few of them are, however, taken into consideration, and others are briefly indicated in the scholia. I have not found any indication of this class of problems having been further pursued: and indeed, if extended to quadrilaterals, *etc.*, it would be difficult to enunciate and limit the problem; and the cases which would arise would be discouragingly numerous. It appears to me, however, to be an interesting task to carefully examine the cases, together with the *determinations* and *indeterminations* of the problem as regards the triangle and quadrilateral.

Ottajano then gives a demonstration of the theorem of Lexell, to which reference was made in a former page (p. 148), and which I intended to bring into these papers. On the whole, it has appeared better, however, to place it amongst the "Exercises"; where, of course, in due time the contributors to that department will have obtained independent demonstrations.

In a given circle to inscribe a rectilineal figure of any given number of sides, which being produced may pass through as many points, however situated.



SOLUTION.

Let the points A, B, C, D, *etc.*, be given, and likewise the circle SR: it is required to inscribe therein the polygon MNOPQ, *etc.*, whose sides, if produced, shall pass through the given points.

1. Join AB, and through M draw MR parallel to AB, and join ROH.
2. Join HC, and through R draw RS parallel to HC, and join SPK.
3. Join the points KD by means of the right line DKX, in the same manner, through S draw ST parallel to DKX and join TQX.
4. Join EXY, and through T draw TU parallel to it, and also join UMY.

And so on in the same manner if there be more given points.

Thus we shall have the angles MRS, RST, STU, *etc.*, of the polygon MRSTU, *etc.*, as also the angles AHK, HKX, KXY, *etc.*, respectively equal to them. If then the number of sides is even MR, RS, ST, TU, *etc.*, of this polygon (which is the case when the figure to be inscribed has an odd number of sides) containing given angles, the first side MR with the second RS, with the third ST, with the fourth TU, *etc.*, the whole arc MRSTU, *etc*, will be given, and hence UM which subtends it, and the point M, which is sufficient to determine all the others necessary to inscribe the figure required.

If the figure MNOPQ, *etc.*, to be inscribed have an even number of sides, then the number of right lines MR, RS, ST, TU, *etc.*, will be odd, and MU will not be the subtending line of a given arc; but the number of the right lines RS, ST, TU, *etc.*, being even, the arc RSTU, *etc.*, will be given, and hence the angle RMU; and producing YMU to F, the angle YFA will be given because of the parallels RM, AB, hence also in this case the point M will be determined. Q.E.D.

MALFATTI, 1784.

Malfatti commences by an enunciation of the general problem, and adds a a note on its history. He then gives the two following lemmas, which are respectively necessary in the construction of the odd-sided and even-sided polygons.

LEMMA I. *Through two given points to draw lines meeting in the circumference of a given circle, so that the line joining the other intersections with the circle shall be parallel to a given line.*

This is tantamount to inscribing a triangle in a given circle, two of whose sides pass through given points, and whose third side is parallel to a given line. It is too simple to need further remark.

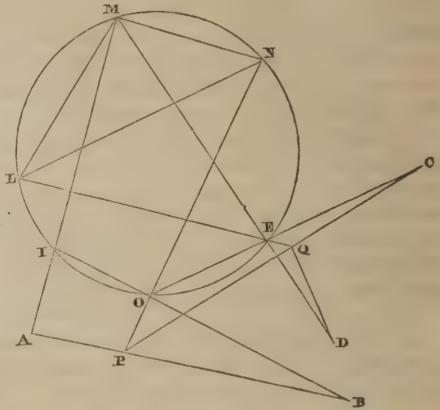
LEMMA II. *Through two given points to inflect lines to a point in the circumference of a given circle, so that they shall contain a given angle.*

I. *In a given circle, to inscribe a triangle, the sides of which shall pass through three given points.*

The analysis only is given, reducing the problem to *Lem. i.* Essentially the process is coincident with that of Ottajano's general solution adapted to this particular case: but it forms, as will be seen from the two following problems, a part of Malfatti's own system of investigation.

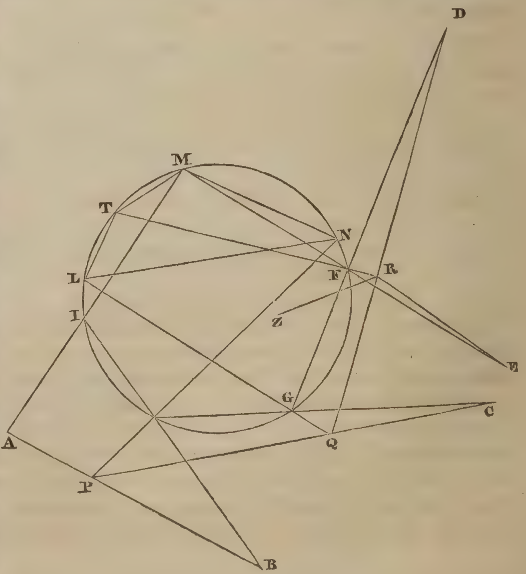
II. *Given a circle* LMN, *and four points* A, B, C, D, *to inscribe in the circle a quadrilateral* OIME, *so that the sides being produced shall meet the given points.*

In the construction of the preceding problem the two points A, B, are joined by the right line AB, and supposing it done, from M draw the chord MN parallel to AB, then from N through O draw the right line NOP, which meets AP in P. Following the same reasoning as Castillon, we shall have AB.BP = the square of the tangent which is drawn from B to the circle. P is then the point required. Then join P and C, and from N draw the chord NL parallel to PC, and join LE, which meets the line CP in Q. Whence we have the rectangle PCQ = to the square of the second tangent from C, and thus we get the point Q. Now draw from the fourth point D the right line DQ, and the angles MNL, MEL, which are subtended by the same arc ML, are equal. But since MN is parallel to BP, and NL to PC, therefore the angle MNL=BPC, which is manifest. The present problem becomes then reduced to *Lemma* ii: given two points Q, D, and a circle, to determine in the circumference a point E such that by joining DE, QE, the angle DEQ shall be given.



III. *Given a circle* LMNO, *and five points* A, B, C, D, E, *to inscribe in the circle a pentagon* MIOGF, *such that its sides, if necessary, being produced, shall meet the five given points.*

Join the points AB, and draw the chord MN parallel to AB, and from N through O the right line NOP. The rectangle ABP is = to the square of the tangent from B. Thus P is a given point. Join P to C, and from N draw the chord NL parallel to PC, and draw the secant LGQ, which meeting PC in Q, makes the rectangle PCQ equal to the square of the tangent from C; and Q is thus given. From Q to the fourth point D draw QD, and from L the chord LT parallel to QD, then from T through F the right line TF, which meets QD in R. We shall have the rectangle QDR equal to the square of the tangent from D, and the point R will be given. Lastly, to the fifth point E draw the right line RL, and the chord TM. In the quadrilateral inscribed in the



circle LTMN the sum of the angles LTM, MNL makes two right angles. But the angle MNL is given equal to BPC. Then there will also be given the angle LTM. Hence, since RD is parallel to LT, if we draw RZ which makes with RD the given angle LTM, to this TM must be drawn parallel, and the problem will be solved with the aid of Lemma i.; whilst the points R, E, being given, a point F must be found in the periphery of the circle such, that with the secants EFM, RET they comprehend a chord DM parallel to the given line, RZ.

General Scholium. This invariable method which we have hitherto practised, shews, that the number of given sides being odd, the solution of the problem depends on the first lemma, and being even, on the second: which with an easy change which we may make will also take place on the hypothesis that either all or part of the given points may be situated within the given circle. And by such a method this celebrated problem will be completely solved in all its generality.

In 1809, Lhuilier published his *Elemens d'Analyse Géométrique et Algébrique*, at Paris. In this work, besides the essential principles of his paper in the Berlin *Mémoires*, already described, he succinctly indicates very neat solutions of the two following problems, by purely geometrical considerations.

I. *In a circle to inscribe a triangle the sides of which are parallel to lines given in position, or, pass through given points.* (p. 277.)

This divides itself into four cases, each of which in succession Lhuilier reduces, by analysing the problem, into a dependence on the preceding one.

(1.) The three sides parallel to given lines:
(2.) Two sides parallel to given lines; the third through a given point:
(3.) One side parallel to a given line; the other two through given points:
(4.) All the sides through given points.

II. *In a given circle to inscribe a quadrilateral figure the sides of which are parallel to given lines or pass through given points.* (p. 282).

This divides itself into five cases, which like the former problem are successively analysed, and thereby reduced to their respectively antecedent ones,—except the second to the first.

(1.) The four sides parallel to given lines:
(2.) Three sides parallel to given lines; the fourth through a given point:
(3.) Two sides parallel to given lines; the other three through given points:
(4.) One side parallel to a given line; the other three through given points:
(5.) All the sides through given points.

The course taken by Lhuilier is remarkable for its neatness and concentration of conception; but it involves no new principle, and differs in nothing geometrically essential from the process of Fuss for the triangle, and Ottajano and Malfatti for the quadrilateral. It is due to the distinguished Genevese geometer to say that he has looked with some care into the *determinations* and into the *indeterminate* cases of the problem. This is a point which has been generally too much neglected; and Lhuiller has only partially succeded in the determination of these cases. He shews, for instance, that his first case of the quadrilateral is inevitably impossible or indeter-

minate; but he says nothing of a corresponding circumstance which occurs in his fifth case. Neither does he take notice of the determinations which are essential to several of the cases of the second problem:—for instance, in the fifth case, where all the given points are *within* the circle.

A new era in the continental history of the problem now opens to us, with a hint of Brianchon's, which was followed out by Gergonne and the eminent correspondents of his *Annales des Mathematiques.* During this discussion the system of "poles and polars" was reduced into a regular form, and the scattered properties of that remarkable line and point, which had been discovered (from Lahire downwards) were combined together into one uniform theory—a theory which, for geometrical power and beauty in discussing the conic sections, is without a rival. This, too, is equally true, whether we establish the theory through the medium of transversals or of coordinates. With Brianchon I propose to commence the next section of these "Historical Notices;" and in the final section to give the *English* History of this problem.

REMARK ON THE GENERAL EQUATION OF THE SECOND DEGREE.

[*Professor De Morgan, University College, London.*]

If two lines (intersecting in O) be drawn in the plane of a curve of the second order, and if the *central conjugate* of *each* line be drawn (on the diameter which bisects chords parallel to either line) and produced to meet *the other*, and if the two points of intersction be B and B′: then BB′ is parallel to the polar line of O. If the centre be at an infinite distance, a line parallel to the axis must be known. If O be on the curve this is an easy way of drawing a tangent at O, in every case.

This theorem may be, of course, demonstrated in various ways, one of which will presently appear. My present object is to point out its meaning, which is rather remarkable.

The general equations of lines of the second order.

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

is reduced to one of its defective forms by transformation of coordinates, which, particularly when the coordinates are oblique, is rather laborious. No meaning is usually given to the coefficients, or rather to their ratios. To treat them in terms of the coordinates of five points, to *any axes*, has always been of more trouble than profit: and to use three straight lines, without restriction, is forbidden; for the data of three straight lines involve *six* points. But the use of three straight lines with a relation between them may be made very effective: and the relation is contained in the preceding theorem.

Let there be axes of coordinates meeting at O, and inclined at any angle. Take any two points B and B′, one in each axis, and from these two points draw lines, the line drawn from the point in either axis being intended for the central conjugate of the other axis. Choose for the polar line to the origin O (not any line but) a line parallel to BB′; and one curve of the second order is determined, and only one. This is the converse of the preceding theorem.

Let B be on the axis of y, and B′ on that of x. Let the intersection of the axis of y and its central conjugate be A, and that of x, C. Let the polar line of the origin meet the axes of y and x in D and E. Accordingly, the equation of AB′ is $2ay + bx + d = 0$, that of CB is $by + 2cx + e = 0$,

that of DE is $dy+ex+2f=0$. Let the six distances from the origin, OA, OB, OD, on the axis of y, OC, OB′, OE, on the axis of x, be denoted by A, B, D, C, B′, E. The equations of the straight lines then give

$$A=-\frac{d}{2a},\quad B'=-\frac{d}{b},\quad D=-\frac{2f}{d},$$

$$C=-\frac{e}{2c},\quad B=-\frac{e}{b},\quad E=-\frac{2f}{e};$$

whence $\frac{B}{B'}=\frac{D}{E}$, or OB : OB′ : :: OD : OE, or BB′ and DE are parallel. And the equation of the curve is

$$\frac{y^2}{AD}+\frac{2xy}{BE \text{ or } B'D}+\frac{x^2}{CE}-\frac{2y}{D}-\frac{2x}{E}+1=0,$$

or if $D=\lambda B$, $E=\lambda B'$

$$\frac{y^2}{AB}+\frac{2xy}{BB'}+\frac{x^2}{CB'}-\frac{2y}{B}-\frac{2x}{B'}+\lambda=0.$$

We can now easily interpret what is meant by the vanishing of one or more coefficients. Suppose, for instance, the form degenerates into $ay^2+cx^2+f=0$. We have then $D=\infty$, $E=\infty$, or the polar line of the origin is at an infinite distance; that is, the origin is at the centre. But AD and CE are finite; whence $A=0$, $C=0$, or either axis is the central conjugate of the other. And B, B′, are indeterminate. Consequently, the curve is referred to conjugate diameters.

The other cases will give as little trouble.

The criterion of distinction is worth notice. We see that b^2-4ac agrees in sign with $\frac{1}{BB'DE}-\frac{1}{ACDE}$, or $1-\frac{BB'}{AC}$. If then of the four lines A, C, B, B′, one or three be negative,—that is, if of the four *points* A, C, B, B′, three lie on the boundary of one of either pair of opposite axis-angles, and one on the boundary of the other, the curve must be an hyperbola. But if none, or two, or four, be negative, that is, if all lie on the boundary of one angle, or two on one and two on its opposite, the curve may be an hyperbola, ellipse, or parabola, distinguishable as follows. If BC and AB′ be parallel, the curve is a parabola: otherwise, it is an ellipse or hyperbola according to the mode in which, B or B′ remaining fixed, the other must be moved to make AB′ and CB parallel. If, B remaining fixed, B′ must be removed *towards* the origin, we have an ellipse; if *from* the origin, an hyperbola.

I need not lengthen this communication by further consequences of these relations, nor by stating the mode in which a corresponding method may be applied to the general equation of surfaces of the second degree.

On the theorem which this method suggested, not finding it any where, I applied to M. Chasles, who, though he did not remember having seen it, found it, as I expected he would, to be a case of one of the many general theorems which he himself has given in his Memoir on Geometry, (p. 692) published with his well-known "Aperçu Historique," is the following theorem: Any triangle being placed in any manner with respect to a surface of the second degree, the polar planes of the three vertices meet the opposite sides in three points which are in one straight line. If, M. Chasles remarks, the triangle be in one of the principal planes, and two of its points be at an infinite distance, the theorem just named becomes that which I have given above.

ON DIAMETERS OF CURVES OF THE THIRD AND HIGHER ORDERS.

[*Mr. Fenwick.*]

The term "diameter" was for a long time restricted to the circle and the conic sections, to designate every straight line passing through the centre. The diameter of a curve in general, however, is the line, straight or curved, which passes through the middle points of a series of parallel chords, or it is the locus of the middle points of such chords. Assuming this as our definition of a diameter, the problem to determine the equation of such, in reference to a given curve, may be thus enunciated :—

Given the equation of a curve and the angular coefficient of a straight line, to find the locus of the middle points of all chords parallel to the line in reference to the given curve.*

One method of investigation is based on the property, *that if x_1 and x_2 be the abscissæ of two points in a straight line, and x the abscissa of the middle point between the two, then $x_1 + x_2 = 2x$.*

I shall illustrate this method by finding the equation of a diameter with respect to the general equation of the third order, viz.,

$$ay^3 + (b+b_1x)\,y^2 + (c+c_1x+c_2x^2)\,y + d + d_1x + d_2x^2 + d_3x^3 = 0 \ldots\ldots (1).$$

Denote the three points A, B, C, in which any chord

$$y = px + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

meets the curve (1), by

$$(x_1, y_1),\ (x_2, y_2) \text{ and } (x_3, y_3)\ ;$$

and let

$$(X_1, Y_1),\ (X_2, Y_2) \text{ and } (X_3, Y_3),$$

be the middle points of the chords AB, AC, and BC.

Then we evidently have

$$x_1+x_2 = 2X_1 \ldots (3), \qquad x_1+x_3 = 2X_2 \ldots (4), \qquad x_2+x_3 = 2X_3 \ldots (5).$$

Hence we readily get from (3, 4, 5),

$$\begin{aligned} x_1 + x_2 + x_3 &= X_1 + X_2 + X_3 \\ &= 2X_3 + x_1 = 2X_2 + x_2 = 2X_1 + x_3 \ldots (6). \end{aligned}$$

Eliminating y between (1) and (2), the coefficient of x^2 in the resulting equation is found to be

$$\frac{q(3ap^2 + 2b_1p + c_2) + bp^2 + c_1p + d_2}{ap^3 + b_1p^2 + c_2p + d_3}.$$

It consequently follows from (6), and the theory of roots, that

$$2X_3 + x_1 = -\frac{q(3ap^2+2b_1p+c_2) + bp^2 + c_1p + d_2}{ap^3 + b_1p^2 + c_2p + d_3} \ldots\ldots\ldots (7).$$

Again, because (X_3, Y_3) is a point in (2), and (x_1, y_1) a point in (2) and (1), we have

$$Y_3 = pX_3 + q \ldots\ldots (8), \qquad y_1 = px_1 + q \ldots\ldots (9),$$

$$ay_1^3 + (b+b_1x_1)y_1^2 + (c+c_1x_1+c_2x_1^2)y_1 + d + d_1x_1 + d_2x_1^2 + d_3x_1^3 = 0 \ldots (10).$$

Now for different positions of the chord BC, p is constant and x_1, y_1, q, variable ; if, therefore, we eliminate these (x_1, y_1, q) from (7, 8, 9, 10), the resulting equation will contain X_3, Y_3, (the variable or current coordinates of the middle point of BC), and constant quantities, and hence this will be

* The parameter a of the equation $y = ax + b$, has been very properly named the *angular*, and b the *linear*, coefficient, of the equation of the line.

the equation of the locus of the middle points of all chords parallel to (2), in reference to (1).

By (8) and (9),

$$q = Y_3 - pX_3, \text{ and } y_1 = Y_3 - p(X_3 - x_1.$$

Hence by (7),

$$x_1 = \frac{(pX_3 - Y_3)(3ap^2 + 2b_1p + c_2) - (bp^2 + c_1p + d_2)}{ap^3 + b_1p^2 + c_2p + d_3} - 2X_3$$

$$= \frac{X_3(ap^3 - c_2p - 2d_3) - Y_3(3ap^2 + 2b_1p + c_2) - (bp^2 + c_1p + d_2)}{ap^3 + b_1p^2 + c_2p + d_3}.$$

The substitution of these in (10) will give an equation between X_3 and Y_3 of the third order. An equation of the same order will also result, if we use either of the other two values of $x_1+x_2+x_3$, in (6). Hence, *the diameters of curves of the third order are in general curves* of the same order.*

This method applies to curves of the second and third, but not (it would seem) to those of higher orders.

Another method, applicable to all curves, is based on this obvious property, *that if we transform the origin of coordinates to the middle point of any chord, the two extremities of that chord in reference to the new axes, necessarily acquire equal coordinates with opposite algebraical signs.*

We will illustrate this method by finding the equation of a diameter of the curve

$$y = x^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Let AB, A′B′, *etc.*, be a series of parallel chords in reference to (11), denoted by the equation

$$y = px + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

of which p is constant and q variable. Then removing the origin to (h,k), the middle point of any one of the parallel chords as A′B′, the equations (11) and (12) become (x and y being the current coordinates in reference to the new axes),

$$y + k = (x + h)^4, \quad \text{and } y = px.$$

Eliminating y between these, there results the equation

$$x^4 + 4hx^3 + 6h^2x^2 + (4h^3 - p)x + h^4 - k = 0 \ldots\ldots (13).$$

We must now introduce the condition that (h, k) is the *middle of the chord* A′B′, that is, the relation between h and k must be such that (13) may assume the form

$$(x^2 - m^2)(x^2 + a_1x + b_1) = 0;$$

for A′B′ would then meet the curve in two points equally distant from the origin, inasmuch as the equation (13) would be satisfied by $x^2 - m^2 = 0$, or $x = \pm m$.

Dividing the equation (13) by $x^2 - m^2$, the remainder is

$$(4h^3 - p + 4m^2h)x + m^2(6h^2 + m^2) + h^4 - k.$$

Hence as the remainder is to vanish for all possible values of x, we must have

$$4h^3 - p + 4m^2h = 0, \text{ and } m^2(6h^2 + m^2) + h^4 - k = 0 \ldots . (14).$$

Moreover, if we eliminate m^2 from these, the resulting equation will give

* Those diameters of curves that are themselves *curves*, are sometimes named *diametral curves.*

a relation between h and k, the coordinates of the middle point of any one of the chords, and hence it will be the equation required.

From the former of the equations (14),

$$m^2 = \frac{p - 4h^3}{4h}.$$

Substituting this in the latter, we get for that diameter of (11) which bisects all chords parallel to (12), the equation

$$64h^6 - 16ph^3 + 16kh^2 - p^2 = 0,$$

which is a curve of the sixth order.

When we apply this method to curves of the *second order*, we are saved the trouble of dividing by a factor of the form $x^2 - m^2$, and the subsequent elimination, for the equation in x corresponding to (13), will, in such case, be only of the *second degree*. That its two roots may be equal and of contrary sign, it will be necessary and sufficient that the first power of x be zero; hence the general equation of a diameter will immediately result from equating to zero the coefficient of x in the transformed equation, as we find in all our treatises.

March 24*th*, 1848.

SOLUTIONS OF MATHEMATICAL EXERCISES.

CXXXVIII. *Mr. Thomas Weddle, Mr. Stoton's Academy, Wimbledon.*

Let $F(x,y,z)=0$, or simply, $F=0$ be the equation of any surface, and put

$$\frac{dF}{dx}=U, \frac{dF}{dy}=V, \frac{dF}{dz}=W, \frac{d^2F}{dx^2}=u, \frac{d^2F}{dy^2}=v, \frac{d^2F}{dz^2}=w,$$

$$\frac{d^2F}{dydz}=u', \frac{d^2F}{dxdz}=v', \text{ and } \frac{d^2F}{dxdy}=w',$$

then according as the expression

$$(vw - u'^2)U^2 + (uw - v'^2)V^2 + (uv - w'^2)W^2 + 2(v'w' - uu')VW$$
$$+2(u'w' - vv')UW + 2(u'v' - ww')UV$$

is positive or negative, the surface in the immediate vicinity of the point (x,y,z) will be situated entirely on one side of the tangent plane at that point, or partly on one side and partly on the other.

[SOLUTION. *Mr. Weddle, the proposer.*]

It is shewn in *Gregory's Solid Geometry*, p. 233, that if the section of the surface

$$ux_1^2 + vy_1^2 + wz_1^2 + 2u'y_1z_1 + 2v'x_1z_1 + 2w'x_1y_1 = \text{constant} \ldots. (1),$$

by the plane, $$Ux_1 + Vy_1 + Wz_1 = 0 \ldots\ldots\ldots\ldots (2),$$

be an ellipse, the surface $F = 0$, in the immediate vicinity of the point (x,y,z) will lie only on one side of the tangent plane, but if the section be an hyperbola, the surface will lie partly on one side and partly on the other. Now, by the criterion (10), given in my paper on "Plane and Circular Sections of Surfaces of the Second Degree," the section will be an ellipse or hyperbola according as the expression

$$(vw - u'^2)U^2 + (uw - v'^2)V^2 + (uv - w'^2)W^2 + 2(v'w' - uu')VW$$
$$+2(u'w' - vv')UW + 2(u'v' - ww')UV \ldots\ldots\ldots (3),$$

is positive or negative; and hence the truth of the exercise is manifest.

If (3) were equal to zero, the section of (1) made by (2) would be a parabola, or rather two parallel straight lines, and the inference would be

virtually the same as when the section is an ellipse, only one of the principal radii of curvature at the point (xyz) would be infinite. As (3) equated to zero is the differential equation to developable surfaces (*Camb. Math. Journal,* vol. iv., p. 154,) it appears that in developable surfaces, the surface near each point lies entirely on one side of the tangent plane at that point.

Note.—The criterion (3) is true whether the axes be rectangular or oblique.

Messrs. S. Barry, City School, London, and R. H. Wright, M. A., London, sent also good solutions.

CXL. *Lieut. A. R. Clarke, R. E.*

If lines drawn from one of the centres of the escribed circles of a plane triangle, or from the centre of the inscribed circle, to the other three, represent in magnitude and direction three forces, then the line joining this centre and that of the circumscribing circle will represent in magnitude and direction one fourth of the resultant.

[FIRST SOLUTION. *Mr. Thomas Weddle, Mathematical Master, Mr. Stoton's Academy, Wimbledon: and similarly by Mr. W. H. Levy, Shalbourn.*]

Let Q $OO_1O_2O_3$ be the centres of the circumscribed, inscribed, and escribed circles of the triangle ABC; draw the diameter GQH of the circumscribed circle perpendicular to BC, meeting OO_1 in G, and O_2O_3 in H; then shall OO_1 and O_2O_3 be bisected in G and H respectively. The force QO is equivalent to the two QG and GO; and the force QO_1 to the two QG and GO_1; hence since GO and GO_1 are equal and opposite, the forces QO and QO_1 are equivalent to 2QG; similarly the two forces QO_2 and QO_3 are equivalent to 2QH, which is equal and opposite to 2QG; hence the forces QO, QO_1, QO_2, and QO_3 are in equilibrium; that is

If four forces be represented in direction and magnitude by the four lines drawn from the centre of the circumscribed circle of a triangle, to the centres of the inscribed and escribed circles, they shall keep each other in equilibrium.

Moreover the force QO_1 is equivalent to QO and OO_1; the force QO_2 to QO and OO_2; and the force QO_3 to QO and OO_3: hence the four forces QO, QO_1, QO_2, and QO_3 are equivalent to the forces OO_1, OO_2, OO_3, and 4QO; hence the latter forces are in equilibrium, for by the above theorem the former are so; and therefore 4OQ is the resultant of the forces OO_1, OO_2, and OO_3.

In a similar manner it may be shown that the forces QO, QO_1, QO_2, and QO_3 are equivalent to the forces O_1O, O_1O_2, O_1O_3, and $4QO_1$; and therefore $4O_1Q$ is the resultant of the forces O_1O, O_1O_2, and O_1O_3. Hence the very neat properties enunciated are true.

The following theorem, which occurred to me some years ago, may be deduced from the property given above (see *Poinsot's Elements de Statique,* p. 218), but I shall give an independent investigation of it.

The centre of gravity of four equal bodies placed at the centres of the inscribed and escribed circles of a triangle, coincides with the centre of the circumscribed circle.

Let P denote the weight of each of the bodies; the weights P and P at O and O_1 produce the same effect as a weight 2P at G, and the weights P and P at O_2 and O_3 produce the same effect as a weight 2P at H; also the weights 2P and 2P at G and H produce the same effect as the weight 4P at Q; hence Q is the centre of gravity of the four equal bodies.

[SECOND SOLUTION. *Messrs. Nicholas Smyth, Galway, and Peter Elliott, Lamesley.*]

Referring to the diagram in the *Lady's and Gentleman's Diary*, for 1843, p. 79, and taking OO_1, OO_2, OO_3 to represent the three forces, both in direction and magnitude, we have by resolving these forces parallel to two axes, one the side of the triangle AC, and the other at right angles to it, resultant parallel to $AC = EE_1 + EE_2 - EE_3 = 4EK_2$,
because the pairs of points (E_1,E_3) ; (A,C) ; (E,E_2) are equidistant from K_2,
and resultant perp. to

$$\begin{aligned} AC &= (O_2E_2+OE)-(O_3E_3-OE)-(O_1E_1-OE) \\ &= 3OE + O_2E_2 - (O_3E_3 + O_1E_1) \\ &= 4OE + (O_2E_2 - OE) - (O_3E_3 + O_1E_1) \\ &= 4OE + 2G_2K_2 - 2K_2H_2 = 4(OE - QK_2). \end{aligned}$$

But EK_2 and $OE - QK_2$ are the components of the force represented by OQ when resolved parallel and perpendicular to AC; hence 4OQ represents in direction and magnitude the resultant of the three forces represented by OO_1, OO_2, OO_3.

The property may be established in a similar manner for the other cases enunciated in the exercise.

Mr. Andrew Roy, of Dundee Academy, sent a good solution.

CXLI. *Lieut. A. R. Clarke, R. E.*

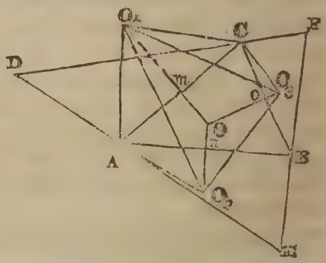
Let ABC be any plane triangle, and let straight lines be drawn through the angular points, making each an angle a with the sides, and forming by their intersection a triangle DEF. Let O_1, O_2, O_3 be the centres; and R_1, R_2, R_3 the radii of the circles circumscribing the triangles BCF, DCA, and ABE; then if R be the radius of the circle circumscribed about the triangle ABC, and Δ, Δ_1 the areas of the triangle ABC and $O_1O_2O_3$, we shall have

$$(1) \quad 16\Delta\Delta_1 = a^2b^2 + b^2c^2 + c^2a^2.$$

$$(2) \qquad R^3 = R_1R_2R_3.$$

[SOLUTION. *Mr. Peter Elliott, Lamesley; and similarly by Mr. William Mortimer, Seedley, Manchester.*]

The figure being constructed as directed, and O being the centre of the the circle circumscribed about the triangle ABC; draw OO_1, OO_2, OO_3 cutting the sides in m, n, o respectively.



Then angle $CBF = ACD = BAE = a$, and because O_3 is the centre of the circle described about the triangle BCF, the angle $CO_3o = CFB = BCD - CBF = BCD - ACD = BCA$. Also,

$$O_3o = \tfrac{1}{2}a\,\frac{\cos C}{\sin C}, \quad Oo = \tfrac{1}{2}a.\frac{\cos A}{\sin A};$$

therefore,

$$OO_3 = \tfrac{1}{2}a\left\{\frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}\right\} = \tfrac{1}{2}a\,\frac{\sin(A+C)}{\sin A \sin C} = \tfrac{1}{2}a.\,\frac{\sin B}{\sin A \sin C}.$$

Similarly, $OO_1 = \tfrac{1}{2}b.\dfrac{\sin C}{\sin A \sin B}$, and $OO_2 = \tfrac{1}{2}c.\dfrac{\sin A}{\sin B \sin C}$: hence,

recollecting that $\sin O_1OO_3 = \sin C$, *etc.*, we have

$$\Delta_1 = O_1OO_3 + O_2OO_1 + O_3OO_2$$
$$= \frac{1}{8}\left\{ab\,\frac{\sin C}{\sin^2 A} + bc\,\frac{\sin A}{\sin^2 B} + ca\,\frac{\sin B}{\sin^2 C}\right\}.$$

Also $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$; therefore by using the appropriate value of Δ in taking the product, we get

$$16\Delta\Delta_1 = a^2b^2\frac{\sin^2 C}{\sin^2 A} + b^2c^2\frac{\sin^2 A}{\sin^2 B} + c^2a^2\frac{\sin^2 B}{\sin^2 C}$$
$$= a^2b^2\frac{c^2}{a^2} + b^2c^2\frac{a^2}{b^2} + c^2a^2\frac{b^2}{c^2} = a^2b^2 + b^2c^2 + c^2a^2.$$

Again, since $BFC = ACB$ we have $R_1 = \frac{a}{2\sin C}$, and similarly

$$R_2 = \frac{b}{2\sin A},\ R_3 = \frac{c}{2\sin B};\ \text{therefore } R_1R_2R_3 = \frac{abc}{8\sin A\sin B\sin C}.$$

Also, $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$; hence it is obvious that

$$R^3 = R_1R_2R_3.$$

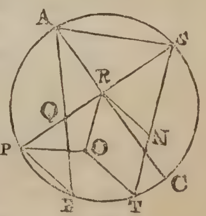
The solutions of Mr. A. Roy, Dundee; Lieut. Clarke, the proposer; and A. D. B., North Tawton, Devonshire, were very elegant, and omitted with much regret.

CXLII. *By Samuel Square.*

From a given point in the circumference of a given circle, to draw a chord intersecting two sides of a given inscribed triangle, so that the rectangle of its extreme segments may be equal to a given square.

[SOLUTION. *By Samuel Square, the proposer.*]

Let ABC be the triangle, and P the given point. Draw PB and make the circumference PT equal to the circumference PA; draw TO parallel to PB, and find TO such that the rectangle contained by PB and TO may be equal to the given square. Join PO, and on it describe a segment of a circle to contain an angle equal to ABP, and let it cut AC in R. Draw PQRS and it is the chord required.



Draw OR, ST, and also draw RN parallel to OT or PB; then by construction PB is parallel to RN: therefore angle SPB is equal to angle SRN; also angle PRO is equal to angle PBA, by construction, and PBA is equal to PST, since the circumference PT is equal to the circumference PA; therefore the triangle PQB is similar to the triangle RSN, and ON is evidently a parallelogram; therefore,

PQ : PB :: RN : RS, but RN is equal to OT; hence the rectangle PQ, RS is equal to the rectangle PB, OT, and therefore it is equal to the given square.

Mr. Nicholas Smyth, sent an excellent geometrical solution, and Mr. A. Roy favoured us with a very good analytical one. The proposer's second solution was very neat.

CXLIII. *Lieut. St. Andrew St. John, R. E.*

The centres of two spheres, radii r_1, r_2, are at the extremities of a line $2a$, on which as a diameter, a circle is described: find a point in the circumference of this circle from which the greatest portion of spherical surface is visible.

[FIRST SOLUTION. *Dr. Burns, Rochester; and similarly by Messrs. W. S. Ashton, Mr. Hoole's Academy, Blackburn; Henry Goodwyn, Mr. Stoton's Academy, Wimbledon; W. H. Levy, Shalbourn; and A. Roy.*]

If x and y be the distances of the required point from the centres of the two spheres, then *(Hall's Diff. and Integral Calculus*, p. 101),

$$u = 2\pi r_1\left(r_1 - \frac{r_1^2}{x}\right) + 2\pi r_2\left(r_2 - \frac{r_2^2}{y}\right) \ldots (1), \text{ a maximum,}$$

and $$y = (4a^2 - x^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots (2).$$

Substituting this value of y in (1) and differentiating with respect to x, we have

$$r_1^3(4a^2 - x^2)^{\frac{3}{2}} = r_2^3 x^3.$$

Extracting the cube root, squaring and transposing,

$$(r_1^2 + r_2^2)x^2 = 4a^2 r_1^2, \text{ or } x = \frac{2ar_1}{\sqrt{(r_1^2 + r_2^2)}};$$

hence by (2), $$y = \frac{2ar_2}{\sqrt{(r_1^2 + r_2^2)}}.$$

Consequently the greatest portion of spherical surface is visible when the distances of the point from the centres of the spheres are

$$\frac{2ar_1}{\sqrt{(r_1^2 + r_2^2)}} \text{ and } \frac{2ar_2}{\sqrt{(r_1^2 + r_2^2)}}.$$

[SECOND SOLUTION. *Mr. Nicholas Smyth, Galway.*]

If x denote the part of $2a$ intercepted between the centre of the circle and a perpendicular from the required point, then

$$2\pi\left\{r_1^2 - \frac{r_1^3}{(2a^2 + 2ax)^{\frac{1}{2}}}\right\} \text{ and } 2\pi\left\{r_2^2 - \frac{r_2^3}{(2a^2 - 2ax)^{\frac{1}{2}}}\right\},$$

will express the spherical segments visible, and since by the question the sum of these is to be a *minimum*, the expression

$$r_1^3(a + x)^{-\frac{1}{2}} + r_2^3(a - x)^{-\frac{1}{2}},$$

must be a *maximum*. Differentiating and equating to zero we get

$$\frac{r_1}{(a+x)^{\frac{1}{2}}} = \frac{r_2}{(a-x)^{\frac{1}{2}}}, \text{ or } x = \frac{r_1^2 - r_2^2}{r_1^2 + r_2^2}a.$$

Hence the perpendicular from the required point meets the line $2a$ at the distances

$$\frac{2ar_2^2}{r_1^2 + r_2^2} \text{ and } \frac{2ar_1^2}{r_1^2 + r_2^2},$$

from the centres of the spheres.

[THIRD SOLUTION. *Mr. P. Elliott, Lamesley, Durham; and similarly by Lieut. A. R. Clarke, R. E.*]

Let B and A be the centres of the given spheres (r_1, r_2), BCA a semicircle described on BA, C the place of the observer, and $\angle$ ABC $= \theta$:

Then CB $= 2a\cos\theta$ and CA $= 2a\sin\theta$.

Also $r_1 - \dfrac{r_1^2}{2a\cos\theta} =$ height of segment of sphere (r_1) seen from C,

and $r_2 - \dfrac{r_2^2}{2a\sin\theta} = \ldots\ldots\ldots\ldots (r_2)\ldots\ldots\ldots$

the sum of the surfaces of these segments being a maximum, we evidently have, therefore,

$$\frac{r_1^{\,3}}{\cos\theta} + \frac{r_2^{\,3}}{\sin\theta} = u, \text{ a minimum}\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\therefore\ \frac{du}{d\theta} = \frac{r_1^{\,3}\sin\theta}{\cos^2\theta} - \frac{r_2^{\,3}\cos\theta}{\sin^2\theta} = 0, \text{ or } \tan\theta = \frac{r_2}{r_1}.$$

Whence the position of C is determined.

Also, $$\frac{d^2u}{d\theta^2} = r_1^{\,3}\left(\frac{1+\sin^2\theta}{\cos^3\theta}\right) + r_2^{\,3}\left(\frac{1+\cos^2\theta}{\sin^3\theta}\right),$$

a function essentially positive between the limits $\theta = 0$ and $\theta = \frac{\pi}{2}$; therefore the substitution of the value of θ, found above, in (1), renders u a minimum.

The proposer also sent a good solution.

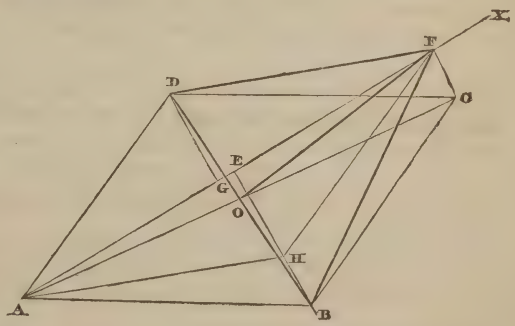
CXLIV. *Mr. John Riddle.*

If AC be the diagonal of a parallelogram ABCD, and AX a line through A in any direction in space, then shew, by *plane trigonometry*, that

$$AC.\cos CAX = AB \cos BAX + AD \cos DAX.$$

[First Solution. *Messrs. W. S. Ashton, Mr. Hoole's Academy, Blackburn, and A. Roy, Dundee.*]

Draw BE, CF, and DG perpendicular to AX. Parallel to BC in the plane BCFE, draw FH, and join AH.



Then by right angled triangles we have

$AF = AC \cos CAX$;

$AE = AB \cos BAX$;

$AG = AD \cos DAX$.

In the triangles ADG and EFH, we have by parallels $AD = FH$, and the angle $AGD = FEH$, being right angles, also the angle $GAD = EFH$,

$\therefore AG = EF$;

hence adding AE to both sides, we have

$$AG + AE = EF + AE = AF;$$

that is, $$AC \cos CAX = AB \cos BAX + AD \cos DAX.$$

[Second Solution. *Mr. W. Mortimer, Seedley, Manchester; and similarly by A. D. B., North Tawton, Devonshire, and Mr. R. H. Wright, M. A., London*]

Join BD meeting AC in O, and draw the other lines as in the figure. Then because the angle AFC is a right angle, $OF = OA = OC$. Now $AB^2 + AD^2 = 2(AO^2 + DO^2) = 2(FO^2 + DO^2) = DF^2 + BF^2$. Consequently,

$$2AF^2 = (AF^2 + AB^2 - BF^2) + (AF^2 + AD^2 - DF^2), \text{ or}$$

$$2AF.AC \cos CAX = 2AF.AB \cos BAX + 2AF.AD \cos DAX.$$

Hence we have

$$AC \cos CAX = AB \cos BAX + AD \cos DAX.$$

Good solutions were also received from Dr. Burns, Rochester; Lieut. A. R. Clarke, R. E.; Mr. J. Riddle, the proposer; and Mr. Nicholas Smyth, Galway.

CXLV. *Dr. August Weigand.*

AB is a given straight line, and C a given point in it; on AB as hypothenuse describe an isosceles right angled triangle ADB, and let a parabola be described about the triangle ADB having its vertex in the angular point D; through C draw CH perpendicular to AB meeting the parabola in H, then CH will be an harmonic mean between AC and CB.

[First Solution. *Dr. Burns; and similarly by Mr. W. S. Ashton, Blackburn.*]

Draw DE perpendicular to AB, then since the triangle ADB is isosceles and right angled, $AE = EB = ED$; and by a property of the parabola $EA^2 : EC^2 :: ED : ED - CH$; therefore

$$EA^2 : EA :: EC^2 : EA - CH, \text{ or } EA : 1 :: EC^2 : EA - CH;$$

and $EA^2 - EC^2 = AC.CB = EA.CH$, hence $BC : BC - CH :: EA : EC$; also $AC : CH - AC :: EA : EC$, consequently $BC : AC :: BC - CH : CH - AC$, and CH is therefore an harmonic mean between AC and BC.

[Second Solution. *Messrs. P. Elliott, Lamesley, and John Stafford, Newcastle.*]

Let E be the middle of AB, then by the ordinary properties of the parabola, we have

$$DE(=EB) : EB :: EB : P, \quad \therefore P = EB,$$

and,

$$2P(=AC+CB) : 2AC :: CB : CH;$$

$$\therefore \quad AC - CB : AC + CB :: CH - CB : CB,$$

$$\therefore \quad 2AC : 2CB :: CH : 2CB - CH,$$

$$\therefore \quad AC : CH :: CB : 2CB - CH,$$

$$\therefore \quad AC : CB :: AC - CH : CH - BC.$$

Hence AC, CH and CB are in harmonic progression.

[Third Solution. *Mr. W. H. Levy, Shalbourn.*]

Bisect AB in E, and join ED. Then by *Hutton*, prop. i. cor. 5, of the parabola, $AC.CB : EB^2 :: CH : ED$;

$$\text{hence } CH = \frac{ED.AC.CB}{EB^2}.$$

But it is obvious that $DE = EB$, therefore

$$CH = \frac{AC.CB}{EB} = \frac{2AC.CB}{2EB} = \frac{2AC.CB}{AC+CB};$$

hence the truth of the theorem.

Correct and elegant solutions were received from Lieut. A.R. Clarke, R.E.; and Messrs. H. Goodwyn, Mr. Stoton's Academy, Wimbledon; J. B. Hollinsworth, Mr. Hoole's Academy, Blackburn; W. Mortimer, Seedley, Manchester; David Maver, Aberdeen; R. W. Wright, M.A., London; Stephen Watson, Shotley Bridge; N. Smyth, Galway; and A. D. B.

CXLVI. *Mr. Weddle.*

If the squares of the sides of a triangle be in arithmetical progression, the lines drawn from the angles to a point within the triangle so as to make equal angles with each other, are in arithmetical progression.

[First Solution. *Mr. W. S. Ashton, Mr. Hoole's Academy, Blackburn; Dr. Burns, Rochester; Messrs. P. Elliott, Lamesley; Henry Goodwyn, Mr. Stoton's Academy, Wimbledon; David Mavor, Bon accord School, Aberdeen; Nicholas Smyth, Galway; John Stafford, St. Ann's School, Newcastle; S. Watson, Castle Side, Shotley Bridge; and R. H. Wright, London.*]

Let a, b, c denote the three sides of the triangle, and x, y, z the three lines drawn from the angles to the point within it, so as to make equal angles with each other; then we have

$$a^2 = x^2 + y^2 - 2xy\cos 120° = x^2 + xy + y^2 \dots\dots\dots\dots (1),$$
$$b^2 = x^2 + z^2 - 2xz\cos 120° = x^2 + xz + z^2 \dots\dots\dots\dots (2),$$
$$c^2 = y^2 + z^2 - 2yz\cos 120° = y^2 + yz + z^2 \dots\dots\dots\dots (3).$$

Now since a^2, b^2, c^2 are in arithmetical progression, the difference between the first and second of these must be equal to the difference between the second and third; hence taking (2) from (1), and (3) from (2), we get

$$y^2 - z^2 + x(y - z) = x^2 - y^2 + z(x - y),$$

or, since $y^2 - z^2 = (y + z)(y - z)$, and $x^2 - y^2 = (x + y)(x - y)$,

$$(y + z + x)(y - z) = (x + y + z)(x - y);$$
$$\therefore\ \ y - z = x - y, \text{ and } x + z = 2y;$$

that is, x, y, z are in arithmetical progression.

[SECOND SOLUTION. *By A.D.B., North Tawton, Crediton; Mr. A. Roy, Dundee; Mr. S. Watson (second solution), and Mr. Weddle, the proposer.*]

Let a, b, c, as usual, denote the sides of the triangle, and x, y, z the other lines mentioned in the question. Since x, y, z make equal angles with one another, each of these angles must $= 120°$;

$$\therefore\ \ y^2 + z^2 - 2yz\cos 120° = a^2,$$

or $\qquad y^2 + z^2 + yz = a^2 \dots\dots\dots\dots\dots\dots (1).$

Similarly, $\qquad x^2 + z^2 + xz = b^2 \dots\dots\dots\dots\dots\dots (2),$

and $\qquad x^2 + y^2 + xy = c^2 \dots\dots\dots\dots\dots\dots (3).$

Since, by hypothesis, the squares of the sides of the triangle are in arithmetical progression, we have $a^2 + c^2 = 2b^2$; hence (1, 2, 3),

$$x^2 + 2y^2 + z^2 + xy + yz = 2(x^2 + z^2 + xz),$$

and this on reduction becomes,

$$y^2 + \tfrac{1}{2}(x + z)y = \tfrac{1}{2}(x + z)^2.$$

The two roots of this quadratic (in y) are $\frac{1}{2}(x + z)$, and $-(x + z)$; and rejecting the latter, we have

$$y = \tfrac{1}{2}(x + z).$$

Hence, x, y, and z are in arithmetical progression.

NOTE. The second root, $-(x + z)$, is rejected because it is inconsistent with (1) and (3), unless the triangle be equilateral; for the difference of (1) and (3) may be written

$$(x - z)(x + y) = c^2 - a^2,$$

and if $y = -(x + z)$ or $x + y + z = 0$, we must have $a = c$, and therefore $a = b = c$.

When, however, the triangle is equilateral, not only is $y = \frac{1}{2}(x + z)$ admissible, but likewise $x + y + z = 0$; only in the latter case, it is not the three lines x, y, and z that make equal angles with each other, but two of them and the third *produced*. Indeed it is not very difficult to see that the condition, $x + y + z = 0$, expresses the following well-known theorem:

If lines be drawn from the angles of an equilateral triangle to any part in the circumference of the circumscribed circle, the sum of two of these lines will be equal to the third.

Mr. Roy adds the following,

Cor. The line which is an arithmetical mean between the other two, is the one drawn from the angle opposite that side whose square is a mean between the squares of the other two sides.

Messrs. W. H. Levy and W. Mortimer, favored us with good solutions.

CXLVII. *Mr. S. Barry, City School, London.*

Find the locus of the middle points of equal chords in an ellipse.

[FIRST SOLUTION. *Messrs. Nicholas Smyth, Galway, and P. Elliott, Lamesley.*]

Let the equation of the given ellipse be

$$a^2y_1^2+b^2x_1^2=a^2b^2;$$

$2c$ the length of the constant chord; x,y the coordinates of its variable middle point, and m,n such quantities as make $(x-m, y+n)$, $(x+m, y-n)$ the coordinates of its extremities.

Then we evidently have

$$m^2+n^2=c^2 \ldots\ldots\ldots\ldots (1),$$

$$a^2(y+n)^2+b^2(x-m)^2=a^2b^2,$$

$$a^2(y-n)^2+b^2(x+m)^2=a^2b^2.$$

The sum and difference of the last two equations give

$$a^2(y^2+n^2)+b^2(x^2+m^2)=a^2b^2 \ldots\ldots\ldots\ldots (2),$$

$$a^2yn-b^2xm=0 \ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we have $m^2=\dfrac{a^4c^2y^2}{b^4x^2+a^4y^2}$, $\quad n^2=\dfrac{b^4c^2x^2}{b^4x^2+a^4y^2}$.

These values substituted for m^2 and n^2 in (2), with a little reduction, give

$$a^6y^4+b^6x^4+a^2b^2(a^2+b^2)x^2y^2+a^2b^4(c^2-b^2)x^2+a^4b^2(c^2-a^2)y^2=0,$$

for the equation of the required curve.

By making x and y successively equal zero, we get

$$y=\pm\frac{b}{a}(a^2-c^2)^{\frac{1}{2}}, \text{ and } x=\pm\frac{a}{b}(b^2-c^2)^{\frac{1}{2}};$$

for the intersections with the coordinate axis.

So long as a is small in comparison with b the locus will be an oval not very dissimilar to the given ellipse, but as c approaches towards an equality with b, it assumes a double knobbed shape, the curve being equally divided by the major axis of the ellipse.

When $c=b$, the locus is a double oval or species of lemniscata connected at the origin.

When $c > b$ and $< a$ the locus will be two equal and similar open curves, concave towards the major axis.

[SECOND SOLUTION. *Messrs. W. Mortimer, Seedley, and S. Barry, the proposer.*]

Let $2a$, $2b$ be the axes of the ellipse, $2c$ the length of the chord, x', y' and x'', y'' the coordinates of its extremities in any position; and x, y those of its middle points. Then we have

$$a^2y'^2+b^2x'^2=a^2b^2 \ldots\ldots\ldots\ldots (1),$$

$$a^2y''^2+b^2x''^2=a^2b^2 \ldots\ldots\ldots\ldots (2),$$

$$(y'-y'')^2+(x'-x'')^2=4c^2 \ldots\ldots\ldots\ldots (3),$$

$$2\mathrm{X}=x'+x'', \quad 2\mathrm{Y}=y'+y'' \ldots\ldots\ldots\ldots (4).$$

From (1) and (2) we get

$$a^2(y'^2-y''^2)+b^2(x'^2-x''^2)=0,$$

or by (4) $$a^2\mathrm{Y}(y'-\mathrm{Y})+b^2\mathrm{X}(x'-\mathrm{X})=0 \ldots\ldots\ldots\ldots (5).$$

Also from (3) and (4) we obtain

$$(y'-\mathrm{Y})^2+(x'-\mathrm{X})^2=c^2 \ldots\ldots\ldots\ldots (6).$$

Whence by (5) and (6) we get

$$y'=\mathrm{Y}+\frac{b^2c\mathrm{X}}{\sqrt{(a^4\mathrm{Y}^2+b^4\mathrm{X}^2)}},\quad x'=\mathrm{X}-\frac{a^2c\mathrm{X}}{\sqrt{(a^4\mathrm{Y}^2+b^4\mathrm{X}^2)}}.$$

Substituting these values in (1), the following equation results:

$$a^2\mathrm{Y}^2+b^2\mathrm{X}^2+a^2b^2c^2.\frac{b^2\mathrm{X}^2+a^2\mathrm{Y}^2}{b^4\mathrm{X}^2+a^4\mathrm{Y}^2}-a^2b^2=0:$$

or $(a^2\mathrm{Y}^2+b^2\mathrm{X}^2)(a^4\mathrm{Y}^2+b^4\mathrm{X}^2)+a^2b^2c^2(a^2\mathrm{Y}^2+b^2\mathrm{X}^2)-a^2b^2(a^4\mathrm{Y}^2+b^4\mathrm{X}^2)=0$

If $a=b$, the ellipse becomes a circle, and the equation reduces to

$$a^6(\mathrm{Y}^2+\mathrm{X}^2)^2=a^6(a^2-c^2)(\mathrm{Y}^2+\mathrm{X}^2),\text{ or }\mathrm{X}^2+\mathrm{Y}^2=a^2-c^2,$$

which is the equation to a circle whose radius is $(a^2-c^2)^{\frac{1}{2}}$.

[THIRD SOLUTION. *Dr. Burns, Rochester.*]

Let a and b be the semi-axes of the ellipse,

a_1 and b_1 any conjugate semi-diameters,

2 c the given chord, and

u the distance from the centre of the ellipse to the point of bisection,

then $$u^2=\frac{a_1^2}{b_1^2}(b_1^2-c^2);\ \text{now } a_1^2=\frac{a^2(1-e^2)}{1-e^2\cos^2\theta},\text{ and since}$$

$$a_1^2+b_1^2=a^2+b^2,\quad\therefore\ b_1^2=a^2+b^2-a_1^2=\frac{a^2(1-2e^2\cos^2\theta+e^4\cos^2\theta)}{1-e^2\cos^2\theta}.$$

Hence $$u^2=\frac{\dfrac{a^2(1-e^2)}{1-e^2\cos^2\theta}}{\dfrac{a^2(1-2e^2\cos^2\theta+e^4\cos^2\theta)}{1-e^2\cos^2\theta}}\times\left(\frac{a^2(1-2e^2\cos^2\theta+e^4\cos^2\theta)}{1-e^2\cos^2\theta}-c^2\right)$$

$$=(1-e^2)\left\{\frac{a^2}{1-e^2\cos^2\theta}-\frac{c^2}{1-2e^2\cos^2\theta+e^4\cos^2\theta}\right\};$$

$$\therefore\ u=\sqrt{\left(1-e^2\right)\left\{\frac{a^2}{1-e^2\cos^2\theta}-\frac{c^2}{1-2e^2\cos^2+e^4\cos^2\theta}\right\}},$$

the equation to the locus of the point of bisection, the centre of the ellipse being the pole.

Mr. A. Roy sent an excellent solution.

MATHEMATICAL EXERCISES—*(continued.)*

155. *By* Δ.

Let P, Q be given points in the circumference of a circle given in magnitude and position, and QM, QN are chords given in position. It is required to draw a line PRST meeting QN in R, QM in S, and the circumference in T, so that RS may have to ST a given ratio.

156. *Mr. Robert Harley, Blackburn.*

It is required to determine, by "De Gua's Criterion" for the detection of imaginary roots, how many are contained in the cubic equation $x^3-ax\pm b=0$, where a and b are any real positive quantities.

157. *By the same.*

Three circles are described in a plane triangle to touch the inscribed circle and two of the sides: find the relation that subsists among the radii of these four circles.

158. *W. H. Levy, Shalbourn.*

Let $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$ be the radii of the circles inscribed in the six triangles formed by joining the middle points of the sides of any plane triangle with the opposite angles; r_1, r_2, r_3 the radii of the escribed circles, and r_4, r_5, r_6 the radii of the circles inscribed in the triangles formed by joining the contiguous corners of the squares described on the three sides; then will

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}+\frac{1}{\rho_5}+\frac{1}{\rho_6}=\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}+\frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}.$$

159. *Lieut. A. R. Clarke, R. E.*

If a tetrahedron whose six edges are denoted by a, b, c, d, e, f, be cut by three planes so that each section is a rhombus, and if m_1, m_2, m_3 be their sides; then will

$$\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}.$$

160. *By Lexell.*

ABCD is a quadrilateral inscribed in a circle (whose centre is O); and the opposite sides are produced to meet, viz., AB and CD in E, and BC and AD in F; join EO, FO, meeting the circle in R and S; and take

OP : OR :: OR : OE and OQ : OS :: OS : OF.

Then if EQ, FP be drawn, they will be perpendicular to FO, EO.

161. *By ———.*

Give a proof of Euler's construction of the triangle inscribed in a circle, whose sides pass through three given points, (see p. 146).

162. *By θ.*

Find the surface which is the locus of the point of intersection of three planes at right angles to each other, each of which touches in a point the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

163. *Mr. Thomas Weddle, Wimbledon.*

Join the angular points of a tetrahedron, and the points in which an inscribed surface of the second degree touches the faces of the tetrahedron; if three of these lines pass through a point, the fourth will pass through the same point.

164. *By δ.*

The radii of three concentric spheres are R, R$'$, R$''$, it is required to find the equation of the surface which is the locus of the point of intersection of three planes at right angles to each other, one of which constantly touches one of the three given spheres.

165. *Mr. R. H. Wright, London.*

Two equal cones of equal densities float in a fluid whose density is likewise given, and unequal weights are placed on their horizontal bases over their centres of gravity; find how far the one cone will sink below the other. Also, if d be the original draft of the fluid for each, w the weight of each cone, m_1 and m_2 very small weights applied as above, δ the difference in draft of fluid; show that

$$\delta=\frac{d}{3w}(m_2-m_1).$$

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ON ANALYTICAL GEOMETRY OF THREE DIMENSIONS RELATIVE TO OBLIQUE AXES.

[*Mr. Thomas Weddle, Wimbledon, Surrey.*]

The present is the first of a series of papers in which I propose to discuss, relative to oblique axes, such topics as in our ordinary works on Analytical Geometry of Three Dimensions are given for rectangular axes only. It is true that this has been partially undertaken in several papers in the *Cambridge Mathematical Journal*, and subsequently in *Gregory's Solid Geometry*; but the discussions in these works are confined to the Straight Line and Plane, and even these have not been considered in so extensive and systematic a manner as they will be in my first and second chapters; and the discussions in my remaining chapters have not been entered upon at all relative to oblique axes. Throughout the whole of these papers I shall employ symmetrical equations, being convinced that the advantage of symmetry in this branch of mathematics can scarcely be over-rated, and it is a steady adherence to this which gives Gregory's excellent treatise* (alluded to above) no inconsiderable portion of its value. I shall not attempt any great amount of originality, and shall not hesitate to adopt (with a suitable acknowledgement† of course) the investigations of others, modified for oblique axes, when I think that they are the best for my purpose.

I have found it convenient to use some new terms, such as *coefficient of cubature*, *projecting coefficients*, *&c*. These may possibly not be the best that might be employed, but some terms seem very much called for to denote certain quantities that constantly appear in these inquiries, and I leave it to others to invent better, if these be objectionable.

* As my obligations to this work are extensive, I may perhaps be allowed to add that I cordially agree with a distinguished mathematician when he affirms that it "may safely be pronounced the best analytical treatise on Geometry of Three Dimensions at present to be found in the English language;" and I would therefore strongly recommend it to the student who may wish to turn his attention to this interesting branch of mathematics.

I may be permitted to observe, too, that Mr. Walton has overcome the difficulties of completing the work in so able a manner, and has entered so fully into the spirit in which Mr. Gregory commenced it, as little to require that 'indulgence' which, in the circumstances, he naturally requests of his readers.

† As it would be tedious to give a reference for every little thing, I must refer, once for all, to the works mentioned above, for what is not original in my first and second chapters.

CHAPTER I.

On the Straight Line.

The angles which the axes make with each other will be called the *angles of ordination*, and the cosines of these angles *the cosines of ordination;* so that if f, g, h denote the cosines of the angles which the axes of y and z, of z and x, of x and y, make with each other, f, g, h will be the cosines of ordination. Also, if $\Gamma = \{1-f^2-g^2-h^2+2fgh\}^{\frac{1}{2}}$, Γ will be termed the *coefficient of cubature.**

PROPOSITION I. *To find the equations to a straight line.*

Let $(\alpha\beta\gamma)$ be a given point, and (xyz) a variable one, in the straight line; also let r denote the distance between these two points.

If r be projected on the axis of x by lines drawn parallel to the plane of yz, then, whatever may be the length of r, it will always have a constant ratio to its projection, and the same is true with respect to the projections on the other axes. Now the projections of r on the axes of x, y and z are evidently $x-\alpha$, $y-\beta$ and $z-\gamma$; hence, if l, m, n denote the constant ratios of these projections to r, we have

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \ldots\ldots\ldots\ldots\ldots (1),$$

and these are the equations to the straight line.

Since lr, mr, nr are the projections of r on the axes by straight lines drawn parallel to the coordinate planes, I shall denominate l, m, n the *coefficients of projection*, or the *projecting coefficients* of the straight line (1), and lr, mr and nr the *coordinate projections* of r on the axes.

Moreover, since the angles which a parallel to the straight line (1), drawn from the origin, makes with the axes of x, y, and z, determine the *direction* of (1), they will be called the *directing angles* of the straight line (1), and the cosines λ, μ, ν of these angles, the *directing cosines* of (1). Also, λr, μr, and νr are the *orthogonal projections* of r on the axes of x, y, z respectively.

PROP. II. *To find the equations to a straight line passing through the given points* $(x_1\, y_1\, z_1)$ and $(x_2\, y_2\, z_2)$.

The straight line passes through the point $(x_1y_1z_1)$, and its equation may therefore be denoted by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}.$$

But since it also passes through $(x_2y_2z_2)$, we must have

$$\frac{x_2-x_1}{l} = \frac{y_2-y_1}{m} = \frac{z_2-z_1}{n}.$$

* If V denote the volume contained by any surface referred to oblique axes, we shall have $V = \Gamma \int\int\int dx\, dy\, dz$, between the proper limits. Hence the term *coefficient of cubature.*

When the *cosines* of ordination are given, Γ may be most readily calculated by means of

$$\Gamma = \{1-f^2-g^2-h^2+2fgh\}^{\frac{1}{2}};$$

but when the *angles*, $\epsilon_1\epsilon_2\epsilon_3$, of ordination are given, it is most convenient to apply the following formula, which may easily be shown to be equivalent to the preceding,

$$\Gamma = 2\{\sin\Sigma.\sin(\Sigma-\epsilon_1).\sin(\Sigma-\epsilon_2).\sin(\Sigma-\epsilon_3)\}^{\frac{1}{2}},$$

where $2\Sigma = \epsilon_1+\epsilon_2+\epsilon_3$.

Divide the former equations by the latter,

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\ldots\ldots\ldots\ldots\ldots(2),$$

which are the equations required.

Prop. III. *To find the condition that two straight lines may intersect, and the position of the point of intersection.*

Let the equations to the straight line be

$$\frac{x-a}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\ldots\ldots\ldots\ldots(3),$$

and
$$\frac{x-a_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}=r_1\ldots\ldots\ldots\ldots(4),$$

which are equivalent to

$$x=a+lr,\qquad y=\beta+mr,\qquad z=\gamma+nr\ \ldots\ldots\ldots(5),$$

and $$x=a_1+l_1r_1,\qquad y=\beta_1+m_1r_1,\qquad z=\gamma_1+n_1r_1\ \ldots\ldots\ldots(6).$$

Now these equations exist simultaneously at the point of intersection; eliminate, therefore, xyz from (5, 6), and we have

$$\left.\begin{aligned} a-a_1+lr-l_1r_1&=0\\ \beta-\beta_1+mr-m_1r_1&=0\\ \gamma-\gamma_1+nr-n_1r_1&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Eliminate the variables r and r_1 from (7),

$$(m_1n-mn_1)(a-a_1)+(ln_1-l_1n)(\beta-\beta_1)+(l_1m-lm_1)(\gamma-\gamma_1)=0\ldots.(8),$$

which is the condition that the lines (3) and (4) may intersect.

To find the position of the point of intersection, determine the value of r or r_1, from (7), and substitute it in (5) or (6), and we have the coordinates of the point required.

Prop. IV. *To determine the distance between two given points* $(x_1y_1z_1)$ and $(x_2y_2z_2)$.

Let r denote the distance between the given points, and $\lambda\mu\nu$ the directing cosines of r.

Take the orthogonal projections of r, and the broken line $(x_2-x_1)+(y_2-y_1)+(z_2-z_1)$, (both of which terminate in the given points) on each of the axes;

$$\therefore\quad\left.\begin{aligned} \lambda r&=\ \ (x_2-x_1)+h(y_2-y_1)+g(z_2-z_1)\\ \mu r&=h(x_2-x_1)+\ \ (y_2-y_1)+f(z_2-z_1)\\ \nu r&=g(x_2-x_1)+f(y_2-y_1)+\ \ (z_2-z_1)\end{aligned}\right\}\ldots\ldots\ldots\ldots(9).$$

Also take the orthogonal projection of the same broken line on r,

$$\therefore\quad r=\lambda(x_2-x_1)+\mu(y_2-y_1)+\nu(z_2-z_1)\ \ldots\ldots\ldots\ldots(10).$$

Multiply the equations (9) by x_2-x_1, y_2-y_1, z_2-z_1 respectively, add, and reduce by (10),

$$\therefore\ r^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+$$
$$2f(y_2-y_1)(z_2-z_1)+2g(x_2-x_1)(z_2-z_1)+2h(x_2-x_1)(y_2-y_1)\ldots(11).$$

Prop. v. *To find the relation among the projecting coefficients of a straight line* (1).

By (11) we have

$$r^2=(x-a)^2+(y-\beta)^2+(z-\gamma)^2+ 2f(y-\beta)(z-\gamma)+2g(x-a)(z-\gamma)+2h(x-a)(y-\beta).$$

Eliminate $x-a$, $y-\beta$, and $z-\gamma$ by (1), and divide by r^2,

$$\therefore\ l^2+m^2+n^2+2fmn+2gln+2hlm=1 \ldots\ldots\ldots\ldots (12),$$

which is the relation required.

Prop. vi. *Given the equations to a straight line, to find the projecting coefficients.*

Let the equations to the straight line be

$$\frac{x-a}{S}=\frac{y-\beta}{T}=\frac{z-\gamma}{U} \ldots\ldots\ldots\ldots (13),$$

then if l, m, n be the projecting coefficients and W a quantity to be determined, we shall have

$$Wl=S, \qquad Wm=T, \qquad Wn=U \ldots\ldots\ldots\ldots (14).$$

Substitute these values of l, m, n in (12),

$$\therefore\ W^2=S^2+T^2+U^2+2fTU+2gSU+2hST \ldots\ldots (15).$$

This equation determines W, and then $l\,m\,n$ are known by (14).

Prop. vii. *Given the projecting coefficients, to find the directing cosines.*

Instead of x_2-x_1, y_2-y_1 and z_2-z_1 in (9), write $x-a$, $y-\beta$ and $z-\gamma$, and then eliminate the latter by (1),

$$\therefore\ \left.\begin{aligned} \lambda &= l+hm+gn \\ \mu &= hl+m+fn \\ \nu &= gl+fm+n \end{aligned}\right\} \ldots\ldots\ldots\ldots (16).$$

Similarly from (10) we have,

$$\lambda l+\mu m+\nu n=1 \ldots\ldots\ldots\ldots (17).$$

We can also find the directing cosines $\lambda\mu\nu$ when the equations (13) to the straight line are given, by substituting the values of lmn from (14, 15) in (16).

Prop. viii. *Given the directing cosines, to determine the projecting coefficients.*

Solving the equations (16), we have

$$\left.\begin{aligned} \Gamma^2 l &= (1-f^2)\lambda+(fg-h)\mu+(fh-g)\nu \\ \Gamma^2 m &= (fg-h)\lambda+(1-g^2)\mu+(gh-f)\nu \\ \Gamma^2 n &= (fh-g)\lambda+(gh-f)\mu+(1-h^2)\nu \end{aligned}\right\} \ldots\ldots (18).$$

Prop. ix. *To find the relation among the directing cosines.*

This is at once effected by multiplying the equations (18) by $\lambda\mu\nu$ respectively, adding and reducing by (17),

$$\therefore\ (1-f^2)\lambda^2+(1-g^2)\mu^2+(1-h^2)\nu^2 +2(gh-f)\mu\nu+2(fh-g)\lambda\nu+2(fg-h)\lambda\mu=\Gamma^2 \ldots\ldots (19).$$

PROP. X. *To find the angles (cosines $= e, i, u$) which the axes of x, y and z make with the normals* to the coordinate planes of yz, xz and xy respectively.*

Since each normal is perpendicular to two of the axes, we must evidently substitute $e, 0, 0$; $0, i, 0$; and $0, 0, u$, successively, for $\lambda\ \mu\ \nu$ in (19); we thus have

$$e^2 = \frac{\Gamma^2}{1-f^2},\quad i^2 = \frac{\Gamma^2}{1-g^2},\quad u^2 = \frac{\Gamma^2}{1-h^2} \ldots\ldots\ldots (20).$$

PROP. XI. *To determine the angles (cosines $=\Lambda\text{MN}$) which the straight line* (1) *makes with the normals to the coordinate planes.*

Take the orthogonal projections of r and of the broken line $(x-a)+(y-\beta)+(z-\gamma)$ on the normal to the plane of yz, $\therefore\ r\Lambda = (x-a)e$; hence (1)

$$\Lambda = le,\qquad \text{M} = mi,\qquad \text{N} = nu \ldots\ldots\ldots\ldots\ldots (21).$$

PROP. XII. *To find the angle* (θ) *of inclination of two straight lines.*

Let (3) and (4) be the equations to the two lines, and $\lambda\mu\nu$ and $\lambda_1\mu_1\nu_1$ their directing cosines.

Take the orthogonal projections of r and the broken line $(x-a)+(y-\beta)+(z-\gamma)$, on the straight line (4), and we get $r\cos\theta = \lambda_1(x-a)+\mu_1(y-\beta)+\nu_1(z-\gamma)$; divide this by r, and reduce by (3), $\therefore\ \cos\theta = \lambda_1 l+\mu_1 m+\nu_1 n$. Similarly we have $\cos\theta = \lambda l_1+\mu m_1+\nu n_1$,

$$\therefore\ \cos\theta = \lambda l_1+\mu m_1+\nu n_1 = \lambda_1 l+\mu_1 m+\nu_1 n \ldots\ldots\ldots\ldots (22).$$

Eliminate the directing cosines from (22) by means of (16),

$$\therefore\ \cos\theta = ll_1+mm_1+nn_1+f(mn_1+m_1n)+g(l_1n+ln_1)+h(lm_1+l_1m) \,..\, (23).$$

Also eliminating the projecting coefficients from (22) by (18), we get

$$\Gamma^2\cos\theta = (1-f^2)\lambda\lambda_1+(1-g^2)\mu\mu_1+(1-h^2)\nu\nu_1+(gh-f)(\mu\nu_1+\mu_1\nu)$$
$$+(fh-g)(\lambda_1\nu+\lambda\nu_1)+(fg-h)(\lambda\mu_1+\lambda_1\mu) \ldots\ldots\ldots\ldots (24).$$

By (12) and (23), we have

$$\sin^2\theta = 1-\cos^2\theta = \{l^2+m^2\ldots.+2hlm\}\{l_1^2+m_1^2\ldots.+2hl_1m_1\}$$
$$-\{ll_1+mm_1\ldots.+h(lm_1+l_1m)\}^2,$$

which reduces to

$$\sin^2\theta = (1-f^2)(m_1n-mn_1)^2+(1-g^2)(ln_1-l_1n)^2+(1-h^2)(l_1m-lm_1)^2$$
$$+2(gh-f)(l_1m-lm_1)(ln_1-l_1n)+2(fh-g)(m_1n-mn_1)(l_1m-lm_1)$$
$$+2(fg-h)(ln_1-l_1n)(m_1n-mn_1) \ldots. (25).$$

Also $\Gamma^4\sin^2\theta = \Gamma^4-\Gamma^4\cos^2\theta = (19, 24)$

$$\{(1-f^2)\lambda^2\ldots.+2(fg-h)\lambda\mu\}\{(1-f^2)\lambda_1^2\ldots.+2(fg-h)\lambda_1\mu_1\}$$
$$-\{(1-f^2)\lambda\lambda_1\ldots.+(fg-h)(\lambda\mu_1+\lambda_1\mu)\}^2,$$

which after a tedious reduction, takes the comparatively simple form

$$\Gamma^2\sin^2\theta = (\mu_1\nu-\mu\nu_1)^2+(\lambda\nu_1-\lambda_1\nu)^2+(\lambda_1\mu-\lambda\mu_1)^2+2f(\lambda_1\mu-\lambda\mu_1)(\lambda\nu_1-\lambda_1\nu)$$
$$+2g(\mu_1\nu-\mu\nu_1)(\lambda_1\mu-\lambda\mu_1)+2h(\mu_1\nu-\mu\nu_1)(\lambda\nu_1-\lambda_1\nu) \ldots\ldots\ldots (26).$$

* These normals are supposed to be drawn on the same side of the coordinate planes as the positive axes, and hence e, i, and u will be positive.

If the equations to the straight lines be

$$\frac{x-a}{S} = \frac{y-\beta}{T} = \frac{z-\gamma}{U} \quad\text{.................(27)},$$

$$\frac{x-a_1}{S_1} = \frac{y-\beta_1}{T_1} = \frac{z-\gamma_1}{U_1} \quad\text{.................(28)};$$

then instead of (23), we shall, (14), have

$$WW_1\cos\theta = SS_1+TT_1+UU_1+f(TU_1+T_1U)+g(SU_1+S_1U)+h(ST_1+S_1T) \quad\text{....(29)};$$

and instead of (25), $W^2W_1^2\sin^2\theta =$

$$(1-f^2)(T_1U-TU_1)^2+(1-g^2)(SU_1-S_1U)^2+(1-h^2)(S_1T-ST_1)^2$$
$$+2(gh-f)(SU_1-S_1U)(S_1T-ST_1)+2(fh-g)(T_1U-TU_1)(S_1T-ST_1)$$
$$+2(fg-h)(SU_1-S_1U)(T_1U-TU_1) \quad\text{....(30)},$$

where W has the same value as in (15), and W_1 is the same function of $S_1T_1U_1$ that W is of STU.

PROP. XIII. *Required the condition that two straight lines* (3, 4) *may be perpendicular to each other.*

Here $\cos\theta = 0$, and $\sin\theta = 1$, and hence the condition may be expressed in any one of the following ways:

$$\lambda l_1+\mu m_1+\nu n=0, \qquad \lambda_1 l+\mu_1 m+\nu_1 n=0 \quad\text{............(31)}.$$

$$ll_1+mm_1+nn_1+f(m_1n+mn_1)+g(ln_1+l_1n)+h(l_1m+lm_1)=0 \quad\text{..(32)}.$$

$$(1-f^2)\lambda\lambda_1+(1-g^2)\mu\mu_1+(1-h^2)\nu\nu_1+(gh-f)(\mu_1\nu+\mu\nu_1)+(fh-g)(\lambda\nu_1+\lambda_1\nu)$$
$$+2(fg-h)(\lambda_1\mu+\lambda\mu_1)=0 \quad\text{........(33)}.$$

$$(1-f^2)(m_1n-mn_1)^2+(1-g^2)(ln_1-l_1n)^2+(1-h^2)(l_1m-lm_1)^2$$
$$+2(gh-f)(l_1m-lm_1)(l_1n-l_1n)+2(fh-g)(m_1n-mn_1)(l_1m-lm_1)$$
$$+2(fg-h)(ln_1-l_1n)(m_1n-mn_1)=1 \quad\text{..(34)}.$$

$$(\mu_1\nu-\mu\nu_1)^2+(\lambda\nu_1-\lambda_1\nu)^2+(\lambda_1\mu-\lambda\mu_1)^2+2f(\lambda_1\mu-\lambda\mu_1)(\lambda\nu_1-\lambda_1\nu)$$
$$+2g(\mu_1\nu-\mu\nu_1)(\lambda_1\mu-\lambda\mu_1)+2h(\mu_1\nu-\mu\nu_1)(\lambda\nu_1-\lambda_1\nu)=\Gamma^2 \quad\text{....(35)}.$$

$$SS_1+TT_1+UU_1+f(TU_1+T_1U)+g(SU_1+S_1U)+h(ST_1+S_1T)=0 \quad\text{...(36)}.$$

$$(1-f^2)(T_1U-TU_1)^2+(1-g^2)(SU_1-S_1U)^2+(1-h^2)(S_1T-ST_1)^2$$
$$+2(gh-f)(SU_1-S_1U)(S_1T-ST_1)+2(fh-g)(T_1U-TU_1)(S_1T-ST_1)$$
$$+2(fg-h)(SU_1-S_1U)(T_1U-TU_1)=W^2W_1^2 \quad\text{..................(37)}.$$

PROP. XIV. *To find the conditions that two straight lines may be parallel.*

These are evidently

$$l=l_1, \quad m=m_1, \quad n=n_1 \quad\text{..................(38)},$$

or

$$\lambda=\lambda_1, \quad \mu=\mu_1, \quad \nu=\nu_1 \quad\text{..................(39)},$$

or

$$\frac{S}{S_1}=\frac{T}{T_1}=\frac{U}{U_1} \quad\text{....................(40)}.$$

NOTE.—The conditions are inadvertently given in Gregory's Solid Geometry (p. 44) in too complicated a form: namely,

$$\frac{l+gn+hm}{l_1+gn_1+hm_1}=\frac{m+fn+hl}{m_1+fn_1+hl_1}=\frac{n+fm+gl}{n_1+fm_1+gl_1},$$

which (16), are nearly equivalent to (39), and from which (38) may easily be deduced.

PROP. XV. *To find the length of the perpendicular from a given point $(x_1y_1z_1)$ on a given straight line.*

Denote the points $(\alpha\beta\gamma)$, $(x_1y_1z_1)$, and the point where the perpendicular meets the straight line by P, Q and R respectively; also put $PQ=d$, $PR=r$, and $QR=r_1$. We evidently have

$$r_1^2=d^2-r^2.$$

Also the orthogonal projection of the broken line $(x_1-\alpha)+(y_1-\beta)+(z_1-\gamma)$ on the straight line (1) is evidently r,

$$\therefore\ r=\lambda(x_1-\alpha)+\mu(y_1-\beta)+\nu(z_1-\gamma)\dots\dots\dots\dots\dots(41).$$

Also (11), $\quad d^2=(x_1-\alpha)^2+\dots+2h(x_1-\alpha)(y_1-\beta)$;

$$\therefore r_1^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2+2f(y_1-\beta)(z_1-\gamma)+2g(x_1-\alpha)(z_1-\gamma)$$
$$+2h(x_1-\alpha)(y_1-\beta)-\{\lambda(x_1-\alpha)+\mu(y_1-\beta)+\nu(z_1-\gamma)\}^2\dots\dots\dots(42),$$

which determines r_1, the perpendicular required.

It is evident from (1) that if $(x_2y_2z_2)$ denote the foot of the perpendicular, then will

$$x_2=\alpha+lr,\qquad y_2=\beta+mr,\qquad z_2=\gamma+nr\dots\dots\dots(43),$$

the value of r being as in (41).

Hence since the perpendicular passes through the points $(x_1y_1z_1)$ and and $(x_2y_2z_2)$, its equations (2) are

$$\frac{x-x_1}{lr+\alpha-x_1}=\frac{y-y_1}{mr+\beta-y_1}=\frac{z-z_1}{nr+\gamma-z_1}\dots\dots\dots(44).$$

Hence also we can find the distances $d'\ d''\ d'''$ of any point $(x_1y_1z_1)$ from the axes of x, y and z. In (42) put $\alpha=\beta=\gamma=0$, $\lambda=1$, $\mu=h$, and $\nu=g$, then

$$d'^2=x_1^2+y_1^2+z_1^2+2fy_1z_1+2gx_1z_1+2hx_1y_1-(x_1+hy_1+gz_1)^2.$$

$$\left.\begin{array}{ll} \therefore & d'^2=(1-h^2)y_1^2+(1-g^2)z_1^2-2(gh-f)\,y_1z_1 \\ \text{Similarly} & d''^2=(1-h^2)x_1^2+(1-f^2)z_1^2-2(fh-g)\,x_1z_1 \\ \text{and} & d'''^2=(1-g^2)x_1^2+(1-f^2)y_1^2-2(fg-h)x_1y_1 \end{array}\right\}\dots(45)^*.$$

PROP. XVI. *To find the perpendicular distance between two straight lines* (3, 4) *not in the same plane.*

Let $l'\ m'\ n'$ be the projecting coefficients, $\lambda'\ \mu'\ \nu'$ the directing cosines of r' the perpendicular required. By (31) we evidently have

$$\lambda'l+\mu'm+\nu'n=0,$$
$$\lambda'l_1+\mu'm_1+\nu'n_1=0.$$

From which we deduce

$$\frac{\lambda'}{m_1n-mn_1}=\frac{\mu'}{ln_1-l_1n}=\frac{\nu'}{l_1m-lm_1}=q\text{ (suppose).}$$

* By means of (45) the moments of inertia of any body (whose surface is referred to oblique axes) may be found relative to the axes of coordinates, and by (42) the moment of inertia may be found relative to any straight line.

Substitute these values of λ' μ' ν' in (19), and reduce by (25),

$\therefore\quad q = \pm\dfrac{\Gamma}{\sin\theta}$, θ being the inclination of the straight lines (3, 4),

$$\therefore\quad \frac{\lambda'}{m_1n - mn_1} = \frac{\mu'}{ln_1 - l_1n} = \frac{\nu'}{l_1m - lm_1} = \pm\frac{\Gamma}{\sin\theta} \ldots\ldots (46).$$

Moreover, a moment's consideration will make it evident that r' is the orthogonal projection of the broken line $(a-a_1)+(\beta-\beta_1)+(\gamma-\gamma_1)$ on the straight line perpendicular to both (3) and (4),

$$\therefore\quad r' = \lambda'(a-a_1)+\mu'(\beta-\beta_1)+\nu'(\gamma-\gamma_1),$$

or, (46),

$$r' = \pm\frac{\Gamma}{\sin\theta}\Big\{(m_1n-mn_1)(a-a_1)+(ln_1-l_1n)(\beta-\beta_1)+(l_1m-lm_1)(\gamma-\gamma_1)\Big\} \ldots\ldots (47).$$

The directing cosines λ' μ' ν' are known by means of (46), and thence the projecting coefficients l' m' n' may be found from (18); but the latter may be expressed in a manner analogous to (46).

By (31) we have

$$\lambda l' + \mu m' + \nu n' = 0,$$
$$\lambda_1 l' + \mu_1 m' + \nu_1 n' = 0.$$

$$\therefore\quad \frac{l'}{\mu_1\nu - \mu\nu_1} = \frac{m'}{\lambda\nu_1 - \lambda_1\nu} = \frac{n'}{\lambda_1\mu - \lambda\mu_1} = s \text{ (suppose).}$$

Substitute these values of l' m' n' in (12), and reduce by (26),

$$\therefore s = \pm\frac{1}{\Gamma\sin\theta};$$

$$\therefore\quad \frac{l'}{\mu_1\nu - \mu\nu_1} = \frac{m'}{\lambda\nu_1 - \lambda_1\nu} = \frac{n'}{\lambda_1\mu - \lambda\mu_1} = \pm\frac{1}{\Gamma\sin\theta} \ldots\ldots\ldots (48).$$

ON THE THEORY OF SYMMETRIC FUNCTIONS.

[*James Cockle, Esq., M. A., Barrister-at-Law.*]

There are properties of Symmetric Functions which — not merely of speculative interest — play an important part in the Theory of Equations. For instance, it is the occurrence of what I have proposed* to term *critical* forms of those functions which prevents us from reducing the general solution of equations of the *sixth*, to that of equations of the *fifth* degree†. On the other hand, there are similar forms which facilitate the discussion of certain problems.

In what follows it will be seen that I avail myself of the labours‡ of Mr. Jerrard on Symmetric Functions, without adhering implicitly to his notation.

Let

$$F^n(x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

denote the general equation of the n^{th} degree in x; also let

$$y = P + Qx^q + etc. \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

* Philosophical Magazine, ser. iii, vol. xxviii., p. 191. † Ibid, p. 395.

‡ Mathematical Researches, by George B. Jerrard, A.B.

then if we eliminate x between (1) and (2), we shall have a result of the form

$$y^n+p_1y^{n-1}+p_2y^{n-2}+\ldots+p_ry^{n-r}+\ldots+p_n=0\ldots.(3).$$

Now Mr. Jerrard has shown* that (in such a case as the present)

$$p_r=\frac{(-1)^r}{1.2\ldots r}\int_n(\mathrm{P}o+\mathrm{Q}q+etc.)^r\ \ldots\ldots\ldots.(4),$$

the development of which last expression, or of similar ones, will be seen on referring to Mr. Jerrard's *Researches* †.

I shall apply the relation (4) to the investigation of some interesting properties of certain symmetric functions; and, first, let us consider the expression

$$2np_2-(n-1)p_1^2\ ‡,$$

which we may denote by a. Then we have

$$a=n\int_n(\mathrm{P}o+\mathrm{Q}q+etc.)^2-(n-1)\{\int_n(\mathrm{P}o+\mathrm{Q}q+etc.)\}^2,$$

or, as we may write it,

$$a=\mathrm{AP}^2+2\mathrm{BQP}+etc.,$$

where

$$\mathrm{A}=n\int_n o^2-(n-1)\{\int_n o^2\},$$

and

$$\mathrm{B}=n\int_n o.q-(n-1)\int_n o\int_n q\,;$$

but ||

$$\int_n o^2=n(n-1),\text{ and }\int_n o=n,$$

hence,

$$\mathrm{A}=n^2(n-1)-(n-1)n^2=0:$$

also §,

$$\int_n o.q=\int_n o\int_n q-\int_n q=(n-1)\int_n q\,;$$

hence,

$$\mathrm{B}=n(n-1)\int_n q-(n-1)n\int_n q=0,$$

and P^2 and QP disappear from a. So if, in the expression for y, $\mathrm{Q}'x^{q'}$ be any term subsequent to $\mathrm{Q}x^q$ we might show, in the same manner, that Q′P disappears from a, and we may hence infer that a is *altogether free from* P, whatever be the number of terms in the expression for y, and a is, consequently, the *critical* function of the second degree.

Let us next consider the function

$$6n^2p_3-(n-1)(n-2)p_1^3\ ¶,\text{ or }\beta,$$

where

$$p_3=-\tfrac{1}{6}\int_n(\mathrm{P}o+\mathrm{Q}q+etc.)^3,$$

and

$$p_1=-\int_n(\mathrm{P}o+\mathrm{Q}q+etc.).$$

Change the sign of β, and denote the result by

$$\mathrm{A}'\mathrm{P}^3+3\mathrm{B}'\mathrm{QP}^2+etc.,$$

* Math. Res., p. 20. † Ibid, pp. 10, *et seq.* ‡ See Mathematician, vol. i. p. 115, art. 8.

|| Jerrard, Math. Res., p. 5. In my text above a small italic (o) is used instead of the arithmetical zero (0). The latter must be understood to be expressed by the former.

§ Jerrard, Math. Res., p. 57, note; or see p. 5, art. II.

¶ See Mathematician, vol. i. p. 115, art. 8.

where

$$A' = n^2 \textstyle\int_n o^3 - (n-1)(n-2)\{\int_n o\}^3,$$

and

$$B' = n^2 \textstyle\int_n o^2.q - (n-1)(n-2)\{\int_n o\}^2 \int_n q.$$

Now †

$$\textstyle\int_n o^3 = n(n-1)(n-2),$$

and §§

$$\begin{aligned}\textstyle\int_n o^2 q &= \{\textstyle\int_n o\}^2 \textstyle\int_n q - 3 \textstyle\int_n o \textstyle\int_n q + 2 \textstyle\int_n q \\ &= (n^2 - 3n + 2)\textstyle\int_n q \\ &= (n-1)(n-2)\textstyle\int_n q;\end{aligned}$$

hence

$$A' = n^3(n-1)(n-2) - (n-1)(n-2)n^3 = 0,$$

and

$$B' = n^2(n-1)(n-2)\textstyle\int_n q - (n-1)(n-2)n^2 \textstyle\int_n q = 0,$$

consequently, P^3 and QP^2 disappear from β, and we may infer (in the same manner as in the case of a) that β does not contain either P^3 or P^2.

§§ Math. Res., p. 57, note, confirming art. II. of p. 5.

† Math. Res., p. 5.

(To be continued.)

2, *Church-Yard-Court, Temple*, 3rd *July*, 1848.

ON THE TRANSFORMATION OF ALGEBRAIC EQUATIONS.

[*James Cockle, Esq., M. A., Barrister-at-Law.*]

In the first volume of this work* I have given four papers "On the Transformation of Algebraic Equations."

The first paper (Vol. I., pp. 82—84) treats of the *Method of Symmetric Products.*†

The second (Ib. pp. 113—116) treats partly (arts. 1 and 2) of the method just named, and partly (arts. 2 to 9) of the *Method of Vanishing Groups.*‡

* See Mathematician, Vol. I, pp. 82, 113, 194, and 297. The reader will be pleased to make the following corrections of typographical errors in that volume :—

Page 83, last line, *for* $x\lambda'$ *read* $x^{\lambda'}$; p. 114, art. 4, line 5, *for* h^{2b-1} *read* $h^2{}_{b-1}$; p. 195, art. 5, line 5, *for* $x^{2\lambda}x^{3\lambda'}$ *read* $x_2{}^{\lambda}x_3{}^{\lambda'}$; p. 298, art. 4, line 4, *for* ϕ_{n-2}, ϕ_{n-1} *read* $\phi_{n-2}\ \phi'_{n-1}$.

† As to the Method of Symmetric Products, consult my additional remarks upon the subject in the Philosophical Magazine, ser. iii. vol. xxvi., (pp. 383—384)—[The reader will have the kindness to refer to the "Errata and Addenda" of the last mentioned volume, (see p. viii. of its introductory part), and, having used my correction of the text of that volume, to substitute, for the statement respecting the formation of ϕ at the top of p. 83 of Vol. I. of the Mathematician, the corrected text of the Philosophical Magazine. I have to apologize to the reader for the somewhat numerous oversights committed in my four papers in the first volume of this work.] — consult also vol. xxvii., (p. 127, art. 4, and pp. 293—294, art. 7); and vol. xxviii., (p. 133, art. 6) of the same work and series; and also the Mechanics' Magazine, vol. xliv. (p. 405 and p. 406); vol. xlv., (p. 36); vol. xlvi., (pp. 492 and 517); vol. xlviii., (p. 538).

‡ For further information as to the application of the Method of Vanishing Groups to the theory of equations, see Philosophical Magazine, s. iii., vol. xxvii., pp. 125—127, and pp. 292,

This paper concludes (p. 116) with a notice of a certain soluble form of equations.

The third paper (Ib. pp. 194—197) treats in its first part (arts. 1 to 5) of the Method first above mentioned. Its latter portion (arts. 6 to 8) is occupied with the subject of cubics. The arithmetical process given in article 8 will probably be found of practical use in ascertaining the nature of the roots of a given cubic equation.

Articles 1 to 8, of the fourth paper, (Ibid, pp. 297—301) are occupied with the Method of Symmetric Products; and article 8 also with a generalization and application to certain algebraic expressions of the "arithmetical process" just mentioned; and article 9 with some remarks on a modification § of Mr. Jerrard's Method, on a property of a generalized form for the root of an equation, and on the limits || of the application of the Method of Vanishing Groups.

With reference to articles 4 and 5 of this last paper, I would observe that, since the respective coefficients of x_1, $x_2 \ldots$, x_n, are the same, in ϕ, as those of x'_1, $x'_2 \ldots x'_n$ respectively in ϕ', we may at once infer that

$$m_1 = m'_1 = m''_1 = etc.,$$

and also that

$$\mu_1 = \mu'_1 = \mu''_1 = etc.,$$

and hence, by the aid of the conditions

$$\Sigma(m_1) = \Sigma(\mu_1) = etc.,$$

we obtain

$$m_1 = \mu_1 = etc.,$$

so we obtain

$$m_2 = \mu_2 = etc.,$$

$$etc. \qquad etc.$$

$$m_{n-1} = \mu_{n-1} = etc.;$$

whence,

$$m_1 + m_2 + \ldots + m_{n-1} = \mu_1 + \mu_2 + \ldots + \mu_{n-1} = etc.$$

and the symmetry of π' is proved.

2, *Church-Yard-Court, Temple, 4th July*, 1848.

293; vol. xxviii., pp. 132—133, 190—191, and 395; vol. xxx., pp. 28—30; as to the limits of the Method, see vols. xxviii. and xxx. (pages just cited) and vol. xxix., pp. 181—182; and also Mechanics' Magazine, vol. xliv., pp. 404, 405; vol. xlv., p. 36; and vol. xlvi., pp. 104, 105, and p. 422: as to the ulterior development of the method, see Phil. Mag., s. iii., vol. xxviii., pp. 395—396, and vol. xxix., pp. 182—183: Cambridge and Dublin Mathematical Journal, vol. ii., pp. 267—273; and vol. iii., pp. 179—181; and the Mathematician, vol. ii., p. 97. The application of the Method of Vanishing Groups to Analytical Geometry will be seen on referring to my "Chapters" on that subject in vols. xlv., xlvi., xlvii., xlviii., of the Mechanics' Magazine. The actual and practical "Process" of the Method (when applied to quadratic functions) will be found at pp. 504—505, of vol. xlvii., of the Mechanics' Magazine; and I have given an "Account" of the Method at pp. 114—119 of s. iii. of the Philosophical Magazine.

§ This modification however, as I have since found, becomes inoperative owing to the occurrence of critical functions.

|| But as to these limits see the places cited in note ‡.

NEW METHOD OF RESOLVING A COMPLETE CUBIC EQUATION WITHOUT REMOVING THE SECOND TERM BY TRANSFORMATION.

[*Dr. Rutherford.*]

The following method of resolving a cubic equation may not be without interest to some of the readers of the Mathematician, especially as it applies equally to every form of a cubic, whether complete in all its terms, or deficient of either its second or third terms.

Let the proposed cubic equation be of the form $x^3 + ax^2 + bx + c = 0$, and let us assume

$$x^3+ax^2+bx+c=\frac{m^3(x+y)^3-(x+z)^3}{m^3-1}\ldots\ldots\ldots\ldots(1).$$

Developing the second side of (1), we get

$$x^3+ax^2+bx+c=x^3+\frac{3(m^3y-z)}{m^3-1}x^2+\frac{3(m^3y^2-z^2)}{m^3-1}x+\frac{m^3y^3-z^3}{m^3-1}.$$

Equating the coefficients of the same powers of x on both sides, gives

$$\frac{m^3y-z}{m^3-1}=\frac{a}{3};\quad \frac{m^3y^2-z^2}{m^3-1}=\frac{b}{3};\quad \frac{m^3y^3-z^3}{m^3-1}=c.$$

For the sake of symmetry, let a' and b' be substituted for $\frac{1}{3}a$ and $\frac{1}{3}b$ respectively; then from these three equations, we derive

$$m^3=\frac{z-a'}{y-a'}=\frac{z^2-b'}{y^2-b'}=\frac{z^3-c}{y^3-c}\ldots\ldots\ldots\ldots(2).$$

Equating the first and second of these values of m^3, and likewise the first and third; and simplifying the results by dividing by a common factor $y-z$, we get

$$yz+b'=a'(y+z)\ldots\ldots\ldots\ldots(3),$$

$$yz(y+z)+c=a'(y^2+yz+z^2)\ldots\ldots\ldots\ldots(4).$$

Multiply (3) by $y+z$, and from the product subtract (4), then

$$a'yz+c=b'(y+z)\ldots\ldots\ldots\ldots(5).$$

From (3) and (5) we obtain

$$y+z=\frac{a'b'-c}{a'^2-b'},\text{ and } yz=\frac{b'^2-a'c}{a'^2-b'}\ldots\ldots\ldots\ldots(6).$$

Hence the values of y and z, found in the usual manner by quadratics, are

$$\left.\begin{aligned} y&=\frac{a'b'-c+\sqrt{\{(a'b'-c)^2+4(a'^2-b')(a'c-b'^2)\}}}{2(a'^2-b')}\\ z&=\frac{a'b'-c-\sqrt{\{(a'b'-c)^2+4(a'^2-b')(a'c-b'^2)\}}}{2(a'^2-b')}\end{aligned}\right\}\ldots\ldots(7).$$

The values of y and z being found, we thence get the value of m from the equation

$$m^3=\frac{z-a'}{y-a'}\ldots\ldots\ldots\ldots(8).$$

And since the first side of (1) is equal to 0, it is obvious that the numerator of the second side will also be equal to 0, and hence

$$m^3(x+y)^3=(x+z)^3, \text{ or } m(x+y)=x+z;$$

$$\therefore x=\frac{z-my}{m-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

But from (8) we get $z=a'+m^3(y-a')$, and substituting this value for z in (9), we eliminate z, and find

$$x=\frac{a'+m^3y-m^3a'-my}{m-1}=\frac{my(m^2-1)-a'(m^3-1)}{m-1},$$

or $\quad x=m(m+1)y-a'(m^2+m+1)$;

$\therefore \quad x=m(m+1)(y-a')-a'\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$

If we had eliminated y between (8) and (9) instead of z, we should have found

$$x=\frac{(m+1)(z-a')}{m^2}-a'\ldots\ldots\ldots\ldots\ldots(11).$$

We have thus obtained an expression for determining the value of x, *without* transforming the complete equation to another which shall want the second power of the unknown quantity. This transformation must always be effected before CARDAN's method of solution can be applied, but by the preceding method all transformation is dispensed with, and its application to the solution of numerical equations is both simple and elegant. If the form of the equation be $x^3+ax-b=0$; then the preceding method gives the following simplified results, viz.,

$$y+z=-\frac{3b}{a}\ldots\ldots(\alpha), \qquad yz=-\frac{a}{3}\ldots\ldots(\beta),$$

$$m=\left(\frac{z}{y}\right)^{\frac{1}{3}}\ldots\ldots(\gamma), \qquad x=m(m+1)y\ldots\ldots(\delta).$$

If the form of the equation be $x^3+ax^2-b=0$; we have

$$y+z=\frac{9b}{a^2}\ldots\ldots(\alpha'), \qquad yz=\frac{3b}{a}\ldots\ldots(\beta'),$$

$$m^3=\frac{z-a'}{y-a'}\ldots\ldots(\gamma'), \qquad x=m(m+1)(y-a')-a'\ldots(\delta').$$

These results are very remarkable, and we shall only add an example or two by way of application.

Ex. 1. Given the cubic equation $x^3-6x^2+18x-22=0$, to find x.

Here $\quad a'=-2, \quad b'=6, \quad c=-22$, and therefore by equations (6),

$$y+z=-5, \quad \text{and } yz=4;$$

whence we get $y=-1$ and $z=-4$; therefore

$$m^3=\frac{z-a'}{y-a'}=-2, \text{ and } m=-\sqrt[3]{2};$$

whence $\quad x=m(m+1)(y-a')-a'=-\sqrt[3]{2}(-\sqrt[3]{2}+1)+2$

$$=2+\sqrt[3]{4}-\sqrt[3]{2}.$$

Ex. 2. Given the equation $x^3+9x=6$, to find x.

Here $a=9$, and $b=6$, therefore by the formulæ $(\alpha,\beta,\gamma,\delta)$

$$y+z=-2 \text{ and } yz=-3;$$

whence $y=1$ and $z=-3$; therefore $m=-\sqrt[3]{3}$,

and $x=m(m+1)y=-\sqrt[3]{3}(-\sqrt[3]{3}+1)=\sqrt[3]{9}-\sqrt[3]{3}$.

R. M. Academy, 24th August, 1848.

PROPOSITIONS ON THE TETRAHEDRON.

[*Lieut. A. R. Clarke, R. E.*]

To determine some of the relations among the inscribed and escribed spheres of a given irregular tetrahedron.

Let $\Delta_1\Delta_2\Delta_3\Delta_4$ be the faces; r the radius of the inscribed sphere; $r_1r_2r_3r_4$ the radii of the escribed spheres (viz. four spheres, each of which touches one face and the other three produced). Let $p_1p_2p_3p_4$ be the perpendiculars from the opposite points on $\Delta_1\Delta_2\Delta_3\Delta_4$ respectively. Let V be the volume; 3S the surface of the solid.

Then we have the following equations, viz.

$$\left.\begin{aligned}\Delta_1p_1=\Delta_2p_2=\Delta_3p_3=\Delta_4p_4=3V\\ \Delta_1+\Delta_2+\Delta_3+\Delta_4=3S\end{aligned}\right\}\ \ldots\ldots\ldots\ldots (a).$$

1°. To find expressions for the radii r, r_1, r_2, *etc.*

If the centre of the inscribed sphere be joined with the angular points of the tetrahedron, we shall have the solid divided into four others, and

$$r(\Delta_1+\Delta_2+\Delta_3+\Delta_4)=3V;$$

$$\therefore\ r=\frac{V}{S}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Similarly, we get

$$r_1(-\Delta_1+\Delta_2+\Delta_3+\Delta_4)=3V;$$

or $$r_1(3S-2\Delta_1)=3V;$$

$$\left.\begin{aligned}\therefore\quad r_1&=\frac{V}{S-\frac{2}{3}\Delta_1}\\ \text{and}\quad r_2&=\frac{V}{S-\frac{2}{3}\Delta_2}\\ r_3&=\frac{V}{S-\frac{2}{3}\Delta_3}\\ r_4&=\frac{V}{S-\frac{2}{3}\Delta_4}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

2°. Having found expressions for the radii, we now proceed to determine some relations among them and the perpendiculars.

Taking the sum of the reciprocals of (c), we get

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{r}\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

Also by (a)

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}=\frac{1}{r}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(e).$$

By (c) and (a)

$$\left.\begin{aligned}-\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}&=\frac{4}{p_1}\\ \frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}&=\frac{4}{p_2}\\ \frac{1}{r_1}+\frac{1}{r_2}-\frac{1}{r_3}+\frac{1}{r_4}&=\frac{4}{p_3}\\ \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}&=\frac{4}{p_4}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(f).$$

And by (d), (e)

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{p_1}+\frac{2}{p_2}+\frac{2}{p_3}+\frac{2}{p_4}\ldots\ldots\ldots\ldots(g).$$

$$\left.\begin{aligned}\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}&=\frac{1}{r}\\ -\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}&=\frac{1}{r_1}\\ \frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}&=\frac{1}{r_2}\\ \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}+\frac{1}{p_4}&=\frac{1}{r_3}\\ \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-\frac{1}{p_4}&=\frac{1}{r_4}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(h).$$

From this set we immediately obtain the following,

$$\left.\begin{aligned}\frac{1}{r}-\frac{1}{r_1}&=\frac{2}{p_1}\\ \frac{1}{r}-\frac{1}{r_2}&=\frac{2}{p_2}\\ \frac{1}{r}-\frac{1}{r_3}&=\frac{2}{p_3}\\ \frac{1}{r}-\frac{1}{r_4}&=\frac{2}{p_4}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(k),$$

$$\left.\begin{aligned}\frac{1}{r_1}+\frac{1}{r_2}&=\frac{2}{p_3}+\frac{2}{p_4}\\ \frac{1}{r_2}+\frac{1}{r_3}&=\frac{2}{p_1}+\frac{2}{p_4}\\ \frac{1}{r_3}+\frac{1}{r_4}&=\frac{2}{p_1}+\frac{2}{p_2}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(l),$$

etc.

$$\left.\begin{aligned}\frac{1}{r_1}-\frac{1}{r_2}&=\frac{2}{p_2}-\frac{2}{p_1}\\ \frac{1}{r_2}-\frac{1}{r_3}&=\frac{2}{p_3}-\frac{2}{p_2}\\ \frac{1}{r_3}-\frac{1}{r_4}&=\frac{2}{p_4}-\frac{2}{p_3}\end{aligned}\right\}\cdots\cdots\cdots\cdots(m).$$

etc.

From the last two sets we get

$$\left.\begin{aligned}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_3}+\frac{1}{r_4}\right)&=4\left(\frac{1}{p_1}+\frac{1}{p_2}\right)\left(\frac{1}{p_3}+\frac{1}{p_4}\right)\\ \left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(\frac{1}{r_3}-\frac{1}{r_4}\right)&=4\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\left(\frac{1}{p_3}-\frac{1}{p_4}\right)\end{aligned}\right\}\ldots(n).$$

etc. *etc.*

Hence, dividing

$$\left.\frac{(r_1-r_2)(r_3-r_4)}{(r_1+r_2)(r_3+r_4)}=\frac{(p_1-p_2)(p_3-p_4)}{(p_1+p_2)(p_3+p_4)}\right\}\cdots\cdots(p).$$

etc. *etc.*

Subtracting the squares of (m) from the squares of (l) we find expressions for $\frac{1}{r_1r_2}$, $\frac{1}{r_2r_3}$, *etc.*; and hence the following sets,

$$\left.\begin{aligned}\frac{1}{r_1r_2}+\frac{1}{r_3r_4}&=\frac{4}{p_1p_2}+\frac{4}{p_3p_4}\\ \frac{1}{r_1r_3}+\frac{1}{r_2r_4}&=\frac{4}{p_1p_3}+\frac{4}{p_2p_4}\\ \frac{1}{r_1r_4}+\frac{1}{r_2r_3}&=\frac{4}{p_1p_4}+\frac{4}{p_2p_3}\end{aligned}\right\}\cdots\cdots\cdots\cdots(q),$$

$$\left.\begin{aligned}\frac{1}{r_1r_2}-\frac{1}{r_3r_4}&=2\left(-\frac{1}{p_1^2}-\frac{1}{p_2^2}+\frac{1}{p_3^2}+\frac{1}{p_4^2}\right)\\ \frac{1}{r_1r_3}-\frac{1}{r_2r_4}&=2\left(-\frac{1}{p_1^2}+\frac{1}{p_2^2}-\frac{1}{p_3^2}+\frac{1}{p_4^2}\right)\\ \frac{1}{r_1r_4}-\frac{1}{r_2r_3}&=2\left(-\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}-\frac{1}{p_4^2}\right)\end{aligned}\right\}\cdots\cdots(r).$$

By adding (q),

$$\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_2r_4}+\frac{1}{r_3r_4}$$
$$=4\left(\frac{1}{p_1p_2}+\frac{1}{p_2p_3}+\frac{1}{p_1p_3}+\frac{1}{p_1p_4}+\frac{1}{p_2p_4}+\frac{1}{p_3p_4}\right)\ldots(s).$$

The sum of the squares of (k) gives

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}=4\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}+\frac{1}{p_4^2}\right)\ldots(t).$$

From (q) by division

$$\left.\begin{aligned}\frac{r_1r_2+r_3r_4}{r_1r_3+r_2r_4}&=\frac{p_1p_2+p_3p_4}{p_1p_3+p_2p_4}\\ \frac{r_1r_2+r_3r_4}{r_1r_4+r_2r_3}&=\frac{p_1p_2+p_3p_4}{p_1p_4+p_2p_3}\\ \frac{r_1r_3+r_2r_4}{r_1r_4+r_2r_3}&=\frac{p_1p_3+p_2p_4}{p_1p_4+p_2p_3}\end{aligned}\right\}\dots\dots\dots\dots(u).$$

From (f) by multiplication,

$$\left.\begin{aligned}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^2-\left(\frac{1}{r_3}-\frac{1}{r_4}\right)^2&=\frac{16}{p_3p_4}\\ \left(\frac{1}{r_1}+\frac{1}{r_3}\right)^2-\left(\frac{1}{r_2}-\frac{1}{r_4}\right)^2&=\frac{16}{p_2p_4}\\ \left(\frac{1}{r_2}+\frac{1}{r_3}\right)^2-\left(\frac{1}{r_1}-\frac{1}{r_4}\right)^2&=\frac{16}{p_1p_4}\end{aligned}\right\}\dots\dots\dots\dots(v).$$

etc.

Writing out the squares at full length, and substituting from (q),

$$\left.\begin{aligned}\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{1}{r_3^2}-\frac{1}{r_4^2}&=8\left(\frac{1}{p_3p_4}-\frac{1}{p_1p_2}\right)\\ \frac{1}{r_1^2}-\frac{1}{r_2^2}+\frac{1}{r_3^2}-\frac{1}{r_4^2}&=8\left(\frac{1}{p_2p_4}-\frac{1}{p_3p_1}\right)\\ -\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}-\frac{1}{r_4^2}&=8\left(\frac{1}{p_1p_4}-\frac{1}{p_2p_3}\right)\end{aligned}\right\}\dots\dots(w).$$

By (k), and reducing

$$\frac{8}{p_1p_2p_3}+\frac{1}{r_1r_2r_3}=\frac{1}{r}\left(\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_2r_3}+\frac{1}{r_1r}-\frac{1}{r^2}\right)\dots(x).$$

etc.

If $\Sigma\left(\frac{1}{p_1p_2p_3}\right)$ denote the sum of the products of the reciprocals of perpendiculars taken three and three; $\Sigma\left(\frac{1}{r_1r_2r_3}\right)$ the same function of $r_1r_2r_3r_4$; $\Sigma\left(\frac{1}{r_1r_2}\right)$ the sum of products two and two,

$$8\,\Sigma\left(\frac{1}{p_1p_2p_3}\right)+\Sigma\left(\frac{1}{r_1r_2r_3}\right)=\frac{2}{r}\left\{\Sigma\left(\frac{1}{r_1r_2}\right)-\frac{1}{r^2}\right\}.$$

A great many other very neat properties may be found.

PROP. II.

Four spheres are described each having its centre in one face of the tetrahedron and touching the other three.

Let $\rho_1\rho_2\rho_3\rho_4$ be their radii; then it will be easily seen that

$$\rho_1(\Delta_2+\Delta_3+\Delta_4)=3\mathrm{V};$$

hence
$$\left.\begin{aligned}\frac{1}{\rho_1}&=\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}\\ \frac{1}{\rho_2}&=\frac{1}{p_1}+\frac{1}{p_3}+\frac{1}{p_4}\\ \frac{1}{\rho_3}&=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_4}\\ \frac{1}{\rho_4}&=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\end{aligned}\right\}\dots\dots\dots\dots(a).$$

By means of this equation and (h),

$$\left.\begin{aligned}\frac{1}{\rho_1}&=\frac{1}{p_1}+\frac{1}{r_1}\\ \frac{1}{\rho_2}&=\frac{1}{p_2}+\frac{1}{r_2}\\ \frac{1}{\rho_3}&=\frac{1}{p_3}+\frac{1}{r_3}\\ \frac{1}{\rho_4}&=\frac{1}{p_4}+\frac{1}{r_4}\end{aligned}\right\}\dots\dots\dots\dots(\beta),$$

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}=\frac{3}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)=\frac{3}{r}\dots\dots(\gamma),$$

$$\left.\begin{aligned}\frac{1}{\rho_1}-\frac{1}{\rho_2}&=\frac{1}{p_2}-\frac{1}{p_1}\\ \frac{1}{\rho_2}-\frac{1}{\rho_3}&=\frac{1}{p_3}-\frac{1}{p_2}\end{aligned}\right\}\dots\dots\dots\dots(\delta).$$

etc:

From (a) and (r),

$$\left.\begin{aligned}\frac{1}{\rho_1\rho_2}-\frac{1}{\rho_3\rho_4}+\frac{1}{p_1p_2}-\frac{1}{p_3p_4}&=\frac{1}{2}\left(\frac{1}{r_1r_2}-\frac{1}{r_3r_4}\right)\\ \frac{1}{\rho_1\rho_3}-\frac{1}{\rho_2\rho_4}+\frac{1}{p_1p_3}-\frac{1}{p_2p_4}&=\frac{1}{2}\left(\frac{1}{r_1r_3}-\frac{1}{r_2r_4}\right)\\ \frac{1}{\rho_1\rho_4}-\frac{1}{\rho_2\rho_3}+\frac{1}{p_1p_4}-\frac{1}{p_2p_3}&=\frac{1}{2}\left(\frac{1}{r_1r_4}-\frac{1}{r_2r_3}\right)\end{aligned}\right\}\dots(\epsilon).$$

Numerous other neat relations may also be obtained.

PROP. III.

To determine an expression for the radius of the circumscribing sphere of a given tetrahedron.

Let A, B, C, D be the angular points opposite $\Delta_1\Delta_2\Delta_3\Delta_4$ respectively. Let the sides of the triangle ABC be a, b, c, and let $DA=d$, $DB=e$, $DC=f$. Let M be the middle point of BC, O_1O_4 the centres of the circumscribing circles of $\Delta_1\Delta_4$, through which draw perpendiculars meeting in O the centre of the sphere required, then $OB = R = \sqrt{OM^2 + \frac{1}{4}a^2}$.

Now if α be the dihedral angle at the edge BC, it may be easily shewn that

$$MO = \frac{O_1O_4}{\sin\alpha}.$$

$$MO_1 = a\,\frac{e^2+f^2-a^2}{8\Delta_1}, \quad MO_4 = a\,\frac{b^2+c^2-a^2}{8\Delta_4}.$$

Also the perpendicular from D on Δ_4 is

$$\frac{2\Delta_1}{a}\sin\alpha\,;$$

$$\therefore \quad V = \tfrac{1}{3}.\frac{2\Delta_1}{a}\sin\alpha.\,\Delta_4 = \tfrac{2}{3}.\frac{\Delta_1\Delta_4}{a}\sin\alpha \,\ldots\ldots\ldots (A),$$

$$\text{or} \qquad \sin\alpha = \frac{3aV}{2\Delta_1\Delta_4}.$$

By spherical trigonometry,

$$\cos\alpha = \frac{\cos ABD - \cos ABC\cos DBC}{\sin ABC\sin DBC}\,;$$

and by substitution, this becomes

$$\cos\alpha = \frac{2a^2(e^2+c^2-d^2)-(e^2+a^2-f^2)(a^2+c^2-b^2)}{16\Delta_1\Delta_4}.$$

Substituting in (A) we get

$$V = \frac{1}{12}\sqrt{\{a^2d^2(-a^2+b^2+c^2-d^2+e^2+f^2)+b^2e^2(a^2-b^2+c^2+d^2-e^2+f^2) +c^2f^2(a^2+b^2-c^2+d^2+e^2-f^2)-a^2b^2c^2-c^2d^2e^2-a^2e^2f^2-b^2d^2f^2\}}.$$

Again;

$$MO^2 = \frac{1}{\sin^2\alpha}\left\{MO_1^2+MO_4^2-2MO_1MO_4\cos\alpha\right\}.$$

Making the proper substitutions and reducing, the result is

$$R = \frac{1}{24V}\left\{2a^2d^2.b^2e^2 + 2a^2d^2.c^2f^2 + 2c^2f^2.b^2e^2 - a^4d^4 - b^4e^4 - c^4f^4\right\}^{\frac{1}{2}},$$

which may be put in the following form:

If $r_1r_2r_3$ be the rectangles of the pairs of opposite edges in any tetrahedron; r half the sum; V the volume of the solid; then the radius of the circumscribing sphere is expressed by

$$\frac{1}{6V}\sqrt{r(r-r_1)(r-r_2)(r-r_3)}.$$

PROP. IV.

Given a solid angle of a tetrahedron and the edges containing it, to find an expression for the area of the opposite face.

Given def and the angles $\alpha\beta\gamma$ contained by them, to find Δ_4.

$$a^2 = e^2 + f^2 - 2ef\cos\alpha,$$
$$b^2 = d^2 + f^2 - 2df\cos\beta,$$
$$c^2 = d^2 + e^2 - 2de\cos\gamma;$$
$$16\Delta_4^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4.$$

Substituting in the fourth equation the values of a, b, c, given in the other three, reducing and putting $(d), (e), (f)$ for the dihedral angles of those edges, the result is

$$\Delta_4^2 = \Delta_1^2 + \Delta_2^2 + \Delta_3^2 - 2\Delta_2\Delta_3\cos(d) - 2\Delta_1\Delta_3\cos(e) - 2\Delta_1\Delta_2\cos(f).$$

If D be a right angle,

$$\Delta_4^2 = \Delta_1^2 + \Delta_2^2 + \Delta_3^2.$$

PROP V.

Required an expression for the distance of an angular point from the centres of the inscribed and escribed circles of the opposite face.

The distance of the inscribed centre of Δ_4 from D is

$$d^2 = \left(\frac{2\Delta_1}{a}\cos\alpha - r\right)^2 + \left(\frac{2\Delta_1}{a}\sin\alpha\right)^2 + \left(\frac{f^2 - e^2 + ac - ab}{2a}\right)^2.$$

Substituting the value of $\cos\alpha$ obtained in (III), the result after a long reduction is found to be

$$d^2 = \frac{ad^2 + be^2 + cf^2}{a+b+c} - \frac{abc}{a+b+c};$$

Similarly, $$d_1^2 = \frac{-ad^2 + be^2 + cf^2}{-a+b+o} + \frac{abc}{-a+b+c},$$

$$d_2^2 = \frac{ad^2 - be^2 + cf^2}{a-b+c} + \frac{abc}{a-b+c},$$

$$d_3^2 = \frac{ad^2 + be^2 - cf^2}{a+b-c} + \frac{abc}{a+b-c}.$$

In a similar way we get (if d_0 represent the distance of the centre of the circumscribing circle of Δ_4 from D),

$$d_0^2 = \frac{a^2d^2(-a^2+b^2+c^2) + b^2e^2(a^2-b^2+c^2) + c^2f^2(a^2+b^2-c^2)}{16\Delta_4^2} - R_4^2.$$

PROP. VI.

The sum of the squares of the distances of the inscribed and escribed circles of a plane triangle from a point in space, is equal to four times the square of the distance of the circumscribing circle together with twelve times the square of its radius.

By the addition of the values of d^2, d_1^2, d_2^2, d_3^2 in (V) we get

$$d^2+d_1^2+d_2^2+d_3^2=4\,\frac{a^2d^2(-a^2+b^2+c^2)+b^2e^2(a^2-b^2+c^2)+c^2f^2(a^2+b^2-c^2)}{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$
$$+\,2R_4(r_1+r_2+r_3-r)$$
$$=4(d_0^2+R_4^2)+2R_4.4R_4;$$
$$\therefore\ d^2+d_1^2+d_2^2+d_3^2=4d_0^2+12R_4^2.$$

If $d_0=0$,

$$d^2+d_1^2+d_2^2+d_3^2=12R_4^2,$$

which is a known relation.

Chatham, August, 1848.

A PLAIN AND SIMPLE DETERMINATION OF THE VARIABLE ELEMENTS OF A DISTURBED PLANET.

[*The Rev. Brice Bronwin.*]

The following method of finding the variation of the elements of a planet appears to me more plain and simple than any with which I am acquainted. The eccentric anomaly w gives us the equations

$$w-e\sin w=\int ndt+\epsilon-\varpi,\quad r=a(1-e\cos w).$$

From the first of these we have

$$(1-e\cos w)\frac{dw}{de}-\sin w=0,\ \text{ or }\ \frac{dw}{de}=\frac{\sin w}{1-e\cos w}=\frac{a\sin w}{r}.$$

The second gives $\dfrac{dr}{de}=ae\sin w\dfrac{dw}{de}-a\cos w=\dfrac{a^2e}{r}\sin^2 w-a\cos w.$

But, v being the true longitude, we have by the nature of the ellipse,

$$a\cos w-ae=r\cos(v-\varpi),\ a\sqrt{1-e^2}\sin w=r\sin(v-\varpi).$$

Therefore $$\frac{dr}{de}=\frac{re\sin^2(v-\varpi)}{1-e^2}-ae-r\cos(v-\varpi)$$

$$=r\left\{\frac{e\sin^2(v-\varpi)}{1-e^2}-\frac{ae}{r}-\cos(v-\varpi)\right\}=-\frac{r\cos(v-\varpi)}{1-e^2}\left\{1+e\cos(v-\varpi)\right\}$$
$$=-a\cos(v-\varpi).$$

From the first of the above relations between w and v we have

$$a\sin w\frac{dw}{de}+a=-\cos(v-\varpi)\frac{dr}{de}+r\sin(v-\varpi)\frac{dv}{de}.$$

Substituting for $\dfrac{dw}{de}$, $\dfrac{dr}{de}$, and $\sin w$, and dividing by $\sin(v-\varpi)$, there results

$\left(\frac{r}{1-e^2}+a\right)\sin(v-\omega)=r\frac{dv}{de}$, and $\frac{dv}{de}=\left(\frac{a}{r}+\frac{1}{1-e^2}\right)\sin(v-\omega)=$ $\left(\frac{1}{r}+\frac{\mu}{h^2}\right)a\sin(v-\omega)$, since $h^2=\mu a(1-e^2)$. We have therefore

$$\frac{dr}{de}=-a\cos(v-\omega),\frac{dv}{de}=\left(\frac{1}{r}+\frac{\mu}{h^2}\right)a\sin(v-\omega)\ \ldots\ldots(a).$$

Let θ be the longitude of the node on the fixed plane, θ' its longitude on the plane of the orbit, and i the inclination. Then by making the plane of the orbit revolve a small space about r, we shall easily see that

$$d\theta'=\cos i d\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b).$$

Suppose the disturbing force perpendicular to the plane of the orbit to act by impulses, and make the velocity it communicates at the instant t the sine of an angle, the radius of which is the velocity in the orbit in its altered position; then the change produced in the velocity along the curve by this change of position will be the versed sine of this angle. Consequently it is an infinitesimal of the second order, and to be neglected. In the same manner it will appear that r is not affected by this force. Therefore the motion on the plane of the orbit is the same as if that plane were fixed.

Now x and y being the rectangular coordinates on the plane of the orbit, and R the perturbating function, which being a function of distances only will be the same for every plane, we have

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}+\frac{d\mathrm{R}}{dx}=0,\ \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}+\frac{d\mathrm{R}}{dy}=0\ .\ \ldots(c).$$

We may make the values of x and y given by $\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0$, $\frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0$, to solve these, if we make the arbitraries or elements to vary, and theoretically in an infinite number of ways. But d operating on t only and δ on the elements, make $\delta x=0$, $\delta y=0$. Then the variable ellipse and the orbit will have a contact of the first order, and $\frac{dx}{dt}$, $\frac{dy}{dt}$ will have the same values in each. And we must change $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in (c) into $\frac{d^2x}{dt^2}+\frac{\delta dx}{dt^2}=-\frac{\mu x}{r^3}+\frac{\delta dx}{dt^2}$ and $\frac{d^2y}{dt^2}+\frac{\delta dy}{dt^2}=-\frac{\mu y}{r^3}+\frac{\delta dy}{dt^2}$; which will reduce (c) to

$$\frac{\delta dx}{dt^2}+\frac{d\mathrm{R}}{dx}=0,\quad \frac{\delta dy}{dt^2}+\frac{d\mathrm{R}}{dy}=0\ \ldots\ldots(d).$$

Whatever be the system of coordinates, we have plainly $\delta\mathrm{R}=0$, and therefore relative to any two systems $\Delta\delta\mathrm{R}=0$. Consequently

$$\left.\begin{aligned}\frac{d\mathrm{R}}{da}\delta a+\frac{d\mathrm{R}}{de}\delta e+\frac{d\mathrm{R}}{d\omega}\delta\omega+\frac{d\mathrm{R}}{d\epsilon}\delta\epsilon=0\\ \frac{d\mathrm{R}}{di}\delta i+\frac{d\mathrm{R}}{d\theta}\delta\theta=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(e).$$

Let ϕ be the longitude on the fixed plane, u the latitude. Then R is a function of ϕ and u, or of v, i, and θ. But the latter being independent quantities, we have necessarily

$$\frac{dR}{d\phi}\,\frac{d\phi}{dv}+\frac{dR}{du}\,\frac{du}{dv}=\frac{dR}{dv};$$

$$\frac{dR}{d\phi}\,\frac{d\phi}{d\theta}+\frac{dR}{du}\,\frac{du}{d\theta}=\frac{dR}{d\theta}.$$

Multiply the first by $\cos i$, and add it to the latter. We have

$$\frac{dR}{d\phi}\left(\cos i\,\frac{d\phi}{dv}+\frac{d\phi}{d\theta}\right)+\frac{dR}{du}\left(\cos i\,\frac{du}{dv}+\frac{du}{d\theta}\right)=\cos i\,\frac{dR}{dv}+\frac{dR}{d\theta}.$$

Spherical trigonometry gives us $\tan(\phi-\theta)=\cos i\tan(v-\theta')$, and $\sin u=\sin i\sin(v-\theta')$. Hence we easily find, having regard to (b),

$$\cos i\,\frac{d\phi}{dv}+\frac{d\phi}{d\theta}=1,\quad \cos i\,\frac{du}{dv}+\frac{du}{d\theta}=0.$$

Therefore

$$\frac{dR}{d\phi}=\cos i\,\frac{dR}{dv}+\frac{dR}{d\theta},\quad\text{or}\quad \frac{dR}{d\phi}-\cos i\,\frac{dR}{dv}=\frac{dR}{d\theta}\ldots\ldots(f).$$

Operating with δ on

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}-\frac{2\mu}{r}+\frac{\mu}{a}=0,\quad \delta a=\frac{2a^2}{\mu}\left(dx\,\frac{\delta dx}{dt^2}+dy\,\frac{\delta dy}{dt^2}\right)=$$

$$-\frac{2a^2}{\mu}\left(\frac{dR}{dx}\,dx+\frac{dR}{dy}\,dy\right)=-\frac{2a^2}{\mu}\,\frac{dR}{ndt}\,ndt=-\frac{2na^2}{\mu}\,\frac{dR}{d\epsilon}\,dt\ldots.(1).$$

In like manner from $x\,\dfrac{dy}{dt}-y\,\dfrac{dx}{dt}=h$ we have

$$\delta h=x\,\frac{\delta dy}{dt}-y\,\frac{\delta dx}{dt}=\left(y\,\frac{dR}{dx}-x\,\frac{dR}{dy}\right)dt.$$

But $$\frac{dR}{dx}\,dx+\frac{dR}{dy}\,dy=\frac{dR}{dr}\,dr+\frac{dR}{dv}\,dv.$$

Substitute for dx and dy their values given by $x=r\cos v$, $y=r\sin v$, and compare the coefficients of dr, dv; we thus find

$$\delta h=-\frac{dR}{dv}\,dt=-\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right)dt\ldots\ldots\ldots\ldots(2),$$

for $\epsilon-\varpi$ is in all the periodical terms.

Again $h^2=\mu a(1-e^2)$, $2h\delta h=\mu(1-e^2)\delta a-2\mu ae\delta e$; make

$$k=1-\sqrt{1-e^2},\text{ and }\delta e=\frac{h}{\mu ae}\left(\frac{dR}{d\varpi}+k\,\frac{dR}{d\epsilon}\right)dt\ldots\ldots(3).$$

Marking x, y and h with an accent for the fixed plane,

$$x'\,\frac{dy'}{dt}-y'\,\frac{dx'}{dt}=h'.$$

Whence as before $\delta h'=-\dfrac{dR}{d\phi}\,dt.$

But $h' = \cos ih$, $\cos i\delta h - h \sin i\delta i = \delta h' = -\dfrac{dR}{d\phi}\,dt$, and

$$h \sin i\delta i = \frac{dR}{d\phi}\,dt + \cos i\delta h = \left(\frac{dR}{d\phi} - \cos i\,\frac{dR}{dv}\right)dt = \frac{dR}{d\theta}\,dt \text{ by } (f);$$

$$\text{or} \qquad \delta i = \frac{dR}{d\theta}\,\frac{dt}{h \sin i} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Substituting this in the second of (e), and dividing by $\dfrac{dR}{d\theta}$,

$$\delta\theta = -\frac{dR}{di}\,\frac{dt}{h \sin i} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Differentiating $\dfrac{1}{r} = \dfrac{\mu}{h^2}\Big\{1 + e\cos(v - \omega)\Big\}$, we have

$$\frac{1}{r^2}\,\frac{dr}{dt} = \frac{\mu e}{h^2}\sin(v-\omega)\,\frac{dv}{dt}. \quad \text{Or, as } \frac{dv}{dt} = \frac{h}{r^2}, \ \frac{dr}{dt} = \frac{\mu e}{h}\sin(v-\omega).$$

From these we deduce

$$e\cos(v-\omega) = \frac{h^2}{\mu r} - 1, \quad e\sin(v-\omega) = \frac{h}{\mu}\,\frac{dr}{dt}.$$

Taking the variation of these,

$$\cos(v - \omega)\delta e + e\sin(v - \omega)\delta\omega = \frac{2h}{\mu r}\,\delta h;$$

$$\sin(v - \omega)\delta e - e\cos(v - \omega)\delta\omega = \frac{1}{\mu}\,\frac{dr}{dt}\,\delta h + \frac{h}{\mu}\,\frac{\delta dr}{dt}.$$

Multiply the first by $\sin(v-\omega)$, and the second by $\cos(v-\omega)$, and subtract the one from the other,

$$e\delta\omega = \frac{2h}{\mu r}\sin(v-\omega)\delta h - \frac{1}{\mu}\,\frac{dr}{dt}\cos(v-\omega)\delta h - \frac{h}{\mu}\cos(v-\omega)\,\frac{\delta dr}{dt}$$

$$= \frac{h}{\mu}\left(\frac{1}{r} + \frac{\mu}{h^2}\right)\sin(v-\omega)\delta h - \frac{h}{\mu}\cos(v-\omega)\,\frac{\delta dr}{dt} = \frac{h}{\mu a}\left(\frac{dv}{de}\,\delta h + \frac{dr}{de}\,\frac{\delta dr}{dt}\right)$$

by (a). Therefore, by further substitution,

$$\delta\omega = -\frac{h}{\mu a e}\left(\frac{dR}{dv}\,\frac{dv}{de} + \frac{dR}{dr}\,\frac{dr}{de}\right)dt = -\frac{h}{\mu a e}\,\frac{dR}{de}\,dt \,.\,.\, (6).$$

If we now put for da, de, $d\omega$ their values in the first of (e), and divide by $\dfrac{dR}{d\epsilon}$, we have

$$\delta\epsilon = \frac{2na^2}{\mu}\,\frac{dR}{da} - \frac{hk}{\mu a e}\,\frac{dR}{de} \ldots\ldots\ldots\ldots\ldots (7).$$

Resuming the equations from which we found $\delta\omega$, and putting for $\dfrac{dr}{dt}$ its value,

$$\cos(v - \omega)\delta e + e\sin(v - \omega)\delta\omega = \frac{2h}{\mu r}\,\delta h,$$

$$\sin(v - \omega)\delta e - e\cos(v - \omega)\delta\omega = \frac{e}{h}\sin(v - \omega)\delta h + \frac{h}{\mu}\,\frac{\delta dr}{dt}.$$

Multiply the first by $\cos(v-w)$, and the second by $\sin(v-w)$, and add,

$$\delta e = \left\{\frac{2h^2}{\mu r}\cos(v-w)+e\sin^2(v-w)\right\}\frac{\delta h}{h}+\frac{h}{\mu}\sin(v-w)\frac{\delta dr}{dt}.$$

But $\dfrac{2h^2}{\mu r}\cos(v-w)+e\sin^2(v-w)=\dfrac{2h^2}{\mu r}\cos(v-w)+e-e\cos^2(v-w)=$

$$\frac{1}{e}\left(\frac{h^4}{\mu^2 r^2}-1\right)+e=\frac{1}{e}\left(\frac{h^4}{\mu^2 r^2}-\frac{h^2}{\mu a}\right)=\frac{h^2}{\mu e}\left(\frac{h}{\mu}\frac{dv}{dt}-\frac{1}{a}\right);$$

and $\dfrac{h}{\mu}\sin(v-w)=\dfrac{h^2}{\mu e}\dfrac{dr}{dt}$. Therefore

$$\delta e=\frac{h}{\mu e}\left(\frac{h}{\mu}\frac{dv}{dt}-\frac{1}{a}\right)\delta h+\frac{h^2}{\mu^2 e}\frac{dr}{dt}\frac{\delta dr}{dt}$$

$$=-\frac{h^2}{\mu^2 e}\left(\frac{dR}{dv}dv+\frac{dR}{dr}dr\right)+\frac{h}{\mu ae}\frac{dR}{dv}dt$$

$$=-\frac{h\sqrt{1-e^2}}{\mu ae}\frac{dR}{d\epsilon}dt+\frac{h}{\mu ae}\left(\frac{dR}{d\epsilon}+\frac{dR}{d\omega}\right)dt=\frac{h}{\mu ae}\left(k\frac{dR}{d\epsilon}+\frac{dR}{d\omega}\right),$$

as before.

The preceding appears to me a more plain and simple determination of the elements than any which I have seen.

MATHEMATICAL NOTES.

I. *On a General Formula in the Calculus of Finite Differences.*

[*By Samuel Square.*]

In Professor Hall's admirable article on Finite Differences in the *Encyc. Metro.*, the author having proved by integration by parts the formula for $\Sigma(u_x v_x)$, proceeds to find those for $\Sigma^2(u_x v_x)$, $\Sigma^3(u_x v_x)$, (*see* pp. 88, 89), by a process which (in subsequently proving the same formulæ by Generating Functions) he allows to be one of "successive and *laborious* investigation." It is proposed to give a demonstration of these formulæ apart from the theory of Generating Functions, by the separation of symbols, a principle previously established in Sec. 21.

Assuming

$$\Sigma(u_x v_x)=u_x\,\Sigma v_x-\Delta u_x\,\Sigma^2 v_{x+1}+\Delta^2 u_x\,\Sigma^3 v_{x+2}-\Delta^3 u_x\,\Sigma^4 v_{x+3}+,\ etc.\ldots(1);$$

let ϕ be the symbol of an operation which performed on $\Sigma^n v_{x+n-1}$ converts it into $\Sigma^{n+1} v_{x+n}$. Then

$$\Sigma^2 v_{x+1}=\phi\Sigma v_x,\ \Sigma^3 v_{x+2}=\phi\Sigma^2 v_{x+1}=\phi(\phi\Sigma v_x)=\phi^2\Sigma v_x,\ \Sigma^4 v_{x+3}=\phi^3\Sigma v_x,$$

and so on.

Substituting these in (1) we have

$$\Sigma(u_x v_x)=u_x\,\Sigma v_x-\Delta u_x\,\phi\Sigma v_x+\Delta^2 u_x\,\phi^2\Sigma v_x-\Delta^3 u_x\,\phi^3\Sigma v_x+,\ etc.$$

$$=(1-\Delta.\phi+\Delta^2.\phi^2-\Delta^3.\phi^3,\ etc.)\Sigma'(u_x v_x)$$

$$=\left(\frac{1}{1+\Delta.\phi}\right)\Sigma'(u_x v_x),$$

where Δ applies only to u_x, and Σ' has the accent to remind us that it applies only to v_x.

Raising the symbols of the equivalent operation to powers, we have

$$\Sigma^2(u_x\, v_x) = \frac{1}{(1+\Delta.\phi)^2}\, \Sigma'^2(u_x\, v_x),$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots$$

$$\Sigma^n(u_x\, v_x) = \frac{1}{(1+\Delta.\phi)^n}.\Sigma'^n(u_x\, v_x).$$

Expanding these and making the substitutions necessary for ϕ and its powers, we have

$$\Sigma^2(u_x\, v_x) = u_x\, \Sigma^2 v_x - 2\Delta u_x\, \Sigma^3 v_{x+1} + 3\Delta^2 u_x\, \Sigma^4 v_{x+2} - 4\Delta^3 u_x\, \Sigma^5 v_{x+3} +, \textit{etc.}$$

$$\Sigma^3(u_x\, v_x) = u_x\, \Sigma^3 v_x - 3\Delta u_x\, \Sigma^4 v_{x+1} + 6\Delta^2 u_x\, \Sigma^5 v_{x+2} - 10\Delta^3 u_x \Sigma^6 v_{x+3} +, \textit{etc.}$$

$$\Sigma^n(u_x\, v_x) = u_x\, \Sigma^n v_x - n\Delta u_x\, \Sigma^{n+1} v_{x+1} + \frac{n(n+1)}{1.2}\Delta^2 u_x\, \Sigma^{n+2} v_{x+2} -, \textit{etc.}$$

If we wish to find the form of ϕ, we have

$$\phi\Sigma v_x = \Sigma^2 v_{x+1} = \Sigma^2(v_x + \Delta v_x) = \Sigma^2 v_x + \Sigma^2\Delta v_x = \Sigma^2 v_x + \Sigma v_x,$$

or $\qquad \phi = 1 + \Sigma'.$

Substituting this for ϕ, we get

$$\Sigma^n(u_x v_x) = \{1 + \Delta.(1+\Sigma')\}^{-n}.\Sigma'^n(u_x v_x)\,;$$

the general form of a theorem given by Condorcet for $n = 1$, namely,

$$\Sigma(u_x v_x) = u_x \Sigma v_x - \Delta u_x(\Sigma^2 v_x + \Sigma v_x) + \Delta^2 u_x(\Sigma^3 v_x + 2\Sigma^2 v_x + \Sigma v_x) -, \textit{etc.}$$

II. *On the Common Tangent to Two Circles.*

[*Mr. J. W. Elliott, Greatham.*]

The points C, C′ (see page 75 of the current volume) where the common tangents to two circles meet the line joining their centres A, B are readily determined geometrically.

(1). Let the position of the tangent DD′ be such that the radii AD, BD′ are on the same side of AB. Produce DD′ to meet AB in C; then since AD, BD′ are each at right angles to DD′, they are parallel: therefore (Euc. vi. 2) AD : BD′ : : AC : BC, or

AD — BD′ : AD : : AB : AC.................(1),

which determines the point C.

(2). Let the tangent DD′ pass between the circles and intersect AB in C′; then in this case the radii are parallel, but on contrary sides of AB: therefore AD : BD′ : : AC′ : BC′, or

AD + BD′ : AD : : AB : AC′................(2),

which determines the point C′.

The proportions in (1, 2) are the same as those in (7) page 74.

The following property readily follows when the two circles intersect each other.

Let O be a point of intersection, and OAF, OBF′ diameters. Draw OPP′ to bisect the angle O and meeting the circles (A), (B) in P, P′: then $DD'^2 = FP.\, F'P'$. For

$$FP = 2R\sin\tfrac{1}{2}O, \qquad F'P' = 2r\sin\tfrac{1}{2}O\,;$$

$$\therefore\; FP.\, F'P' = 4Rr\sin^2\tfrac{1}{2}O = AB^2 - (R-r)^2 = DD'^2.$$

AN EASY AND EXPEDITIOUS MODE OF DISTINGUISHING SURFACES OF THE SECOND ORDER.

[*Professor Hearn, R. M. College.*]

[Communicated by Mr. Davies, who adds the following notice.

" This paper was communicated to me by Mr. Hearn in March, 1847, at my request, for a particular purpose. It was not, however, used in the way originally intended; and I have much pleasure (with Mr. Hearn's approbation) in submitting it to the Editors of the Mathematician, for publication in their valuable Journal. T. S. DAVIES."]

Oct. 5, 1848.

A surface of the second order is represented by a general equation of the second order in three variables x, y, z, which are the coordinates of any point in the surface with reference to three planes, oblique or rectangular. Such equation may take the following primitive forms in the several cases here specified;

u, v, w being linear functions of x, y, z, and a a constant quantity.

(1) $u^2+v^2+w^2=a$ denotes $\begin{cases} a > 0 \text{ an ellipsoid,} \\ a = 0 \text{ a point,} \\ a < 0 \text{ imaginary.} \end{cases}$

(2) $u^2+v^2-w^2=a$,, $\begin{cases} a > 0 \text{ hyperboloid of one sheet,} \\ a = 0 \text{ a cone,} \\ a < 0 \text{ hyperboloid of two sheets.} \end{cases}$

(3) $u^2+v^2=a$,, $\begin{cases} a > 0 \text{ elliptic cylinder,} \\ a = 0 \text{ a straight line,} \\ a < 0 \text{ imaginary.} \end{cases}$

(4) $u^2-v^2=a$,, $\begin{cases} a > 0 \text{ hyperbolic cylinder,} \\ a = 0 \text{ two planes,} \\ a < 0 \text{ hyperbolic cylinder.} \end{cases}$

(5) $u^2=a$,, $\begin{cases} a > 0 \text{ two parallel planes,} \\ a = 0 \text{ a single plane,} \\ a < 0 \text{ imaginary.} \end{cases}$

(6) $u^2+v^2+w=0$,, elliptic paraboloid.

(7) $u^2-v^2+w=0$,, hyperbolic paraboloid.

(8) $u^2+v=0$,, parabolic cylinder.

Hence when the *primitive* form of an equation of the second order is found, the species of surface represented by it will immediately be known.

It seems to have been usual with geometers to investigate the *principal* planes, and to form the cubic equation for the squares of the semi-axes, and to conclude the species of the surface from the nature of the roots as to $\pm$ of this equation. This is usually a very troublesome process, and when the species of surface is the only object of investigation, appears to me unnecessarily circuitous.

It seems to me unnecessary to prove that a given equation is not susceptible of more than one of the primitive forms in the preceding synopsis; and that it is in general susceptible of that one form in an infinite variety of ways, because these premises are proved in all elementary works.

The plan I have pursued to arrive at the primitive form is this. Consider the equation $f(x, y, z) = 0$ of the second order as not deficient in any of its ten terms. We may arrange it thus,

$$x^2 + Px + Q = 0,$$

where P and Q are functions of y and z, P linear and Q of the second order.

Complete the square as in quadratics,

$$(x + \tfrac{1}{2}P)^2 + Q - \tfrac{1}{4}P^2 = 0.$$

The part $Q - \frac{1}{4}P^2$, being of the second order in yz, may be written

$$K(y^2 + z_1 y + z_2),$$

z_1 a linear function of z, and z_2 a quadratic function of z.

Completing the square in this, we now have

$$(x + \tfrac{1}{2}P)^2 + K(y + \tfrac{1}{2}z_1)^2 + K(z_2 - \tfrac{1}{4}z_1^2) = 0;$$

and $z_2 - \frac{1}{4}z_1^2$ may be written

$$L(z^2 + Az + B),$$

where A and B are constants. We have therefore the given equation resolved into

$$(x + \tfrac{1}{2}P)^2 + K(y + \tfrac{1}{2}z_1)^2 + KL(z + \tfrac{1}{2}A)^2 + KL(B - \tfrac{1}{4}A^2) = 0.$$

In particular cases it may be that $Q - \frac{1}{4}P^2$ becomes linear, in which case the process terminates after the first operation, and the primitive form is at once identified with (8) of the synopsis. In other cases, that $z_2 - \frac{1}{4}z_1^2$ becomes linear, in which cases the process terminates after the second operation, and the primitive form is either (6) or (7); or if $z_2 - \frac{1}{4}z_1^2$ reduces to a constant, the form is (3) or (4).

It seems unnecessary to particularize further, as the plan is so simple that the reader cannot fail in at once fully comprehending it. After remarking that there are one or two particular cases in which $f(x,y,z)$ is deficient in some of its terms, which require a modification of the method, I proceed to illustrate the method by a few examples.

The usual artifices to avoid fractions and facilitate numerical calculation, are of course introduced.

Ex. 1. Required the surface represented by

$$2x^2 + 4y^2 - 6z^2 + 6xy - 5yz - 2xz + 2x + 6y - z = 8.$$

Multiply by 2 and arrange,

$$4x^2 + (3y - z + 1)4x + 8y^2 - 10yz - 12z^2 + 12y - 2z - 16$$

$$(3y - z + 1)^2 = 9y^2 - 6yz + z^2 + 6y - 2z + 1$$

$$\text{Remainder in } yz = -y^2 - 4yz - 13z^2 + 6y - 17$$

Multiply by -1 and arrange, $y^2 + (2z - 3)2y + 13z^2 + 17$

$$(2z - 3)^2 = 4z^2 - 12z + 9$$

$$\text{Remainder in } z = 9z^2 + 12z + 8$$

$$4$$

Constant $\quad 4$

The equation has therefore been resolved into

$$(2x + 3y - z + 1)^2 - (y + 2z - 3)^2 - (3z + 2)^2 = 4,$$

which (changing signs) accords with form (2) when $a \lessdot 0$. The surface is therefore an hyperboloid of *two* sheets.

It is clear that

$$2x + 3y - z + 1 = 0,$$
$$y + 2z - 3 = 0,$$
$$3z + 2 = 0,$$

are three conjugate planes, and the values of x, y, z *immediately* deducible are the coordinates of the centre. Moreover the equation, when the centre is the origin, is represented by the foregoing merely by suppressing the constant terms in the squares, thus

$$(2x + 3y - z)^2 - (y + 2z)^2 - 9z^2 = 4,$$

or $$4x^2 + 8y^2 - 12z^2 + 12xy - 10yz - 4xz = 4.$$

Ex. 2. Required the surface represented by

$$3x^2 + 2y^2 + 11z^2 - 8yz + 6xz = 1.$$

Arranging,

$$\begin{array}{lr}
 & 3x^2 + 6zx + 2y^2 - 8yz + 11z^2 - 1 \\
 & 3z^2 \\
\hline
\text{Remainder in } yz \quad = & 2y^2 - 8yz + \;\; 8z^2 - 1 \\
 & 0 \\
\hline
\text{Constant} & -1
\end{array}$$

The equation is therefore

$$3(x + z)^2 + 2(y - 2z)^2 = 1,$$

which is form (3) when $a \gtrdot 0$. The surface is therefore an elliptic cylinder, of which

$$x + \;\; z = 0,$$
$$y - 2z = 0,$$

are conjugate planes, and therefore taken as simultaneous, represent the axis of the cylinder.

Ex. 3. What surface is represented by

$$x^2 - 2y^2 - 3yz + 3zx + xy + 4z = 0.$$

Multiply by 4 and arrange,

$$\begin{array}{lr}
 & 4x^2 + (y + 3z)\,4x - 8y^2 - 12yz + 16z \\
(y + 3z)^2 = & y^2 + \;\; 6yz + \;\; 9z^2 \\
\hline
 & -9y^2 - 18yz - \;\; 9z^2 + 16z.
\end{array}$$

The equation is therefore

$$(2x + y + 3z)^2 - 9(y + z)^2 + 16z = 0,$$

which is form (7), and therefore the surface is a hyperbolic paraboloid, of which

$$2x + y + 3z = 0, \text{ and } y + z = 0,$$

are conjugate diametral planes, and $z=0$ the tangent plane at the point in which the diameter determined by the preceding, considered as simultaneous, intersects the surface.

To add further examples seems unnecessary prolixity.

I have now to speak of the excepted cases.

When some of the squares are wanting, we cannot proceed exactly as above.

For instance, $$x^2-4zy=6\,;$$
but by writing it thus,
$$x^2-(z+y)^2+(z-y)^2=6,$$
it agrees with the form (2) when $a>0$, and is therefore an hyperboloid of one sheet.

When all the squares are wanting, as for instance
$$4xy-2xz+yz=1.$$

Add and subtract the term l^2x^2, where l^2 is a coefficient to be determined.

We have the square $\frac{1}{l^2}(l^2x+2y-z)^2$, and the residual function in yz is
$$yz-\frac{(2y-z)^2}{l^2}-1.$$

Determine l^2 so that $l^2yz-(2y-z)^2=-(2y+z)^2$,
$$\therefore\quad l^2+4=-4,\qquad \text{or } l^2=-8.$$

The equation is thus resolved into
$$(2y+z)^2+64x^2-(2y-z-8x)^2=8,$$
and therefore represents an hyperboloid of one sheet.

As a concluding example of the excepted cases take the following:
$$xy+xz+yz-x-y-z+\tfrac{3}{4}=0.$$
Multiplying by 4, adding and subtracting $4x^2$, the terms of the second order are equivalent to
$$4x^2+(y+z)4x+4yz-4x^2,$$
$$\text{or } (2x+y+z)^2-(y-z)^2-(2x)^2.$$

Hence making
$$(2x+y+z+\alpha)^2-(y-z+\beta)^2-(2x+\gamma)^2=\delta,$$
identical with the given equation we have
$$\alpha-\gamma=-1,\qquad \alpha-\beta=-2,\qquad \alpha+\beta=-2,$$
$$\text{or } \alpha=-2,\qquad \beta=0,\qquad \gamma=-1,$$
$$\text{and}\qquad \delta-\alpha^2+\beta^2+\gamma^2=-3,\qquad \therefore\quad \delta=0,$$

The equation is therefore equivalent to
$$(2x+y+z-2)^2-(y-z)^2-(2x-1)^2=0,$$
which agrees with form (2) when $a=0$, and is therefore a cone.

In order to ascertain whether a given surface is one of revolution, take the terms of the second order and add to them the following,

$$x^2 + y^2 + z^2 + 2\cos\alpha . yz + 2\cos\beta . xz + 2\cos\gamma . xy$$

multiplied by an arbitrary λ, then if λ can be so determined that the result shall be $\pm$ a complete square it will be a surface of revolution, otherwise it will not. For the preceding function is the expression for the square of the distance of a point xyz from the origin. The preceding criterion is therefore merely the condition that it may be possible to draw two parallel planes through the mutual intersection of the surface and a concentric sphere. Thus in the last example the terms of the second order are

$$xy + xz + yz.$$

And supposing the coordinate planes rectangular $\cos\alpha = 0$, $\cos\beta = 0$, $\cos\gamma = 0$. By adding $\frac{1}{2}(x^2 + y^2 + z^2)$ we have

$$\tfrac{1}{2}(x + y + z)^2,$$

the surface is therefore one of revolution, consequently it is a *right cone* whose axis is a perpendicular plane $x + y + z = 0$. This criterion extends to all surfaces non-central as well as central.

When the surface is not one of revolution we can always determine λ so as to make the sum equivalent to the sum or difference of two squares. The final equation for λ is a cubic. We can therefore by this very simple process find the cyclic planes, principal planes and axes of the surface. This part of the subject may possibly form another paper should the matter be found interesting to the readers of the *Mathematician*. I will close the present paper with an example of a non-central surface of revolution in which the planes of reference are supposed rectangular.

Example. To find the surface represented by the following equation;

$$10x^2 + 5y^2 + 13z^2 - 12xy + 4xz + 6yz - 4x - 6y + 2z = 0.$$

Multiply by 10 and arrange,

$$100x^2 - 2(6y - 2z + 2)10x + 50y^2 + (60z - 60)y + 130z^2 + 20z$$

Subtract $(6y - 2z + 2)^2 \quad = \quad 36y^2 - (24z - 24)y + 4z^2 - 8z + 4$

Remainder in $yz \quad = \quad 14y^2 + 84(z - 1)y + 126z^2 + 28z - 4$

or dividing by 14, $\quad y^2 + 6(z - 1)y + 9z^2 + 2z - \frac{2}{7}$

Subtract $\{3(z - 1)\}^2 \quad = \quad 9z^2 - 18z + 9$

Remainder in $z \quad = \quad 20z - \frac{65}{7}$

The equation is therefore equivalent to

$$(10x - 6y + 2z - 2)^2 + 14(y + 3z - 3)^2 + 280z - 130 = 0,$$

which represents an elliptic paraboloid.

To ascertain whether it be one of revolution.

To the terms of the second order in the original equation add $k(x^2 + y^2 + z^2)$.

The result,

$$(10 + k)x^2 + (5 + k)y^2 + (13 + k)z^2 - 12xy + 4xz + 6yz,$$

ought if the solid is one of revolution to be $\pm$ a complete square, and in order to this, k must satisfy the following:

$$4(10+k)(5+k)=144, \quad 4(5+k)(13+k)=36, \quad 4(10+k)(13+k)=16,$$

and we find that the value of $k=-14$ does satisfy all. Hence the paraboloid is one of revolution.

Moreover, for this value k the above expression becomes $-(2x+3y-z)^2$, and hence the original equation may also be written thus:

$$14(x^2+y^2+z^2)-(2x+3y-z)^2=4x+6y-2z;$$

or, if
$$\frac{2x+3y-z}{\sqrt{14}}=p, \quad x^2+y^2+z^2=r^2;$$

$$\therefore \; r^2-p^2=2.\,\frac{p}{\sqrt{14}}=\left(\frac{2}{7}\right)^{\frac{1}{2}}.\,p.$$

From which we conclude that the axis of revolution passes through the origin, its equations being

$$-\frac{x}{2}=-\frac{y}{3}=z,$$

or
$$x+2z=0, \qquad y+3z=0,$$

and the latus-rectum of the generating parabola $=\left(\frac{2}{7}\right)^{\frac{1}{2}}$.

GENERAL SOLUTION OF A DIOPHANTINE PROBLEM.

[*Mr. Samuel Bills, Hawton, Newark-on-Trent.*]

To find three square numbers such that the sum of every two of them shall be a square number.

Let x, y, z denote the roots of the required squares, and put

$$x^2+y^2=a^2\,..(1), \qquad x^2+z^2=b^2\,..(2), \qquad y^2+z^2=c^2\,....(3).$$

From (1) and (2) we have $y^2-z^2=a^2-b^2$, or $b^2-z^2=a^2-y^2$...(4).

Assume $b=a-mv$, and $z=y-nv$; then substituting these values in (4) and reducing, we find $v=\dfrac{2am-2ny}{m^2-n^2}$; and thence we get

$$b=a-mv=\frac{2mny-(m^2+n^2)a}{m^2-n^2},$$

$$z=y-nv=\frac{(m^2+n^2)y-2mna}{m^2-n^2}.$$

Now in order that x may be rational, let us assume $a=(p^2+q^2)(m^2-n^2)$, and $y=2pq(m^2-n^2)$, then will $x=(p^2-q^2)(m^2-n^2)$, and we shall have $b=4mnpq-(m^2+n^2)(p^2+q^2)$, and $z=(m^2+n^2)2pq-2mn(p^2+q^2)$.

By substitution in (3) we have

$$m^2n^2p^4-2mn(m^2+n^2)p^3q+2(m^4+m^2n^2+n^4)p^2q^2-2mn(m^2+n^2)pq^3 + m^2n^2q^4=\frac{c^2}{4}\,....\,(5).$$

Assume $\frac{c}{2} = mnp^2 - (m^2+n^2)pq - mnq^2$; then squaring this, and comparing it with (5), we find, after reduction, $p = \frac{4mn}{m^2+n^2}q$. Take $q = m^2+n^2$, then $p = 4mn$, and by substituting these values in the different expressions, we obtain the following results:

$$x = (m^2-n^2)(m^4-14m^2n^2+n^4), \qquad a = (m^2-n^2)(m^4+18m^2n^2+n^4),$$

$$y = 8mn(m^4-n^4), \qquad b = (m^2+n^2)^3,$$

$$z = 2mn(3m^2-n^2)(m^2-3n^2), \qquad c = 2mn(5m^4-6m^2n^2+5n^4);$$

where m and n may be taken at pleasure. Let $m=2$, and $n=1$; then we get $x=117$, $y=240$ and $z=44$. Also $a=267$, $b=125$ and $c=244$.

The above are the least numbers answering the conditions of the question, and other answers may be found at pleasure.

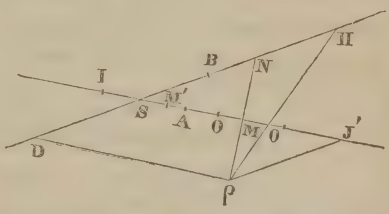
ON THREE PROBLEMS OF APOLLONIUS.

[*From a letter of M. Chasles to Mr. Davies.*]

It may interest you to see how I construct the "three problems of section" of Apollonius, by one uniform process.

1. Section of Ratio. *To draw through the point ρ the straight line ρMN, such that* $\frac{\text{AM}}{\text{BN}} = \lambda$.

Draw ρJ′ parallel to SB, and ρD parallel to SA. The point J′ is determined by ρJ′; and the point I by the proportion $\frac{\text{AI}}{\text{BD}} = \lambda$. Take O the middle of IJ′, and find H by the proportion $\frac{\text{AO}}{\text{BH}} = \lambda$. Join ρH determining the point O′; and take

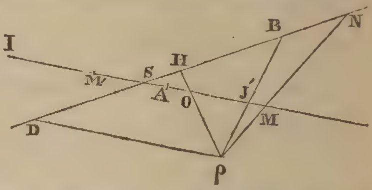


on each side of the point O the segments $\text{OM} = \text{OM}' = \sqrt{\text{OO}'.\text{OJ}'}$. Then the two straight lines ρMN, ρM′N′ fulfil the conditions of the problem.

If the two segments OJ′, OO′ be not on the same side of the point O, the two solutions will be imaginary.

2. Section of space. *To draw ρMN which shall cut off two segments* AM, BN *such that the rectangle* AM . BN $= \nu$ *shall be a given space.*

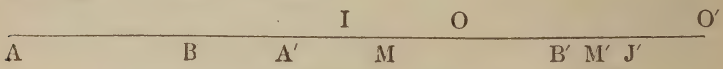
Draw ρB which gives J'; then ρD parallel to SA, which serves to determine the point I by the relation $AI \cdot BD = \nu$. Take O the middle of IJ'; find the point H by the relation $AO \cdot BH = \nu$; and draw ρH which gives the point O'.



Take on each side of the point O the two segments $OM = OM' = \sqrt{OO' \cdot OJ'}$. Then the two straight lines ρMN, $\rho M'N'$ fulfil the conditions of the problem.

The two solutions will be imaginary if the point O' fall on the other side of O.

3. Determinate Section. *To find a point* M *such that* $\dfrac{AM \cdot BM}{A'M \cdot B'M} = \lambda$, *a given ratio.*



Determine the point I by the relation $\dfrac{AI}{A'I} = \lambda$; and the point J' by the relation $\dfrac{BJ'}{B'J'} = \lambda$. Take O the middle of IJ'; and determine the point O' by the relation $\dfrac{BO'}{B'O'} = \lambda \cdot \dfrac{A'O}{AO}$. Then, take two segments $OM = OM' = \sqrt{OO' \cdot OJ'}$; and the two points M, M' resolve the problem.

If the two segments $OO' \cdot OJ'$ have contrary signs, the solutions will be imaginary.

I am not aware that this last problem has been immediately resolved without passing through particular preliminary cases and auxiliary constructions,—such as are exemplified in the work of Simson.

Chartres, 9. 7bre, 1848.

I have given the translation as literally as I could of this extract from M. Chasles's letter. I shall, probably, hereafter offer a few further observations on these subjects; which want of the requisite leisure alone prevents me from doing now. T. S. D.

SOLUTIONS OF MATHEMATICAL EXERCISES.

The following solution of Exercise 147 *by Mr. Thomas Weddle, Wimbledon, was, by an oversight, omitted in its proper place.*

Let $2c$ be the length of the constant chord, the axes of the ellipse being $2a$ and $2b$. Let also $(a\cos\phi, b\sin\phi)$, and $(a\cos\Psi, b\sin\Psi)$ denote the points in which the chord cuts the ellipse, and (x, y) its middle point;

$$4c^2 = a^2(\cos\phi - \cos\Psi)^2 + b^2(\sin\phi - \sin\Psi)^2 \quad\text{........(1)},$$

$$\frac{2x}{a} = \cos\phi + \cos\Psi \quad\text{........(2)},$$

$$\frac{2y}{b} = \sin\phi + \sin\Psi \quad\text{........(3)}.$$

Now $-(\cos\phi + \cos\Psi)(\cos\phi - \cos\Psi) = (\sin\phi + \sin\Psi)(\sin\phi - \sin\Psi)$;

$\therefore$ (2, 3), $-\frac{x}{a}(\cos\phi - \cos\Psi) = \frac{y}{b}(\sin\phi - \sin\Psi)$ which put $= 2vxy$;

$$\therefore \quad \cos\phi - \cos\Psi = -2avy, \text{ and } \sin\phi - \sin\Psi = 2bvx \ldots\ldots\ldots(4).$$

Combining these equations with (2) and (3), we have,

$$\cos\phi = \frac{x}{a} - avy, \quad \text{and } \sin\phi = \frac{y}{b} + bvx.$$

Add the squares of these two equations,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + v^2(a^2y^2 + b^2x^2) = 1 \ldots\ldots\ldots(5).$$

Substitute the values of $\cos\phi - \cos\Psi$, and $\sin\phi - \sin\Psi$ from (4) in (1), and there results

$$v^2 = \frac{c^2}{a^4y^2 + b^4x^2},$$

and (5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{c^2(a^2y^2 + b^2x^2)}{a^4y^2 + b^4x^2} = 1,$$

$$\text{or} \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(1 + \frac{a^2b^2c^2}{a^4y^2 + b^4x^2}\right) = 1 \ldots\ldots\ldots(6);$$

and this is the equation to the locus.

By changing the sign of b^2 this result will be adapted to the hyperbola. Also by writing $x - a$ for x, and pa for b^2, and then making a infinite, we might obtain the locus of the middle points of constant chords in a parabola, but the following independent investigation for this curve may not perhaps be unacceptable.

Let the equation to the parabola be

$$y^2 = 2px \ldots\ldots\ldots(7).$$

Also let $2c$ denote the length of the constant chord, $(x'y')$ and $(x''y'')$ its extremities, and (xy) its middle point;

$$\therefore \quad 4c^2 = (x' - x'')^2 + (y' - y'')^2 \ldots\ldots\ldots(8),$$

$$\text{and} \quad 2x = x' + x'', \quad 2y = y' + y'' \ldots\ldots\ldots(9).$$

From (7) we have

$$2p(x' - x'') = y'^2 - y''^2 = (y' + y'')(y' - y'');$$

$\therefore$ (9), $p(x' - x'') = y(y' - y'')$, which assume $= 2vpy$;

$$\therefore \quad x' - x'' = 2vy, \quad y' - y'' = 2vp \ldots\ldots\ldots(10).$$

Combining these equations with (9) we have,

$$x' = x + vy, \quad y' = y + vp.$$

Substitute these values in the equation $y'^2 = 2px'$,

$$\therefore \quad y^2 + v^2p^2 = 2px \ldots\ldots\ldots(11).$$

Now, by (8) and (10) we have,

$$v^2 = \frac{c^2}{y^2 + p^2}.$$

Hence, (11), the equation to the curve is

$$y^2 + \frac{p^2c^2}{y^2 + p^2} = 2px \dots\dots\dots\dots\dots\dots (12).$$

CXLVIII. *Mr. John Walker, Trin. Coll., Dublin.*

If a and b be the extreme, and c the mean, angles of a harmonic pencil of four lines, prove that

(1) $\cos(a+c)\cos(b+c) = \cos(a+b)$;

(2) $\tan D \tan c - \tan a \tan b = 2\tan a \tan b \tan c \tan D$,

where $D = a + b + c$.

[First Solution. *Lieut. A. R. Clarke, R.E., and Mr. Andrew Roy, Dundee Academy.*]

Let Dm, Dn, Dp, Dq be four lines meeting at the point D and forming a harmonic pencil. In the straight line Dm take any point m, and draw the line $mnpq$ at right angles to Dm intersecting the harmonic pencil in m, n, p, q. Then since the line $mnpq$ is harmonically divided, we have

$$mn : pm - mn :: mq : mq - mp,$$

or $\tan b : \tan(b+c) - \tan b :: \tan(a+b+c) : \tan(a+b+c) - \tan(b+c)$:

$$\therefore \frac{\sin b}{\cos b} : \frac{\sin c}{\cos b\cos(b+c)} :: \frac{\sin(a+b+c)}{\cos(a+b+c)} : \frac{\sin a}{\cos(b+c)\cos(a+b+c)}.$$

Hence $\quad \sin a \sin b = \sin c \sin(a+b+c) = \sin c \sin D \dots\dots\dots (1).$

Now since $D = a+b+c$, we have $\sin D = \sin(a+b+c)$, or

$$\sin D = \sin a \cos(b+c) + \cos a\,(\sin b \cos c + \cos b \sin c),$$

consequently, (1), we have

$$\sin a \sin b = \sin c \sin D = \sin a \sin c \cos(b+c) + \cos a \sin c(\sin b \cos c + \cos b \sin c)$$

$$= \sin a \sin c \cos(b+c) + \cos a \cos c(\sin b \sin c - \cos b \cos c) + \cos a \cos b.$$

Hence $\quad \cos a \cos b - \sin a \sin b = \cos(b+c)(\cos a \cos c - \sin a \sin c),$

or $\quad \cos(a+b) = \cos(a+c)\cos(b+c) \dots\dots\dots (2).$

$$\text{Again } \tan D = \frac{\sin D}{\cos D} = \frac{\sin c \sin D}{\sin c \cos D} = \frac{\sin a \sin b}{\sin c\{\cos(a+b)\cos c - \sin(a+b)\sin c\}}$$

$$= \frac{\sin a \sin b}{\tan c\{\cos(a+b)\cos^2 c - \sin c \cos c \sin(a+b)\}}$$

$$= \frac{\sin a \sin b}{\tan c\{\cos(a+b) - \sin c \sin D\}} = \frac{\sin a \sin b}{\tan c\{\cos(a+b) - \sin a \sin b\}}$$

$$= \frac{\tan a \tan b}{\tan c(1 - 2\tan a \tan b)};$$ consequently, we have

$$\tan D \tan c - \tan a \tan b = 2\tan a \tan b \tan c \tan D \dots\dots (3).$$

[SECOND SOLUTION. *Mr. William Mortimer, Seedley, Manchester; and similarly by A. D. B., North Tawton, near Crediton, Devonshire.*]

By the property of harmonicals we have $\sin D \sin c = \sin a \sin b$. Also

$$2\cos(a+c)\cos(b+c) = \cos(a+b+2c)+\cos(a-b)$$
$$= 2\cos(a+b)\cos^2 c-\cos(a+b)-2\sin(a+b)\sin c\cos c+\cos(a-b);$$

$$\therefore \cos(a+c)\cos(b+c)=\{\cos(a+b)\cos c - \sin(a+b)\sin c\}\cos c+\sin a\sin b$$
$$=\cos D\cos c+\sin D\sin c = \cos(D-c)=\cos(a+b)\ldots(1).$$

Again, we have $\qquad \cos(a+b) = \cos(D-c)$, or

$$\cos a\cos b - \cos D\cos c = \sin a \sin b + \sin D\sin c = 2\sin D\sin c.$$

Multiplying by $\dfrac{\sin a\sin b}{\cos a\cos b\cos c\cos D}$, and recollecting that $\sin a \sin b = \sin D \sin c$, we have finally

$$\tan D\tan c - \tan a\tan b = 2\tan a\tan b\tan c\tan D \ldots\ldots(2).$$

Very good solutions were received from Mr. Peter Elliott, Lamesley, and the Proposer.

CXLIX. *Mr. John Walker, Trinity College, Dublin.*

Sum the series

$$x.1^y + \frac{x(x-1)}{1.2}.2^y + \frac{x(x-1)(x-2)}{1.2.3}.3^y + \ldots + x(x-1)^y + x^y,$$

by the method of the Calculus of Finite Differences, and as an example prove that

$$x.1^5+\frac{x(x-1)}{1.2}.2^5+\ldots+x^5=2^{x-5}\left\{x(x+5)^2-10(x+1)\right\}x^2.$$

[FIRST SOLUTION. *Mr. John Walker, the proposer.*]

Let the sum of the series $=\Sigma$, then will

$$\Sigma = \left\{1+x(1+\Delta)+\frac{x(x-1)}{1.2}(1+\Delta)^2+, etc.\right\}0^y$$
$$=\left\{1+(1+\Delta)\right\}^x 0^y = (2+\Delta)^x 0^y,$$
$$=\left\{2^x+x.2^{x-1}\Delta+\frac{x(x-1)}{1.2}2^{x-2}(\Delta)^2+, etc.\right\}0^y.$$

In the particular case of the question, $y=5$, hence

$$\Delta 0^5 = 1,\ \Delta^2 0^5 = 30,\ \Delta^3 0^5 = 150,\ \Delta^4 0^5 = 240,\ \Delta^5 0^5 = 120;$$

$$\therefore \Sigma = 2^{x-5}x\left\{\begin{array}{r}16+120(x-1)+100(x-1)(x-2)+20(x-1)(x-2)(x-3)\\+(x-1)(x-2)(x-3)(x-4)\end{array}\right\}$$
$$= 2^{x-5}x^2(x^3+10x^2+15x-10)$$
$$= 2^{x-5}\{x(x+5)^2-10(x+1)\}x^2.$$

[SECOND SOLUTION. *Mr. Thomas Weddle.*]

The series is evidently equivalent to

$$\left\{1+x(1+\Delta)+\frac{x(x-1)}{1.2}(1+\Delta)^2\ldots+x(1+\Delta)^{x-1}+(1+\Delta)^x\right\}0^y$$
$$=\left\{1+(1+\Delta)\right\}^x 0^y = \left(2+\Delta\right)^x 0^y.$$

On expansion by the binomial theorem this becomes

$$\left\{2^x + x.\,2^{x-1}.\Delta + \frac{x(x-1)}{1.2}\,.\,2^{x-2}\Delta^2 + \ldots\right\}0^y$$

$$= x2^{x-1}\Delta\, 0^y + \frac{x(x-1)}{1.2}\,.\,2^{x-2}\Delta^2 0^y + \frac{x(x-1)(x-2)}{1.2.3}\,2^{x-3}.\Delta^3\, 0^y + \ldots(1),$$

and this is the sum of the series in terms of the differences of 0^y; but it is evidently no simplification of the given series unless $x \not< y$. The series (1) will terminate after y terms for $\Delta^n 0^y = 0$, when $n \not< y$.

In the particular example proposed, we have $y = 5$; also (*De Morgan's Diff. and Int. Calc*, p. 253), $\Delta 0^5 = 1$, $\frac{\Delta^2 0^5}{1.2} = 15$, $\frac{\Delta^3 0^5}{1.2.3} = 25$, $\frac{\Delta^4 0^5}{1.2.3.4} = 10$, $\frac{\Delta^5 0^5}{1.2.3.4.5.} = 1$, and all the rest $= 0$; hence (1) becomes in this case,

$$2^{x-5}\{x.2^4 + x(x-1).2^3.15 + x(x-1)(x-2).2^2.15$$
$$+ x(x-1)(x-2)(x-3).2.10 + x(x-1)(x-2)(x-3)(x-4)\}$$
$$= 2^{x-5}\{x(x+5)^2 - 10(x+1)\}x^2.$$

The symbolical expression for the sum of the series, namely,

$$(2+\Delta)^x.0^y = 2^x\left(1 + \frac{\Delta}{2}\right)^x.\,0^y,$$

may also be expanded by the exponential theorem in the form,

$$2^x\left\{1 + \log\left(1 + \frac{\Delta}{2}\right).x + \left\{\log\left(1 + \frac{\Delta}{2}\right)\right\}^2 \frac{x^2}{1.2.} + \ldots\right\}0^y;$$

but the calculation of the coefficients of the powers of x would be tedious.

Solutions were sent by Mr. Andrew Roy, Dundee and A. D. B., North Tawton, Crediton, Devonshire.

LC. *Dr. Burns, Rochester.*

AB, BC are two equal straight beams whose length is L, they are united at B and form an acute angle θ; AB rests on the top of an upright wall and is perpendicular to its plane, BC is in the same vertical plane with AB, and C touches the face of the wall; determine the power which applied horizontally at A will sustain a weight w placed on AB at the distance l from the wall, the beams being supposed inflexible and without weight.

[First Solution. *Mr. Henry Goodwyn, pupil at Mr. Stoton's Academy, Wimbledon, and Mr. Peter Elliott, Lamesley.*]

Let D be the point in AB which rests upon the wall, O the point at which w acts, and P the power required; then taking moments about the point C to get rid of the reaction of the wall at C and D, we obtain $w.\text{OD} = \text{P.CD}$; but $\text{OD} = l$, and $\text{CD} = \text{L}\sin\theta$; therefore

$$w.l = \text{P.L}\sin\theta, \quad \text{and } \text{P} = \frac{w\,l}{\text{L}\sin\theta}.$$

[SECOND SOLUTION. *Dr. Burns, the proposer.*]

Let D be the point in AB which is in contact with the top of the wall; then $BD = BC \cos\theta = L \cos\theta$. Also

$$L \cos\theta : l :: w : \frac{wl}{L \cos\theta} = \text{weight sustained at B.}$$

Now if P denote the power required, then we have

$$P : \frac{wl}{L\cos\theta} :: DB : DC :: L\cos\theta : L\sin\theta;$$

$$\therefore \quad P = \frac{wl}{L \sin\theta}.$$

[THIRD SOLUTION. *Mr. Andrew Roy, Dundee Academy.*]

The weight w acts by the lever l in tending to produce rotation round the point C, the effect being wl; and the power which applied horizontally at A to resist this, acts by the lever $L \sin\theta$.

Let P be this power; then we have

$$P.L\sin\theta = wl; \qquad \therefore \qquad P = \frac{wl}{L\sin\theta}.$$

This exercise was answered by Messrs. R. H. Wright, London, and Nicholas Smyth, Galway.

CLII. *Mr. Weddle, Wimbledon.*

Let $\phi(x_1, x_2 \ldots x_n)$ be a homogeneous function of n variables, and of the m^{th} degree, then will

$$m(m-1)\ldots(m-p+1).\phi(x_1, x_2 \ldots x_n) =$$
$$\left\{a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} \ldots + a_n \frac{d}{dx_n}\right\}^p . \phi(x_1, x_2 \ldots x_n);$$

in the right hand member of which $x_1, x_2 \ldots x_n$ are to be written for $a_1, a_2 \ldots a_n$, after performing the operation

$$\left\{a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} \ldots + a_n \frac{d}{dx_n}\right\}^p \text{ on } \phi(x_1, x_2 \ldots x_n.)$$

[FIRST SOLUTION. *Mr. Weddle, the proposer.*]

Expanding by Taylor's theorem, we have

$$\phi(x_1 + a_1 r, x_2 + a_2 r, \ldots . x_n + a_n)$$
$$= \phi(x_1, x_2 \ldots . x_n) + \left\{a_1 \frac{d\phi}{dx_1} + a_2 \frac{d\phi}{dx_2} + \ldots + a_n \frac{d\phi}{dx_n}\right\} r \ldots .$$
$$\ldots . + \frac{r^p}{1.2\ldots p}\left\{a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \ldots + a_n \frac{d}{dx_n}\right\}^p \phi + \ldots . (a),$$

where for brevity ϕ is written for $\phi(x_1, x_2 \ldots x_n)$.

Now let $a_1 = x_1$, $a_2 = x_2 \ldots a_n = x_n$, then the left hand member of this equation is equivalent to

$$\phi(x_1 x_2 \ldots x_n).(1+r)^m$$
$$= \phi(x_1 x_2 \ldots x_n).\left\{1 + mr \ldots + \frac{m(m-1)\ldots .(m-p+1)}{1.2 \ldots \ldots . p} r^p + \ldots\right\} \ldots (\beta).$$

Hence equating the coefficients of the same powers of r in (α) and (β), we have

$$m.\phi(x_1x_2..x_n)=x_1\frac{d\phi}{dx_1}+x_2\frac{d\phi}{dx_2}+\ldots+x_n\frac{d\phi}{dx_n}\ldots\ldots(1),$$

$$m(m-1)\,\phi(x_1x_2..x_n)=x_1^2\frac{d^2\phi}{dx_1^2}+x_2^2\frac{d^2\phi}{dx_2^2}+\ldots+x_n^2\frac{d^2\phi}{dx_n^2}$$

$$+\,2x_1x_2\frac{d^2\phi}{dx_1dx_2}+\ldots+2x_{n-1}\,x_n\frac{d^2\phi}{dx_{n-1}\,dx_n}\,..(2);$$

and generally $m(m-1)\ldots\;(m-p+1)\phi(x_1x_2..x_n)$

$$=\left\{a_1\frac{d}{dx_1}+a_2\frac{d}{dx_2}\ldots\ldots+a_n\frac{d}{dx_n}\right\}^p.\phi(x_1x_2\ldots.x_n)\ldots.(p),$$

in the right hand member of which $x_1\,x_2..x_n$ are to be written for $a_1\,a_2..a_n$, after performing the operation

$$\left\{a_1\frac{d}{dx_1}+a_2\frac{d}{dx_2}+\;\ldots+a_n\frac{d}{dx_n}\right\}^p\text{ on }\phi(x_1\,x_2\ldots x_n).$$

From the formula (1), which is well known, all the others may be deduced in succession. Thus since $\frac{d\phi}{dx_1}$, $\frac{d\phi}{dx_2}\ldots\ldots\frac{d\phi}{dx_n}$ are homogeneous functions of the $(m-1)^{\text{th}}$ degree, (1) gives

$$(m-1)\frac{d\phi}{dx_1}=x_1\frac{d^2\phi}{dx_1^2}+x_2\frac{d^2\phi}{dx_1dx_2}\;\ldots+x_n\frac{d^2\phi}{dx_1dx_n},$$

$$(m-1)\frac{d\phi}{dx_2}=x_1\frac{d^2\phi}{dx_1dx_2}+x_2\frac{d^2\phi}{dx_2^2}\;\ldots.+x_n\frac{d^2\phi}{dx_2dx_n},$$

$$\vdots$$

$$(m-1)\frac{d\phi}{dx_n}=x_1\frac{d^2\phi}{dx_1dx_n}+x_2\frac{d^2\phi}{dx_2dx_n}\;\ldots.+x_n\frac{d^2\phi}{dx_n^2},$$

and from these equations we shall obtain (2) by multiplying them by $x_1\;x_2\ldots x_n$ respectively, adding, and reducing the left hand member by (1). Similarly we could obtain (p) by aid of ($p-1$), and (1).

NOTE. Since proposing this question, I have been referred by a friend to *Hall's Diff. and Int. Calculus*. I find that the formula (2) is there deduced, and the investigation is a good deal similar to the above. The general form (p), however, is not given in Hall, nor am I aware that it has ever been proposed before; it was suggested by some investigations which I published in the Philosophical Magazine for December, 1847, in a paper on "Asymptotic Straight Lines, Planes, Cones, and Cylinders to Algebraical Surfaces."

[SECOND SOLUTION. *Mr. A. Roy, Dundee.*]

Put $w=\phi(x_1,\,x_2,..x_n)$: and since w is a homogeneous function of the variables, put in the usual way, tx_1, tx_2, $tx_3..tx_n$ for x_1, x_2, $x_3..x_n$, and w will take the form $t^m.\,w$. Make $t=1+v$ and w will become $(1+v)^m.\,w$. Hence also x_1, x_2,..x_n will become x_1+vx_1; x_2+vx_2; $x_3+vx_3..x_n+vx_n$. We may thus take vx_1; vx_2; $vx_3..vx_n$ as the increments of $x_1\,x_2\,x_3\ldots x_n$

in the expansion of w by Taylor's Theorem. But w_1 being the new value of w, that theorem gives

$$w_1 = w + \left\{ \frac{d}{dx_1} vx_1 + \frac{d}{dx_2} vx_2 \dots \frac{d}{dx_n} vx_n \right\} w$$
$$+ \frac{1}{1.2} \left\{ \frac{d}{dx_1} vx_1 + \frac{d}{dx_2} vx_2 \dots \frac{d}{dx_n} vx_n \right\}^2 . w$$
$$+ \frac{1}{1.2.3} \left\{ \frac{d}{dx_1} vx_1 + \frac{d}{dx_2} vx_2 \dots \frac{d}{dx_n} vx_n \right\}^3 . w$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

which is equal, as above, to

$$(1+v)^m . w = w + mvw + m.\frac{m-1}{2} v^2 w + m.\frac{m-1}{2}.\frac{m-2}{3} v^3 . w + \text{etc.}$$

Hence as these two expansions are equal, the like powers of v in both must be equal,

$$\text{or} \left\{ x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} + x_3 \frac{d}{dx_3} \dots + x_n \frac{d}{dx_n} \right\}^p w$$
$$= m(m-1)(m-2) \dots (m-p+1) . w.$$

CLIII. *Mr. Fenwick.*

Assuming that the diameter of a curve is the "locus of the middle points of a series of parallel chords," it is required to prove that all diameters of the Witch of Agnesi (the principal axis excepted) are curves of the third order. The equation of the Witch to rectangular coordinates is

$$xy^2 = 4a^2(2a - x).$$

[First Solution. *Mr. Thomas Weddle.*]

Let $$F(xy) = 0 \quad \dots\dots\dots\dots (1),$$

be the equation to a curve of the third degree, and $(\alpha\beta)$ a point in a diametral curve, the equation to the chord through this point will be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = r,$$

$$\text{or} \quad x = \alpha + lr, \quad y = \beta + mr \quad \dots\dots\dots\dots (2);$$

l and m being the directing cosines of the system of parallel chords. Eliminate x and y from (1, 2), and we have

$$F(\alpha + lr, \beta + mr) = 0 \quad \dots\dots\dots\dots (3),$$

and this equation will determine the points in which (1) and (2) intersect. For brevity let F denote $F(\alpha\beta)$, also put

$$F_1 = \frac{dF}{d\alpha} l + \frac{dF}{d\beta} m, \qquad F_2 = \frac{1}{2}\left(\frac{d^2F}{d\alpha^2} l^2 + 2 \frac{d^2F}{d\alpha . d\beta} lm + \frac{d^2F}{d\beta^2} m^2 \right),$$

and $$F_3 = \frac{1}{6}\left(\frac{d^3F}{d\alpha^3} l^3 + 3 \frac{d^3F}{d\alpha^2 . d\beta} l^2 m + 3 \frac{d^3F}{d\alpha . d\beta^2} lm^2 + \frac{d^3F}{d\beta^3} m^3 \right);$$

then (3) will, by Taylor's theorem, become,

$$F + F_1 r + F_2 r^2 + F_3 r^3 = 0 \dots\dots\dots\dots\dots (4).*$$

Now it is evident that, since the point $(\alpha\beta)$ bisects the distance between two of the points in which (1) and (2) intersect, the roots of (4) must be of the form, r_1, $-r_1$ and r_2; hence we must have,

$$F + F_1 r_1 + F_2 r_1^2 + F_3 r_1^3 = 0,$$

$$\text{and} \quad F - F_1 r_1 + F_2 r_1^2 - F_3 r_1^3 = 0,$$

and these equations, (since $r_1 = 0$ is inadmissible) reduce to

$$F + F_2 r_1^2 = 0, \qquad \text{and } F_1 + F_3 r_1^2 = 0.$$

Eliminate r_1^2 between these equations and we have

$$FF_3 = F_1 F_2 \dots\dots\dots\dots\dots\dots\dots (5).$$

This then is the relation that must exist between α and β, but $(\alpha\beta)$ is a point in the diametral curve which bisects the system of parallel chords whose directing cosines are l and m; hence (5) is the equation to this diametral curve, and it is obviously in general of the third degree.

It may happen that the values attributed to l and m, make $F_3 = 0$ (F_3 does not contain α or β), and in this case the system of chords will be parallel to an asymptote. We must here go back to (4) which becomes $F + F_1 r + F_2 r^2 = 0$, and in order that this may be true both for r_1 and $-r_1$ we must have

$$F_1 = 0 \dots\dots\dots\dots\dots\dots\dots\dots (6),$$

which now replaces (5); and it may without much difficulty be shown that (6) generally denotes an hyperbola having one of its asymptotes coincident with the asymptote of the given curve. For take the asymptote for axis of x, then the equation to the curve may be denoted thus,

$$F(xy) = f(xy).\, y + Ax + B = 0,$$

where $f(xy)$ is of the second degree, hence f denoting $f(\alpha\beta)$, and recollecting that now $l = 1$, $m = 0$, we have

$$F_1 = \frac{dF}{d\alpha}\, l + \frac{dF}{d\beta}\, m = \frac{df}{d\alpha}.\, \beta + A = 0,$$

and this equation (since $\frac{df}{d\alpha}$ is of the first degree) usually denotes an hyperbola having the axis of x for an asymptote, the other asymptote being the straight line, $\frac{df}{d\alpha} = 0$. This hyperbola will degenerate into its asymptotes when $A = 0$; and should x enter $f(xy)$, only in the first degree, so that α does not appear in $\frac{df}{d\alpha}$, it will consist of one or two straight lines parallel to the given asymptote.

It may be observed that it is not necessary to calculate F_3 by the formula given above, it will evidently be got by writing l and m for x and y in the terms of the third degree in (1), and rejecting the other terms. Also since

* The expansion of (3) may, of course, be effected independently of Taylor's theorem, by common algebra.

l and m evidently enter (5) and (6) only in the form $l \div m$, we may substitute for them any quantities to which they are proportional, as the coefficients of x and y in the equation, $ax + by = c$, to a straight line to which the system of chords are parallel.

In the present question we have

$$F = \alpha\beta^2 + 4a^2(\alpha - 2a),$$
$$F_1 = (\beta^2 + 4a^2)l + 2\alpha\beta m,$$
$$F_2 = 2\beta lm + \alpha m^2, \text{ and } F_3 = lm^2,$$

hence, (5) the equation to the diametral curve is

$$lm^2\big(\alpha\beta^2 + 4a^2(\alpha - 2a)\big) = \big((\beta^2 + 4a^2)l + 2\alpha\beta m\big)(2\beta lm + \alpha m^2),$$

which reduces to

$$\beta(l\beta + m\alpha)^2 + 4a^2 l(l\beta + m\alpha) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

and this is in general of the third degree in α and β.

The equation $F_3 = 0$, or $lm^2 = 0$, gives $l = 0$, or $m = 0$, and we must now have recourse to (6) instead of (5).

When $l = 0$, $F_1 = 0$, gives $\alpha = 0$, or $\beta = 0$, which denote the two axes; and when $m = 0$, $F_1 = 0$, becomes $\beta^2 + 4a^2 = 0$, which denotes two imaginary parallel straight lines.

It is evident that a similar method of investigation will apply to curves of any degree, only the elimination of $r_1{}^2$ will be more difficult to effect. I must not however enter into further details, lest I should occupy too much space.

The following is another investigation, which however does not seem applicable to curves of a higher degree than the third.

Let $$F(xy) = \phi_3(xy) + \phi_2(xy) + \phi_1(xy) + \phi_0 = 0 \ldots\ldots (8),$$

where $\phi_n(xy)$ denotes a homogeneous function of x and y of the n^{th} degree.

Eliminate x and y from (8) by means of (2), and expand in powers of r;

$$\therefore \quad \phi_3.r^3 + \left(\frac{d\phi_3}{dl}\alpha + \frac{d\phi_3}{dm}\beta + \phi_2\right)r^2 +, \text{ etc. } . = 0^* \ldots (9),$$

where ϕ_n denotes $\phi_n(lm)$.

The equation (9) determines the values of r at the points in which the curve (1) or (8) and the chord (2) intersect, and as we have seen above, the roots of this are of the form r_1, $-r_1$ and r_2, hence by the usual expression for the sum of the roots, we have,

$$r_2 = -\frac{1}{\phi_3}\left(\frac{d\phi_3}{dl}\alpha + \frac{d\phi_3}{dm}\beta + \phi_2\right) \ldots\ldots\ldots\ldots\ldots\ldots (10);$$

Hence substituting this value of r in (2), and eliminating x and y from (1) or (8), by means of the resulting equations, we have

$$F(\alpha + lr_2, \beta + mr_2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

for the equation to the diametral curve; and this of course must only be (5) in another form.

* This expansion also may be obtained by algebra.

Recurring again to the example proposed in the exercise, we have

$$\phi_3 = lm^2 \text{ and } \phi_2 = 0,$$

$$\therefore\ r_2 = -\frac{1}{lm_2}\left(m^2a + 2lm\beta\right) = -\frac{ma + 2l\beta}{lm};$$

$$\therefore\ a + lr_2 = -\frac{2l\beta}{m}, \text{ and } \beta + mr_2 = -\frac{ma + l\beta}{l};$$

hence substituting $-\frac{2l\beta}{m}$, and $-\frac{ma + l\beta}{l}$, for x and y in the equation to the curve, we have

$$-\frac{2l\beta}{m}\left(\frac{ma + l\beta}{l}\right)^2 = 4a^2\left(2a + \frac{2l\beta}{m}\right),$$

$$\text{or, } \beta(ma + l\beta)^2 + 4a^2l(ma + l\beta) = 0,$$

which coincides with (7).

When $\phi_3 (= F_3) = 0$, we must evidently equate the coefficient of r in (9) to zero.

Investigations of the equation to diametral curves may also be found in *Hymer's Conic Sections*, 3rd ed. p. 148, and in the *Camb. Math. Journal*, vol. ii. (first series) p. 76.

I have used the term *diametral curve* in this solution instead of *diameter*, because the latter term is used to denote a straight line which divides "parallel straight lines in such a manner that in each case the sum of the ordinates on one side of the line is equal to the sum of the ordinates on the other," and it would create confusion to use the same term for both kinds of lines. In curves of the second degree, diameters and diametral curves coincide, and are straight lines; if this be considered an objection to the term diametral *curve*, we may use diametral *line* instead.

[SECOND SOLUTION. *Mr. Peter Elliott, Lamesley.*]

The given equation being

$$xy^2 = 4a^2(2a - x) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and}\quad y = px + q \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the equation to a chord which intersects (1) in the points A, B, C, whose coordinates are (x_1y_1), (x_2y_2) and (x_3y_3); hence if (X_1Y_1), (X_2Y_2), (X_3Y_3) be the middle points of AB, AC, BC respectively, then we have

$$x_1 + x_2 = 2X_1;\ x_1 + x_3 = 2X_2;\ x_2 + x_3 = 2X_3;$$

$$\therefore\quad \left.\begin{aligned} x_1 + x_2 + x_3 &= X_1 + X_2 + X_3 = 2X_1 + x_3 \\ &= 2X_2 + x_2 = 2X_3 + x_1 \end{aligned}\right\} \ldots (3).$$

Eliminate y between (1) and (2) and there results

$$x(px + q)^2 = 4a^2(2a - x),$$

in which by (3) and the theory of equations, the coefficient of x^2 is

$$\frac{2q}{p} = -(2X_3 + x_1) \ldots\ldots\ldots\ldots (4).$$

Also (X_3Y_3) being a point in (2) and (x_1y_1) a point in (1) and (2) we therefore have

$$Y_3^2 = pX_3 + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$y_1 = px_1 + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$x_1y_1^2 = 4a^2(2a - x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From (4), (5), and (6) we obtain

$$x_1 = -\frac{2Y_3}{p},$$

$$y_1 = -(pX_3 + Y_3),$$

and substituting these expressions for x_1, y_1 in (7), we finally get

$$-\frac{2Y_3}{p}\left(pX_3 + Y_3\right)^2 = 4a^2\left(2a + \frac{2Y_3}{p}\right),$$

or $$Y_3(pX_3+Y_3)^2 + 4a^2(ap+Y_3) = 0 \ldots\ldots\ldots\ldots (8),$$

for the equation of the required locus, which is that to a curve of the third degree.

If (2) be parallel to the axis of y, then $p = \infty$, and (8) becomes

$$Y_3X_3^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9);$$

but, as the condition of parallelism to the axis of y must give real values to X_3, (9) can only be satisfied by the condition

$$Y_3 = 0,$$

the equation to the axis of x. This being perpendicular to the system of parallel chords which it bisects, is therefore the principal diameter of the curve.

Mr. Elliott sent another good solution.

[THIRD SOLUTION. *Mr. Nicholas Smyth, Galway.*]

Let

$$y = px + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be one of the system of parallel chords in the given curve,

$$xy^2 + 4a^2x = 8a^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Transforming the origin of coordinates to the middle point $(\alpha\beta)$ of (1), the equations of the given curve and line become, respectively,

$$(x+\alpha)(y+\beta)^2 + 4a^2(x+\alpha) = 8a^3 \ldots\ldots\ldots\ldots (3),$$

and $$y+\beta = p(x+\alpha), \text{ or (since } \beta = p\alpha + q), \ y = px \ldots\ldots (4).$$

Combining (3) and (4) gives

$$(x+\alpha)(px+\beta)^2 + 4a^2(x+\alpha) = 8a^3,$$

or $$p^2x^3 + (p^2\alpha + 2p\beta)x^2 + (2p\alpha\beta + \beta^2 + 4a^2)x + \alpha\beta^2 + 4a^2\alpha - 8a^3 = 0 \ldots\ldots (5).$$

This cubic in x from the peculiar position of the origin, has two equal roots with opposite algebraic signs; the other root therefore is,

$$x = -\frac{p\alpha + 2\beta}{p},$$

which being substituted in (5) gives

$$\beta(pa+\beta)^2+4a^2(pa+\beta)=0,$$

for the required diametral curve.

This method of solution is obviously of general application.

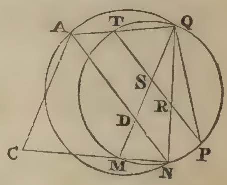
Good solutions were also received from Messrs. W. Mortimer, Seedley, and A. Roy, Dundee.

CLV. *By* Δ.

Let P, Q be given points in the circumference of a circle given in magnitude and position, and QM, QN chords given in position. It is required to draw a line PRST meeting QN in R, QM in S, and the circumference in T, so that RS may have to ST a given ratio.

[First Solution. *Dr. Burns, Rochester.*]

On QN describe a segment of a circle QAN similar to the segment QMNP. Draw NM and produce it; take NM : MC in the given ratio, and draw CA parallel to QM. Join NA cutting QM in D, and QA cutting the circumference QMN in T: draw PT and it is the line required.



Since CA is parallel to QM; therefore

NM : MC :: ND : DA.

Also since the segments QAN and QMP are similar, the angle NAQ is equal to the angle PTQ; hence TR is parallel to NA, and ND : DA :: RS : ST; therefore RS : ST :: NM : MC, that is, in the given ratio.

If the line CA cuts the circumference of the circle QAN in two points, the question admits of two solutions; but if it touches the circle, there will be only one position of the line PRST. The construction may be effected by drawing through N *any* line cutting QM in B, and producing it till NB have to BC the given ratio.

[Second Solution. *Lieut. A. R. Clarke, R.E., and Mr. Andrew Roy, Dundee Academy.*]

Let the arc PQ $=2a$, therefore angle PTQ $=a$, and let NQM $=\beta$, PSQ $=\theta$; then

$$\mathrm{RS}=\mathrm{SQ}\,\frac{\sin\beta}{\sin(\beta+\theta)},\ \text{and}\ \ \mathrm{ST}=\mathrm{SQ}\,\frac{\sin(\theta-a)}{\sin a};$$

$$\therefore\quad \frac{\mathrm{RS}}{\mathrm{TS}}=\frac{\sin a\sin\beta}{\sin(\beta+\theta)\sin(\theta-a)}=r,\ \text{the given ratio.}$$

Consequently, $\sin(\theta+\beta)\sin(\theta-a)$ is given; but

$$\sin(\theta+\beta)\sin(\theta-a)=\tfrac{1}{2}\cos(a+\beta)-\tfrac{1}{2}\cos(2\theta+\beta-a);$$

$$\therefore\quad \cos(2\theta+\beta-a)=\cos(a+\beta)-\frac{2\sin a\sin\beta}{r},$$

or

$$2\theta=a-\beta+\cos^{-1}\left\{\cos(a+\beta)-\frac{2}{r}\sin a\sin\beta\right\}.$$

Whence 2θ, or θ becomes known, and thence the position of the line PRST.

An excellent solution was sent by the proposer; and ingenious geometrical solutions were also received from Masters Wm. S. Ashton, Richard Glenn, and Thomas Johnson, all pupils at Mr. Hoole's Academy, Blackburn.

CLVI. *Mr. Robert Harley, Blackburn.*

It is required to determine by "De Gua's criterion" for the detection of imaginary roots, how many are contained in the cubic equation $x^3 - ax \pm b = 0$, where a and b are any real positive quantities.

[SOLUTION. *Lieut. A. R. Clarke, R.E.*]

The equation containing the roots diminished by a certain quantity d is,

$$(x+d)^3 - a(x+d) \pm b = 0,$$

or
$$x^3 + 3dx^2 + (3d^2 - a)x + d^3 - ad \pm b = 0.$$

Now as d is perfectly arbitrary, we may assume $d = \pm \left(\frac{a}{3}\right)^{\frac{1}{2}}$, or $3d^2 - a = 0$, hence

$$x^3 \pm x^2(3a)^{\frac{1}{2}} \pm \left(\frac{a}{3}\right)^{\frac{1}{2}} \left(\frac{a}{3} - a\right) \pm b = 0,$$

or $$x^3 \pm x^2(3a)^{\frac{1}{2}} \mp \frac{2a^{\frac{3}{2}}}{3\sqrt{3}} \pm b = 0,$$

or $$x^3 \pm x^2(3a)^{\frac{1}{2}} \pm \left(b - \frac{2a^{\frac{3}{2}}}{3\sqrt{3}}\right) = 0.$$

Hence if $b \ngtr \frac{2a^{\frac{3}{2}}}{3\sqrt{3}}$, or $b^2 \ngtr \frac{4}{27} a^3$, we have a cipher-coefficient between two terms having like signs, and therefore by De Gua's criterion, the equation has two imaginary roots when

$$b^2 \ngtr \frac{4a^3}{27}.$$

Some remarks respecting the insufficiency of De Gua's criterion in cases of this nature were received from Master William Hilton Gibb, pupil at Mr. Hoole's academy, Blackburn.

CLVII. *Mr. Robert Harley, Blackburn.*

Three circles are described in a plane triangle to touch the inscribed circle and two of the sides: find the relation that subsists among the radii of these four circles.

[GENERAL SOLUTION. *Lieut. A. R. Clarke, R.E.*]

There can be drawn twelve circles to fulfil the required conditions, four at each angular point of the triangle. Let ABC be the triangle and O the centre of the inscribed circle: draw AO and produce it; then the centres of two of the circles touching the inscribed circle and the two sides AB, AC, or these produced, are in the line through A and O. Also the centres of the two other equal circles which touch the inscribed circle and the sides BA, CA produced, are in the line through A perpendicular to AO. In a similar manner, four circles at each of the other angular points B, C can be drawn.

Lemma. If t_1, t_2, t_3 be the tangents of $\frac{1}{4}$A, $\frac{1}{4}$B, $\frac{1}{4}$C; then will

$$2(1+t_1t_2t_3) = (1+t_1)(1+t_2)(1+t_3).$$

For $\tan\frac{1}{4}(A+B+C)=1=\dfrac{t_1+t_2+t_3-t_1t_2t_3}{1-(t_1t_2+t_2t_3+t_1t_3)}$; therefore we have

$$t_1t_2t_3-(t_1t_2+t_2t_3+t_1t_3)-(t_1+t_2+t_3)+1=0,$$

or
$$2(1+t_1t_2t_3)=(1+t_1)(1+t_2)(1+t_3)\dots\dots\dots\dots (a).$$

Let r be the radius of the inscribed circle, ρ_1 and ρ_1' the radii of the two circles touching the inscribed circle and the sides AC, AB and these produced; then we get

$$\sin\tfrac{1}{2}A=\frac{r-\rho_1}{r+\rho_1},\ \cos\tfrac{1}{2}A=\frac{2\sqrt{r\rho_1}}{r+\rho_1},\ \tan\tfrac{1}{4}A=\left(\frac{1-\cos\frac{1}{2}A}{1+\cos\frac{1}{2}A}\right)^{\frac{1}{2}}=\frac{\sqrt{r}-\sqrt{\rho_1}}{\sqrt{r}+\sqrt{\rho_1}}.$$

Similarly if ρ_2, ρ_3 be the radii of the corresponding circles at B and C, we get

$$\tan\tfrac{1}{4}B=\frac{\sqrt{r}-\sqrt{\rho_2}}{\sqrt{r}+\sqrt{\rho_2}};\ \tan\tfrac{1}{4}C=\frac{\sqrt{r}-\sqrt{\rho_3}}{\sqrt{r}+\sqrt{\rho_3}}.$$

Substituting in the formula (a) deduced in the lemma, and reducing, we have

$$r=\sqrt{\rho_1\rho_2}+\sqrt{\rho_2\rho_3}+\sqrt{\rho_1\rho_3}\dots\dots\dots\dots (\mathcal{A}),$$

which is the relation in the case proposed.

Again if ρ_1', ρ_2', ρ_3' be the radii of the circles described to touch the inscribed circle and the sides produced, we shall get in the same way as before,

$$\tan\tfrac{1}{4}A=\frac{\sqrt{\rho_1'}-\sqrt{r}}{\sqrt{\rho_1'}+\sqrt{r}},\ \tan\tfrac{1}{4}B=\frac{\sqrt{\rho_2'}-\sqrt{r}}{\sqrt{\rho_2'}+\sqrt{r}},\ \tan\tfrac{1}{4}C=\frac{\sqrt{\rho_3'}-\sqrt{r}}{\sqrt{\rho_3'}+\sqrt{r}}.$$

Substituting in the formula (a) deduced in the lemma, and reducing,

$$\frac{1}{r}=\frac{1}{\sqrt{\rho_1'\rho_2'}}+\frac{1}{\sqrt{\rho_2'\rho_3'}}+\frac{1}{\sqrt{\rho_1'\rho_3'}}\dots\dots\dots\dots (B).$$

Comparing $(\mathcal{A})$ and (B) we get

$$\frac{1}{\sqrt{\rho_1'\rho_2'}}+\frac{1}{\sqrt{\rho_2'\rho_3'}}+\frac{1}{\sqrt{\rho_1'\rho_3'}}=\frac{1}{\sqrt{\rho_1\rho_2}+\sqrt{\rho_2\rho_3}+\sqrt{\rho_1\rho_3}}\dots\ (C).$$

Lastly, if ρ', ρ'', ρ''' denote the radii of the pairs of equal circles whose centres are in the lines through A, B, C, perpendicular to the lines bisecting the angles of the triangle; then drawing OE perpendicular to AB, we have

$$\rho_1=AE\cot\tfrac{1}{2}A=r\cot^2\tfrac{1}{2}A;\ \text{therefore}\ \tan\tfrac{1}{2}A=\sqrt{\frac{r}{\rho'}}.$$

Similarly, $\tan\frac{1}{2}B=\sqrt{\frac{r}{\rho''}}$ and $\tan\frac{1}{2}C=\sqrt{\frac{r}{\rho'''}}$; hence, substituting in the trigonometrical formula

$$1=\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B+\tan\tfrac{1}{2}A\tan\tfrac{1}{2}C+\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C,$$

we have the relation

$$\frac{1}{r}=\frac{1}{\sqrt{\rho'\rho''}}+\frac{1}{\sqrt{\rho''\rho'''}}+\frac{1}{\sqrt{\rho'\rho'''}}\dots\dots\dots\dots (D).$$

Equating the values of $\tan\frac{1}{4}A$, $\tan\frac{1}{4}B$, $\tan\frac{1}{4}C$, in the first case, to the values of the same functions of the angles in the second case, we get

$$r = \sqrt{\rho_1\rho_1'} = \sqrt{\rho_2\rho_2'} = \sqrt{\rho_3\rho_3'} \text{.................} (E).$$

Let us next take the escribed circle whose radius is r_1, and let s_1, s_2, s_3 be the radii of the circles between the angles of the triangle and circle (r_1); s_1', s_2', s_3' the radii of the circles described on the opposite sides of (r_1); s', s'', s''' the radii of the circles (corresponding to ρ', ρ'', ρ''') which have their centres in the lines through A, B, C at right angles to AO, BO, CO; then the corresponding relations are,

$$-\frac{1}{r_1} = \frac{1}{\sqrt{s_1s_2}} + \frac{1}{\sqrt{s_1s_3}} - \frac{1}{\sqrt{s_2s_3}} \text{............} (A'),$$

$$-r_1 = \sqrt{s_1's_2'} + \sqrt{s_1's_3'} - \sqrt{s_2's_3'} \text{...............} (B'),$$

$$\therefore \frac{1}{\sqrt{s_1s_2}} + \frac{1}{\sqrt{s_1s_3}} - \frac{1}{\sqrt{s_2s_3}} = \frac{1}{\sqrt{s_1's_2'} + \sqrt{s_1's_3'} - \sqrt{s_2's_3'}} \text{....} (C'),$$

$$-\frac{1}{r_1} = \frac{1}{\sqrt{s's''}} + \frac{1}{\sqrt{s's'''}} - \frac{1}{\sqrt{s''s'''}} \text{..............} (D'),$$

$$r_1 = \sqrt{s_1s_1'} = \sqrt{s_2s_2'} = \sqrt{s_3s_3'} \text{...............} (E').$$

Similar relations may be deduced with respect to the two other escribed circles, and circles described to touch them and the sides.

Messrs. Peter Elliott, Lamesley, W. H. Levy, Shalbourne, and Nicholas Smyth, Galway, give the following relation:

$$\frac{(r-\rho_1)(r-\rho_2)}{\sqrt{\rho_1\rho_2}} + \frac{(r-\rho_1)(r-\rho_3)}{\sqrt{\rho_1\rho_3}} + \frac{(r-\rho_2)(r-\rho_3)}{\sqrt{\rho_2\rho_3}} = 4r.$$

[SECOND SOLUTION. *Master W. S. Ashton, pupil at Mr. Hoole's Academy, Blackburn; and similarly by Masters Richard Glenn, Thomas Johnson, and William Hilton Gibb, pupils at the same Academy.*]

Let ABC be any plane triangle; O the centre of the inscribed circle; O_1, O_2, O_3 the centres, and r_1, r_2, r_3 the corresponding radii of the circles described to touch two sides and the inscribed circle. From the centres O, O_1 draw the radii OD, O_1D_1 perpendicular to the side AB; and join OA, OB, and OC.

Then from the right angled triangles AOD and AO_1D_1, we have

$$AO = \frac{r}{\sin\frac{1}{2}A}; \text{ and } AO_1 = \frac{r_1}{\sin\frac{1}{2}A}.$$

Now $OO_1 = r + r_1$; and $AO = AO_1 + OO_1$; hence

$$\frac{r}{\sin\frac{1}{2}A} = \frac{r_1}{\sin\frac{1}{2}A} + r + r_1; \quad \sin\tfrac{1}{2}A = \frac{r-r_1}{r+r_1};$$

and consequently

$$\cos\tfrac{1}{2}A = \sqrt{1-\sin^2\tfrac{1}{2}A} = \frac{2\sqrt{rr_1}}{r+r_1}.$$

Similarly, we have

$$\sin\tfrac{1}{2}B = \frac{r-r_2}{r+r_2};\ \sin\tfrac{1}{2}C = \frac{r-r_3}{r+r_3};$$

$$\cos\tfrac{1}{2}B = \frac{2\sqrt{rr_2}}{r+r_2};\ \cos\tfrac{1}{2}C = \frac{2\sqrt{rr_3}}{r+r_3}.$$

But $\sin\frac{1}{2}A = \cos\frac{1}{2}(B+C) = \cos\frac{1}{2}B\cos\frac{1}{2}C - \sin\frac{1}{2}B\sin\frac{1}{2}C$.

Wherefore substituting from above we obtain,

$$\frac{r-r_1}{r+r_1} = \frac{4r\sqrt{r_2r_3}}{(r+r_2)(r+r_3)} - \frac{(r-r_2)(r-r_3)}{(r+r_2)(r+r_3)}.$$

This equation, after slight reduction, becomes

$$r^2 - 2r\sqrt{r_2r_3} + r_2r_3 = r_1r_2 + 2r_1\sqrt{r_2r_3} + r_2r_3;$$

that is, $$(r-\sqrt{r_2r_3})^2 = (\sqrt{r_1r_2}+\sqrt{r_1r_3})^2;$$

whence we get $$r = \sqrt{r_1r_2} + \sqrt{r_1r_3} + \sqrt{r_2r_3},$$

which is the relation required.

Solutions were sent by Messrs. R. Harley, A. Roy, J. Stafford, and A. D. B.

CLVIII. *Mr. W. H. Levy, Shalbourne.*

Let $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$ be the radii of the circles inscribed in the six triangles formed by joining the middle points of the sides of any plane triangle with the opposite angles; r_1, r_2, r_3 the radii of the escribed circles, and r_4, r_5, r_6 the radii of the circles inscribed in the triangles formed by joining the contiguous corners of the squares described on the three sides; then will

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}+\frac{1}{\rho_5}+\frac{1}{\rho_6} = \frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}+\frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}.$$

[First Solution. *Mr. Peter Elliott, Lamesley.*]

Let ABC be a plane triangle, AD, BE, CF the lines bisecting the sides in D, E, F; ANMB, BLKC, and CIHA the squares described on the sides AB, BC, CA; and AHN, BML, and CKI the triangles formed by joining the contiguous corners of the squares described upon the sides. Draw DF, which will be parallel to AC, and AH = AC = 2DF. Also AN = AB = 2AF, and angle HAN is equal to angle AFD, each being the supplement of angle BAC; hence the triangles AHN and AFD are equiangular, and also NH = 2AD. Similarly, ML = 2BE and KI = 2CF. Let the triangle ABC be denoted by Δ, then it is well known that the three triangles HAN, BML, and CKI are equal to each other and to the triangle ABC,

$$\therefore\ \frac{1}{2}\Delta = ACF = BCF = BAD = DAC = CEB = EBA.$$

Now since the area of any triangle is equal to the product of the semi-perimeter and the radius of the inscribed circle, it is evident that

$$\frac{\Delta}{2}=\frac{\frac{1}{2}\mathrm{AB}+\mathrm{AC}+\mathrm{CF}}{2}\rho_1\,; \qquad \therefore \frac{1}{\rho_1}=\frac{\frac{1}{2}\mathrm{AB}+\mathrm{AC}+\mathrm{CF}}{\Delta}.$$

Similar expressions are obtained for the reciprocals of the other five radii; hence

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}+\frac{1}{\rho_5}+\frac{1}{\rho_6}=\frac{3(\mathrm{AB}+\mathrm{BC}+\mathrm{CA})+2(\mathrm{AD}+\mathrm{BE}+\mathrm{CF})}{\Delta} \quad \ldots.(1).$$

Again, we have $\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}=\frac{2}{r}=\frac{\mathrm{AB}+\mathrm{BC}+\mathrm{CA}}{\Delta}$ ....(2).

Also,

$$\frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}=\frac{\mathrm{AN}+\mathrm{AH}+\mathrm{HN}+\mathrm{BM}+\mathrm{BL}+\mathrm{ML}+\mathrm{CK}+\mathrm{CI}+\mathrm{KI}}{\Delta}$$

$$=\frac{2(\mathrm{AB}+\mathrm{BC}+\mathrm{CA})+2(\mathrm{AD}+\mathrm{BE}+\mathrm{CF})}{\Delta} \quad \ldots\ldots\ldots.(3).$$

But the right hand member of (1) is equal to the sum of the same members of (2) and (3): therefore

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}+\frac{1}{\rho_5}+\frac{1}{\rho_6}=\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}+\frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}.$$

If the lines AD, BE, and CF intersect in O, and $\rho_1', \rho_2', \rho_3', \rho_4', \rho_5', \rho_6'$ be the radii of the circles inscribed in AOF, BOF, *etc.*, then in a similar manner we obtain

$$\frac{1}{\rho_1'}+\frac{1}{\rho_2'}+\frac{1}{\rho_3'}+\frac{1}{\rho_4'}+\frac{1}{\rho_5'}+\frac{1}{\rho_6'}=\frac{6}{r_4}+\frac{6}{r_5}+\frac{6}{r_6}-\frac{6}{r_1}-\frac{6}{r_2}-\frac{6}{r_3}.$$

[SECOND SOLUTION. *Mr. W. H. Levy, the proposer.*]

Referring to the second solution of question 11 (1375), Gentleman's Diary for 1838, p. 41, it is easily seen that DE $=$ 2AI, and similarly for the other lines joining the contiguous corners of the squares described on the sides. Also the triangle ADE is equal to the triangle ABC; hence, if b_1, b_2, b_3 denote the lines bisecting the sides, and ρ_1, ρ_2 the radii of the circles inscribed in the triangles ABI and ACI, and so on for the others, we have

$$\frac{1}{\rho_1}=\frac{c+\frac{1}{2}a+b_1}{\Delta}, \qquad \frac{1}{\rho_2}=\frac{b+\frac{1}{2}a+b_1}{\Delta},$$

$$\frac{1}{\rho_3}=\frac{c+\frac{1}{2}b+b_2}{\Delta}, \qquad \frac{1}{\rho_4}=\frac{a+\frac{1}{2}b+b_2}{\Delta},$$

$$\frac{1}{\rho_5}=\frac{a+\frac{1}{2}c+b_3}{\Delta}, \qquad \frac{1}{\rho_6}=\frac{b+\frac{1}{2}c+b_3}{\Delta},$$

$$\therefore \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}+\frac{1}{\rho_5}+\frac{1}{\rho_6}=\frac{3(a+b+c)+2(b_1+b_2+b_3)}{\Delta}.$$

Again, $\dfrac{2}{r_4}=\dfrac{b+c+2b_1}{\Delta}$, $\dfrac{2}{r_5}=\dfrac{a+c+2b_2}{\Delta}$, $\dfrac{2}{r_6}=\dfrac{a+b+2b_3}{\Delta}$,

$$\therefore \frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}=\frac{2(a+b+c)+2(b_1+b_2+b_3)}{\Delta};$$ which added to

$$\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}=\frac{2}{r}=\frac{a+b+c}{2}$$ (*Horæ Geom.* prop. VII. cor. 2),

gives

$$\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}+\frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}=\frac{3(a+b+c)+2(b_1+b_2+b_3)}{\Delta};$$

$$\therefore \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}+\frac{1}{\rho_5}+\frac{1}{\rho_6}=\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}+\frac{2}{r_4}+\frac{2}{r_5}+\frac{2}{r_6}.$$

Cor. It is evident from the preceding that

$$\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_5}=\frac{1}{\rho_1}+\frac{1}{\rho_4}+\frac{1}{\rho_6},$$

$$\therefore \frac{1}{\rho_1}+\frac{1}{\rho_4}+\frac{1}{\rho_6}=\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_5}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}+\frac{1}{r_5}+\frac{1}{r_6}.$$

Master Wm. S. Ashton deduces the following properties:

$$\frac{1}{\rho_3}-\frac{1}{r_5}=\frac{1}{r_1}=\frac{1}{\rho_6}-\frac{1}{r_6}$$

$$\frac{1}{\rho_1}-\frac{1}{r_4}=\frac{1}{r_2}=\frac{1}{\rho_5}-\frac{1}{r_6}$$

$$\frac{1}{\rho_2}-\frac{1}{r_4}=\frac{1}{r_3}=\frac{1}{\rho_4}-\frac{1}{r_5};$$

$$\therefore \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}-\frac{1}{\rho_4}-\frac{1}{\rho_5}-\frac{1}{\rho_6}=\frac{2}{r_4}-\frac{2}{r_6}.$$

[THIRD SOLUTION. *Lieut. A. R. Clarke, R.E.*]

Let M be the middle point of AB, and let CM$=m_3$. The corresponding lines from A, B being m_1, m_2. Let ρ_1, ρ_2 be the radii of the circles inscribed in ACM and BCM; then

$$\frac{1}{\rho_1}=\frac{b+\frac{1}{2}c+m_3}{\Delta},\quad \frac{1}{\rho_2}=\frac{a+\frac{1}{2}c+m_3}{\Delta};$$

$$\therefore \quad \frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{a+b+c+2m_3}{\Delta};$$

$$\text{and } \Sigma\frac{1}{\rho}=\frac{3(a+b+c)+2(m_1+m_2+m_3)}{\Delta}.$$

Now the triangles formed by joining the contiguous corners of the squares are each equal to the original triangle, also their exterior sides are easily shewn to be $2m_1, 2m_2, 2m_3$; consequently,

$$\frac{1}{r_6}=\frac{a+b+2m_3}{2\Delta};\quad \frac{1}{r_5}=\frac{a+c+2m_2}{2\Delta};\quad \frac{1}{r_4}=\frac{b+c+2m_1}{2\Delta};$$

Also, $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}=\frac{a+b+c}{2\Delta};$$

and consequently,

$$\Sigma\frac{1}{r}=\frac{3(a+b+c)+2(m_1+m_2+m_3)}{2\Delta};$$

$$\therefore\quad \Sigma\frac{1}{\rho}=2\Sigma\frac{1}{r}. \qquad Q.E.D.$$

If $R_1, R_2, R_3, R_4, R_5, R_6$ and R_7, R_8, R_9 be the circumscribing radii corresponding to the inscribed radii $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$ and r_4, r_5, r_6 respectively; then it may be easily proved that

$$8\sqrt{R_1R_2R_3R_4R_5R_6}=R_7R_8R_9.$$

Mr. Roy's demonstration of the case of the circles being inscribed in the triangles AOF, BOF, *etc.*, was very similar to Mr. Elliott's proof of the other case. Good solutions were sent by Messrs. W. Mortimer, John Stafford, Nicholas Smyth, and A. D. B.

CLIX. *Lieut. A. R. Clarke, R.E.*

If a tetrahedron whose six edges are denoted by a, b, c, d, e, f, be cut by three planes so that each section is a rhombus, and if m_1, m_2, m_3 be their sides; then will

$$\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\,\frac{1}{f}.$$

[First Solution. *Mr. Peter Elliott, Lamesley; and Mr. Andrew Roy, Dundee.*]

Lemma. Every section of a tetrahedron made by a plane parallel to two of its opposite edges, is a parallelogram.

Let ABCD represent a tetrahedron, EFGH a plane section parallel to AD and BC, the points E, F, G, H being in the edges AB, AC, DC, DB respectively. Then EF and GH are each parallel to BC, and therefore to each other. Similarly EH is parallel to FG, and hence EFGH is a parallelogram.

Next, to find the relations that subsist when EFGH is a rhombus, we have AC : AF :: BC : EF; therefore

$$AC - AF : AC :: BC - EF : BC \ldots\ldots\ldots\ldots (1).$$

Also, by reason of the parallels AD and FG, we have

$$\text{AC} - \text{AF} : \text{AC} :: \text{FG or EF} : \text{AD} \ldots\ldots\ldots\ldots (2).$$

Comparing (1) and (2), we obtain

$$\text{BC} - \text{EF} : \text{EF} :: \text{BC} : \text{AD};$$

$$\therefore \quad \text{BC} : \text{EF} :: \text{BC} + \text{AD} : \text{AD};$$

$$\therefore \quad \frac{1}{\text{EF}} = \frac{\text{BC}+\text{AD}}{\text{BC.AD}} = \frac{1}{\text{AD}} + \frac{1}{\text{BC}}.$$

Hence adopting the notation of the exercise, we have

$$\frac{1}{m_1} = \frac{1}{a} + \frac{1}{d}, \quad \frac{1}{m_2} = \frac{1}{b} + \frac{1}{e}, \quad \frac{1}{m_3} = \frac{1}{c} + \frac{1}{f};$$

$$\therefore \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}.$$

[SECOND SOLUTION. *Dr. Burns, Rochester.*]

The opposite sides of a rhombus being parallel, it can easily be proved that two of the sides m_1 are parallel to a and two to d; that two of the sides m_2 are parallel to b and two to e; and that two of the sides m_3 are parallel to c and two to f. Now if x_1, x_2, x_3 be the upper segments of the edges made by the cutting planes, then

$$b : d :: x_1 : m_1 = \frac{dx_1}{b}, \text{ and } a : b :: m_1 : b - x_1;$$

therefore $$m_1 = \frac{ab - ax_1}{b}; \text{ hence } (a+d)x_1 = ab, \text{ and } x_1 = \frac{ab}{a+d}.$$

Substituting for x_1 gives $m_1 = \frac{d}{b}x_1 = \frac{ad}{a+d}$; therefore

$$\frac{1}{m_1} = \frac{1}{a} + \frac{1}{d}.$$

Similarly, $$\frac{1}{m_2} = \frac{1}{b} + \frac{1}{e}, \text{ and } \frac{1}{m_3} = \frac{1}{c} + \frac{1}{f};$$

consequently,

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}.$$

[THIRD SOLUTION. *Mr. Nicholas Smyth, Galway.*]

Let ABCD be the tetrahedron, and EFGH a rhombus whose side is m_1, and formed as enunciated; $BC = a$, and $AD = d$, being the edges in which none of the angles of the rhombus lie, and E, F, G, H being in the edges CD, DB, BA, AC.

By similar triangles we have

$$BC : EF :: CD : DE = \frac{m_1}{a}.CD; \quad \therefore CE = \frac{a-m_1}{a}.CD;$$

$$CE : CD :: EH : AD, \text{ that is}$$

$$a - m_1 : a :: m_1 : d; \quad \therefore \quad m_1 = \frac{ad}{a+d}.$$

Hence $\frac{1}{m_1} = \frac{1}{a} + \frac{1}{d}$, and thus it appears that the reciprocal of the side of the rhombus is equal to the sum of the reciprocals of the pair of edges in which neither of its angles are situated; and therefore, by mere addition, the proposition is established.

MATHEMATICAL EXERCISES—*(continued.)*

166. *Dr. Burns, Rochester.*

Given $\tan^{-1} x^2 + \tan^{-1} x = \tan^{-1} \frac{1}{3}$, to find x.

167. *James Lockhart, Esq., Fasnacloich, Argyleshire.*

In cubic equations of the general form

$$x^3 + ax^2 + bx + c = 0:$$

prove that if $a^2b^2 + 18abc$ is greater than $4ca^3 + 4b^3 + 27c^2$, the roots are real; if less, two roots are imaginary; and if equal, two roots are equal to each other.

168. *Mr. John Walker, Trin. Coll., Dublin.*

If the line joining the centre of the inscribed circle with the vertex of any angle of a triangle be a mean proportional between the diameter of that circle and the radius of the circumscribed circle, prove that the three sides of the triangle are in arithmetical progression.

169. *Mr. Robert Harley, Mathematical Master at Mr. Hoole's Academy, Blackburn.*

Find the centre of a given circle, by means of straight lines only.

170. *Mr. W. H. Levy, Shalbourne.*

Let ρ_1, ρ_2, ρ_3 be the radii of the three circles described in a plane triangle to touch the inscribed circle whose radius is r and two of the sides, and r_1, r_2, r_3 the radii of the escribed circles: then will

$$\frac{2\sqrt{r\rho_1} + 2\sqrt{r\rho_2} + 2\sqrt{r\rho_3}}{\sqrt{(r_1r_2 + r_2r_3 + r_1r_3)}} + \frac{\rho_1}{r_1} + \frac{\rho_2}{r_2} + \frac{\rho_3}{r_3} = 1.$$

171. *Lieut. A. R. Clarke, R.E.*

A semicircle is divided into two parts by an ordinate at right angles to the diameter. In these two spaces circles are described; required the locus of the middle point of the line joining their centres.

172. *By M. Poncelet.*

If a hexagon have its three non-contiguous sides meeting in one point, and the three others meeting in another point; then the three lines which join the pairs of opposite summits will also meet in one point.

173. *Mr J. W. Elliott, Greatham.*

If tangents be drawn from each point of a curve of the second order to any other curve of the same order; then will the chords of contact touch another curve of the second order.

174. *James Cockle, Esq., M.A., Barrister-at-Law.*

Prove that the equation

$$x^{12} + ax^{3} + bx^{2} + \frac{b^{2}}{3a}\,x + d = 0,$$

admits of finite algebraic resolution.

175. *Mr. John Walker, Trinity College, Dublin.*

Find the locus of the intersection of three planes at right angles to each other, each of which touches one of three confocal ellipsoids.

176. *Mr. R. H. Wright, M.A., London.*

A fluid mass revolves round an axis with a given angular velocity, and is attracted to a centre of force in that axis varying as the distance; find the form the fluid will assume, and the equation which represents it.

177. *Mr. Thomas Weddle, Wimbledon.*

From the centre O of a surface of the second degree, and any point P, draw perpendiculars to the polar plane of P, and let the former perpendicular meet the said plane in G, and the latter intersect the diametral plane containing any two principal diameters in H. The rectangle of the perpendiculars OG and PH, is equal to the square of half the third principal diameter.

178. *Mr. Andrew Roy, Dundee Academy.*

A tetrahedron is divided by a plane making a section which is a rhombus; to find in how many different ways this may be done, and the relations between the segments into which it may be divided.

THE MATHEMATICIAN.

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HISTORICAL NOTICES RESPECTING AN ANCIENT PROBLEM.

[*Mr. Davies.*]

(Continued from page 154.)

BRIANCHON, 1810.

The paper of Brianchon, to which reference has already been made, contains several things which are worthy of especial notice. Its professed object is, to shew the value of the perspective projection in the demonstration of theorems respecting plane figures :— a process occasionally resorted to by geometers of our own country, but generally esteemed (whether justly or not, I am not disposed to discuss *here*) to be a species of "geometrical trickery." It is sufficient to say just now, of such 'trickeries,' that they furnish unquestionable evidence of the truth of theorems respecting the conic sections when their correlatives for the circle have been proved; and though *our notions of elegance or geometrical purity may be violated by their use*, we are still *placed a step in advance*, as regards actual knowledge and perfect conviction, by the use of such processes as these. It is a strong incentive to perseverance in searching for a systematic demonstration of a theorem, to know before hand that it is absolutely true. I may ask English Geometers of the highest order (alas! how few of the great English Geometers are left among us! and how are those few rewarded !!) whether they have not often been guided and encouraged to pursue the, apparently, most hopeless research after a demonstration, by observing that in all the carefully-drawn figures they used, the truth they sought after was *visibly* indicated? In such cases, the making of diagrams are so many *experiments;* and in truth we thereby descend from geometrical to physical methods of inquiry: but being convinced of the truth (whatever amount of importance be attached by metaphysicians to the character of its evidence) we remount to first principles, and ultimately obtain a legitimate and logical proof that the truth itself is not accidental and special, but necessary and universal. Projection is, however, possessed of evidence far superior to that afforded by experiment; and in the hands of the Continental Geometers it has been made the basis of the *only* proofs

we yet possess of some of the most general and most interesting of all the propositions at which we have hitherto arrived. In truth, when rigorously examined, our coordinate systems are only *projective systems of research*: at any rate, the Cartesian (or symmetrical system) is strictly so. There is not the slightest *conceptive difference* between the Cartesian system in respect to three dimensions, and the methods of descriptive geometry as delivered by Monge: the whole difference that can be found is simply of the subordinate manipulations—the one *working* symbolically and the other graphically, and each using the machinery which is constrained by the fundamental choice of the one method or the other.

What has been just stated is to be understood as relating to the demonstration of theorems: for in respect to the construction of problems, the aid so derived is generally of very limited and doubtful value. It is often productive, too, of additional and unnecessary complexity, from requiring the description of a conic section where a geometer of the Grecian or English school would have only used the straight line and circle. Still even this drawback has not been without good effects: for it has led foreign geometers (especially the French) to devise modes of constructing the conic sections subject to any five limiting conditions, which are amongst the most remarkable graphic inventions of modern times—rich as modern times have been in such inventions. In this class of researches Brianchon himself is the most distinguished:—though many others have contributed to the perfection of his methods.

Brianchon begins by the enunciation of the following theorem:—

"*There being given any conic section, if we trace at pleasure a polygon, all whose summits except the last one are situated upon the curve, and if we take upon each of its sides a fixed point*, or pole; *when we deform the polygon by turning each point about its pole, so that the summits shall always glide along the curve, the magnitudes of the sides and angles continually change*, and when the poles are distributed in the same straight line, *the last summit will describe a conic section.*" Journal de l'Ecole Polyt., tom. iv. p. 1.

To prove this he adopts two methods. In the first, he projects the systems of polygons and their circumscribing conic sections upon a plane drawn parallel to that which passes through the eye and the line of poles. It is a familiar property of such a perspective that the representation is also a conic section, whilst the perspectives of all the polygons will have their corresponding sides parallel; and he *assumes* (he does not offer the semblance of a proof) that the last summit will always lie in a conic section.

Having thus stated, and, in such way proved, the general proposition, he descends to certain special cases selected for discussion; making this remark initially:—

"It is easy to perceive that by determining the points of intersection of the generated and directing curves, we resolve the following problem,

"*Being given in the plane of any conic section any number of fixed points situated in a straight line, to inscribe in the curve a polygon of the same number of angles, of which each side, produced if necessary, shall pass through one of these fixed points.*"

For anything which he *shews* to the contrary, there might be *four* such polygons inscribed: though he does remark in a foot note that two of the four possible intersections are *étrangers* to the problem proposed; and he says we shall readily account for this circumstance by drawing a figure, and following the movement of the polygon with attention! The fact is, and should have been shewn by Brianchon, as it has been shewn by Poncelet and Gaskin, that the two conic sections are so related that they cannot have more than two points common, and that those are points of contact. Neither does he express the least suspicion that the number of sides being odd or even affects his method of construction: still less the remarkable property developed in the solution of Exercise 154 of the present number of the Mathematician, that taking the number of sides *even* renders the problem either indeterminate or impossible. There appear, too, to be other cases of impossibility and indeterminateness which affect the solution when the number of sides is *odd;* and which, though not hitherto fully discussed, deserve the careful attention of geometers.

Brianchon next gives an elegant construction for the particular case which is considered by *Pappus, vii. prop.* 117: a construction which he professes to have obtained from the general coordinate discussion before spoken of, and applied to the data of the problem in its present shape. It is virtually this:—Describe an arbitrary circle through the two given points G and H to cut the given circle ABM in A, B; draw AB to meet GH in P; and from P draw the tangents PM, PM′ to the circle: then if GM and HM (or GM′ and HM′) be drawn to meet the circle in R and S (or in R′ and S′) the the chord RS (or R′S′) will be parallel to GH, as required. This, it is evident is not essentially different from the construction given by Pappus, or by Leslie in his *Geometrical Analysis*, i. 25: but it is a great practical improvement both in respect to construction and demonstration to introduce the arbitrary circle instead of the third or fourth proportional, as the case may be. It may be as well to remark that P is the "radical centre," (see the *Lady's and Gentleman's Diary*, 1849, *App.* p. 78), of the given circle and two given points, the latter considered as circles of zero-radius.

From this he proceeds to notice two simple properties familiarly known long before his time; and then gives a long historical note upon which comment has already been made. The most important part of his paper in respect to *this subject*, however, is that which immediately follows: for there he shews that if the line of poles cut the given conic section, and the points of intersection be two of the poles themselves, *the locus of the vertices is the* POLAR *of the third* POLE. He then also deduces from this, the construction of the tangents to a conic section from a point without it by means of straight lines only; which as far as I know is the earliest proposal of that method: though, undoubtedly the elements for its investigation were well known long before. The actual construction has been given at p. 215, vol. v., of the *Mathematical Repository*, and in the 12th ed. of *Hutton's Mathematics*, vol. i. p. 388; and had occurred to myself independently of a knowledge of what Brianchon had done, though perhaps nearly twenty years later.

Perhaps the most remarkable part of Brianchon's short paper is his deduction of the property which now bears his name: or at least one which involves it, and the theorem itself is formally given in *cahier* 13 of the *Journal de l'Ecole Polytechnique,*—reference to which is made in this paper dated two years before the cahier just quoted.

He also discusses the genesis of a conic section by the revolution of lines, the method which was the source of so much unseemly discussion between Maclaurin and Braikenridge. He likewise deduces the particular case of Pascal's theorem which forms *Pappus* vii., 139; and he concludes with a proposition which appears to be originally due to him, and which has by subsequent writers been found to be of great value—and in the problem under consideration, too, amongst the rest. It is :—

"*Having any two conic sections arbitrarily disposed upon the same plane; if in the first we inscribe an arbitrary chord subjected only to the condition of being a tangent to the other: then the two tangents to the first conic section at the extremities of this chord will always intersect in a third conic section.*"

I have spoken of this paper very fully (and even in respect to some points which may be deemed irrelevant to my professed object) from its containing the germs of that system of inquiry, which, in the hands of Poncelet, Chasles, Ollivier, and their *collaborateurs*, has grown up into the *Géométrie supérieur* of the modern French school.

Gergonne, Econtre, Servois, and Rochat, 1810-11.

The principle that in the conic sections there is no property which has not also a corresponding one different from itself (which has since been called "the principle of *duality*," and has been extended much further than to lines and surfaces of the second order) began to take a distinct form, when Brianchon shewed the constant relation between the line and point now called the pole and polar; and especially when in his own théorem was found the dual of Pascal's hexagram. The active and penetrating mind of Gergonne readily seized the principle; and in the first number of his *Annales des Mathematiques*, he proposed the following problem, the strict dual of that to which these Notes relate.

"A circle and three lines in the same plane with it being given, it is required to circumscribe a triangle about the circle whose three summits shall be upon the three given lines." *Annales, tom i. p.* 17.

At *p.* 122, Econtre enunciates the problem for the polygon, adapted to any number of given straight lines; and by means of the doctrine of Poles and Polars, shews that it will be constructed by drawing tangents at the angular points of an inscribed polygon of the same name, which passes through the poles of the given lines. He refers for the solution of this problem to *Carnot*, *Géom. de Pos.*, and to Lhuilier, *Elémens de Analyse;* which appear to have been the only works he had consulted in reference to this problem.

Three different constructions are given for finding the *poles* of the sides of the triangle: the last of which is by straight lines only, and not essentially different from Brianchon's.

A few pages further on in the same number (p. 126), Gergonne gives a very remarkable construction for his own form of the problem, and proposes it for the exercise of his correspondents in demonstration.

"Construction. *Let a, b, c be the three given lines, and* A, B, C *the summits of the triangle formed by them. Let there be determined the*

poles α *and* β *of a and b; draw* Aα *to meet a in m, and* Bβ *to meet b in n cutting the circle in t and t′: then if through either of these points we draw a tangent to the circle, terminating in a and b, this tangent will be one of the sides of the required triangle.*"

At page 259 of the same volume, Gergonne reproposes the problem for any conic section, the given lines being three, and assigns the condition that the operation shall be performed by means of *straight lines only:* and finally, at p. 337-42, there are given two solutions of the problem in respect to *three* given lines by Servois and Rochat. Of these, some description is annexed.

Servois commences by the definition of the *pole of a straight line* in reference to a line of the second order; and this is the earliest place in which I have remarked that the word was used in that definite sense. He then gives the same construction for finding the tangent to a conic section that Brianchon had given.*

Servois commences with an analysis of the problem, which he founds partly on the theorem of Pascal and partly on the fundamental (and then known or guessed at) properties of the *pole* or *polar*. It does not, indeed, take the strict form of an analysis, as the ancient and the English Geometers view that process: but it is yet virtually an analysis, and might without much difficulty be rendered formally so. It would perhaps be more correct to designate the investigation as a series of properties subsidiary to either the analysis, or to the demonstration of the construction. It is, however, more my present business to let Servois speak for himself, and in his own way, than to transform his method to suit our own model.

"1. Let the hexagon *rstuxy* be inscribed in a conic section; and produce the sides *rs*, *tu*, *xy* to intersect two and two, forming the triangle *abc*. Draw the diagonals *rx*, *ys*; *su*, *tr*; *tx*, *uy*, intersecting each other: the first pair in *o*, the second in *p*, and the third in *q*. Draw the lines *sx*, *ru*,

* The irregular manner in which the foreign Transactions and Journals have been generally published, renders it extremely difficult to determine the exact sequence as to actual order of time, in which any parts of a system of discovery were made. There is often a discrepancy of *several years* between the date (when dated or otherwise known) of writing a paper and the date of its publication: and as if to increase this confusion, very few of the papers are dated at all. This has not only been the case in the minor societies, but in the greatest continental ones, as the Paris, the Berlin, and the Petersburgh. The Journal of the Polytechnic School is another instance. The *eleventh* volume which contains Brianchon's first paper is dated in the title-page, "Novembre, 1810;" whilst the *thirteenth* which contains his second paper is dated "Avril, 1806." There is also a reference in the first sheet of the 11th volume to a paper which begins at p. 297 of the 13th: but the page not being given in the reference, seems to shew that it had not then been actually printed. That this is not a mere transposition of *papers* is clear from that bearing the title-page date of 1806 being strictly a supplementary one to that bearing a similar date of 1810. Nor is it a mere transposition of the title-pages by the binder of the volumes, for each is marked with the number of the volume to which the *cahier* belongs, viz. "tome iv" and "tome vi." What renders the question still more perplexing is—that Gergonne, in the *Annales* for Mai, 1811, quotes this second paper of Brianchon.

If, therefore, in estimating the order of time in which the several papers appeared, I shall have committed any error, that error may well be pardoned when the confusion of the materials from which I have had to deduce my conclusions is considered. Indeed I have been obliged to rely more upon internal evidence drawn from the natural order which the research may be supposed to have taken, than upon any external evidence furnished by the works I have consulted.

yt; the first meeting *yr* in *k*, the second meeting *st* in *l*, and the third meeting *xu* in *m*. Join *ka*, *lb*, *mc*, which by their intersections form the triangle ABC. Finally draw *ao*, *bp*, *cq*.

"2. It is in the first place evident that *o* is the *pole* of BC; for *ao* is the chord of the tangents drawn from *k*, and *ko* will, therefore, be that of the tangents drawn from *a*. In the same way it follows that *p* and *q* are the poles of AC and AB respectively.

"3. By a known property of the inscribed hexagon (proved before by Servois, and now well known) the three points *k*, *l*, *m* are in a straight line. Moreover, *ao*, *bp*, *cq* are evidently the chords of contact of tangents drawn from *k*, *l*, *m* respectively. Whence these three straight lines intersect in one point *d*, the pole of the straight line *klm*.

"4. In the quadrilateral *tuxy*, the diagonal *uy* is divided harmonically (*Carnot*, *Essai*, prop. vi.) by the lines *tu*, *xy*, *cq*, *cm*; and hence the harmonical sectors *ca*, *cq*, *cb*, *c*A divide harmonically every line which does not pass through *c* (*ib.* prop. vii). For similar reasons the systems *ac*, *ao*, *ab*, *a*C, and *ba*, *bp*, *bc*, *b*A are harmonical sectors.

"5. Let α be the intersection of *ao*, *bc*; and designate by A′ and A″ the intersections of *ao* with AB and AC. [These latter points are not marked in the figure, since they will be proved not to be different from A.] We see from (4) that *ao* is divided harmonically by *ca*, *cq*, *cb*, *c*A, in *a*, *d*, α, A; and, again, the same straight line is divided harmonically by *ba*, *bp*, *bc*, *b*A, in *a*, *d*, α, A″. But when upon the same line two systems of harmonical points have three points coincident, the fourth point of the one system will coincide with the fourth point of the other. [This is proved by Gergonne, in a foot note, pretty much as we should prove it, by division of ratios.] Whence the the point A″ will coincide with A. In the same manner the point A′ upon AB will coincide with A. Wherefore the line *ao* passes through A: and in a similar manner it may be proved that *bp* passses through B, and *cq* through C.

"6. Through any one of the summits, *r*, draw a tangent to the curve terminating at D and E in the lines AC, BC; and draw E*x* terminating in AB at F; and join F*t*. Then, because E*r* is a tangent drawn from the point in BC of which the pole is *o*, the other tangent from E must touch the curve at *x* the extremity of *ro*; and hence EF is that other tangent. In considering this again as the tangent from F in the line AB of which the pole is *q*, we see that the other tangent must touch the curve in *t*, the extremity of the chord *xq*; and hence F*t* is this other tangent. Wherefore ED and F*t* are tangents at the extremity of the chord *rt* which passes through the pole *p* of AC; and hence they meet in one point D in AC. Wherefore F*t* produced passes through D upon AC. Whence the tangents at *r*, *x*, *t* form a triangle whose summits are upon the sides of the triangle ABC; and it is clear that tangents at *s*, *u*, *y* possess the same property.

"7. (Constr.) The curve and the triangle ABC being given, it is required to construct the triangle *abc*.

(1.) Determine any two *o*, *p*, of the poles *o*, *p*, *q*.

(2.) Draw A*o*, B*p* which give respectively by (5) the two points *a*, *b*.

(3.) Draw *ab* which gives, generally, upon the curve two points *r*, *s*.

(4.) Draw through either, *r*, of these two points a tangent to the curve meeting AC, BC in D and E: and then two other tangents from D and E.

These will meet at F in AB. We then have one solution of the problem ; and we get another by operating with respect to S in the same manner.

"8. When the line *rs* touches the curve, the two triangles furnished by the construction become identical; and when it does not meet the circle, the problem is impossible. Also when the three given lines meet in a point, the construction above given is illusory, because *o*, *p*, *q* are then in one line."

The following is the construction of Rochat: but he gives neither analysis nor demonstration, simply contenting himself that "*toutes les constructions qui viennent d'être indiquées* POUVENT SE DÉMONTRER PAR L'ANALYSE GÉOMÉTRIQUE."

Let AB, BC, CA be the three given lines, and SXYV the given conic section: it is required to circumscribe a triangle about the conic section whose angular points shall lie upon the three given lines—*using the ruler only*.

1. Find the *poles o* and *p* corresponding to the lines AB, AC. [The construction for effecting this is indicated, very similar to that of Brianchon, and is omitted here.]

2. Draw C*o*, B*p* to meet BA, AC in Q, R; join QR cutting the conic section in S, T.

3. Draw S*p* meeting the curve again in V and the line AC in U; and through U the arbitrary secant UXY meeting the curve in X and Y; join XV, YS intersecting in Z; and draw ZP intersecting AC in B′.

4. Finally, draw B′S, B′V meeting AB, BC in C′ and A′: and draw A′C′. This will be one of the triangles required.

5. By operating for T in the same manner, we shall obtain the other triangle which solves the problem."

This construction, it is quite clear, differs from that of Gergonne and Servois, in this:—that it includes the linear process for constructing the tangents at S and T by means of the point B′; whereas Gergonne only directed the tangents to be drawn. This is effected by means of the arbitrary secant UXY, and the consequent determination of Z so as to obtain the *polar* of the point U; and it is the only improvement made by Rochat, whilst Servois has furnished the elements for framing a complete demonstration of Gergonne's construction.

This method, ingenious as it is, does not seem capable of being extended to polygons of a greater number of sides: but the ideas suggested by some of the steps have led Poncelet to a construction adapted to any case whatever that is actually consistent in its data. Our attention naturally proceeds to the researches of this great geometer.

The contemplation of figures in space without the aid of any visible representation is not a faculty easy of acquisition; and the habitual reference to diagrams which is created by our ordinary modes of geometrical study rather retards than facilitates its acquisition. It is, however, the peculiar tendency of the Descriptive Geometry to cultivate this faculty: by rendering necessary the conception of the actual figures whose projections are delineated in the *épures*. We hence find it more fully developed amongst the pupils of the Polytechnic School than amongst those of any other school of geometry—of which numerous examples may be cited. As instances, it may be sufficient to mention, Brianchon, Poncelet, Ollivier, and Chasles. The greatest difficulty which such men find in explaining their views is—that of designing

intelligible figures to represent their contemplated ideas; notwithstanding the aid in representation which the Descriptive Geometry affords them. They can describe in language intelligible to each other, and to any one who will take the trouble to cultivate the faculty of contemplating unseen figures, the truth itself and the reasoning by which it is enforced: but such descriptions become altogether unintelligible or unconceptive, and such arguments are entirely forceless, to minds that have not been so trained.*

It was to such training, for instance, that we owe the remarkable system of truths, and their methods of investigation which are developed by Poncelet, in his *Traité des Propriétés Projéctivs des Figures*, 1822. That his results might be proved by methods more analogous to those we usually employ, is not denied:—but could they have been so *discovered?* or is it likely that they would have been even *suspected?* I think not,—for some ages at least.

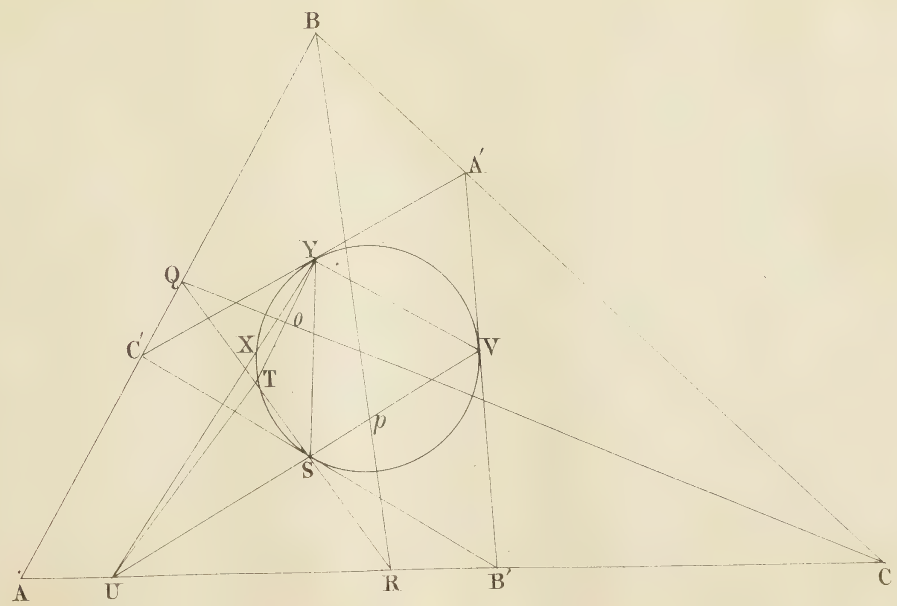
The reasonings in the work just quoted are of a kind, which it would be impossible to render distinct to the mind of a reader in the space to which I am limited; although I have often thought of attempting to do something of the kind—either in continuation of the "Modern Geometry," or in a separate paper. Whether I may be able to command time for the purpose, is very uncertain: but if any gentleman will take it off my hands, he will have my sincere thanks, and will deserve well of English mathematicians.

In the absence of this, I shall here content myself with giving Poncelet's constructions of each of the problems which constitute the dual propositions, the one of the other, as already explained. Of course that of one problem might have been sufficient, as either might at once have been derived from the other by the ordinary constructions which are rendered so simple by the method of poles and polars: but independent constructions are in many respects preferable, as constructions; though they may, and indeed often must repose upon the same general principles.

"THE INSCRIBED POLYGON. *Inscribe at pleasure and successively in the given conic section three portions of polygons, of which the sides (equal in number to that of the given points) respectively pass, and in the assigned order, through these points. Let a, a', a'' be the first extremities of those portions of polygons, and k, k', k'' the last, respectively. Consider these points as the summits of an inscribed hexagon having for pairs of opposite summits a and k, a' and k', and a'' and k''; and a, a', a'' the alternate summits. The three intersections will be (Pascal's theorem) in one straight line; and this line will cut*

* It cannot be doubted, however, that this system of reasoning upon undelineated conceptions is open to great abuse, and liable to create a dangerous vagueness in minds not otherwise well disciplined. Some important step of the reasoning, and even some fatally erroneous assumption, is very liable to creep into the investigation; and it requires great caution to discover the peculiar conditions that render a theorem which is generally true, or a construction which is generally applicable, either illusory or inapplicable to particular cases.

To me it appears that every method of "getting at" a general truth, is legitimate: but however effective such general processes may be in effecting it, it is incumbent upon all enquirers who adopt them to seek a more simple and recognised mode of ultimate demonstration, and to exercise a careful scrutiny into the particular cases which arise from the variations in the hypothesis or the data respectively. It would, for instance, be most interesting and most valuable to give proofs of the remarkable theorems of Poncelet, either founded on the *method of transversals* (which is peculiarly fitted for the purpose) or by means of the *Cartesian System of Coordinates*.



J R. Jobbins, 3 Warwick C^t

the conic section in two points, either of which might be used as the first summit of the required polygon." *Traité*, p. 352.

"THE CIRCUMSCRIBED POLYGON. *Circumscribe at pleasure three portions of polygons which have their summits upon the given lines, and in the same prescribed order. Let a, a', a'' be the first sides of these portions of polygons, and k, k', k'' the last, respectively. Consider these six lines as the sides of a circumscribed hexagon, the opposite sides being a and k, a' and k', a'' and k'', and a, a', a'' as alternate. The three diagonals joining the opposite summits intersect in one point (Brianchon's theorem), and the polar of this point by its intersection with the curve determines two points, each of which might be used as the point of contact of the first side of the required polygon.*" *Traité, p.* 354.

In both cases, when the lines found by the prescribed methods *touch* the curve, the two polygons coalesce in one; and when those lines do not meet the curve, the problem cannot be constructed—or in other words no polygons can exist which fulfil all the conditions.

Poncelet first published these constructions in the *Annales des Math.*, tom. viii, p. 147 (*Mai*, 1817); but without demonstration: and in a great degree incidentally. I am not aware of any continental writer having taken up the subject, beyond what Poncelet himself has done in his *Traité*; the resulting constructions being those given above. Still Poncelet had constructed these problems in a different way in a previous part of his work: though those constructions are wanting in the symmetry and simplicity which so peculiarly distinguish the methods given above.

This closes the account that I proposed to give of the labours of the Continental geometers upon the "Ancient Problem" and its dual; except, indeed, I shall find that I have omitted any attempt of the least importance to solve or extend it, made in writings which have not come to my knowledge. I shall next enter upon the English history of the problem: but I may remark at once that Mr. Gaskin has succeeded in proving Poncelet's construction—or rather been able to deduce it from an independent analysis of his own. See *Mechanics' Magazine, vol.* 48, *p.* 396. On this and on the solution of Exercise 154, in the present number of the *Mathematician*, by the same able geometer, I shall have something to say hereafter.

Littleheath, Charlton, Oct. 30, 1848.

ON ANALYTICAL GEOMETRY OF THREE DIMENSIONS RELATIVE TO OBLIQUE AXES.

[*Mr. Thomas Weddle, Wimbledon. Surrey.*]

(Continued from page 176.)

CHAPTER II.

On the Plane.

PROP. I. *To find the equation to a plane.*

Let p be the length of the *normal*, that is, the perpendicular from the origin on the plane, and $\lambda\mu\nu$ its directing cosines; also let r be the distance of any point (xyz) in the plane from the origin. The orthogonal projection of r on the normal is evidently P, and the orthogonal projection of the broken line $x + y + z$ on the same is, $\lambda x + \mu y + \nu z$,

$$\therefore \lambda x + \mu y + \nu z = p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49),$$

and since this relation holds for every point on the plane, it is the equation to the plane.

I am not aware that this very simple method of obtaining the equation to a plane has been published before, although a very near approach to it is made in *Gregory's Solid Geometry*, p. 30; indeed the very process here given is there made use of, merely however to *simplify* the equation which had been previously got in a different manner. The above method has this advantage (besides simplicity);—it immediately follows (as in the next proposition) from the equation so obtained that (the axes being either rectangular or oblique) the absolute term and the coefficients of the variables in the equation to a plane, are proportional to the length, and the directing cosines, of the normal.

PROP. II. *Given the equation to a plane, to find the length* (p) *of the normal, and its directing cosines* ($\lambda\ \mu\ \nu$).

Let the equation to the plane be

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = \mathrm{D} \qquad (50).$$

Multiply this by $\frac{\Gamma}{\mathrm{K}}$, and equate the coefficients of the resulting equation to those of (49);

$$\therefore p = \frac{\mathrm{D}\Gamma}{\mathrm{K}} \qquad (51),$$

$$\text{and,}\quad \lambda = \frac{\mathrm{A}\Gamma}{\mathrm{K}},\quad \mu = \frac{\mathrm{B}\Gamma}{\mathrm{K}},\quad \nu = \frac{\mathrm{C}\Gamma}{\mathrm{K}} \qquad (52).$$

Substitute these values of λ, μ and ν in (19),

$$\therefore\quad \mathrm{K}^2 = (1-f^2)\mathrm{A}^2 + (1-g^2)\mathrm{B}^2 + (1-h^2)\mathrm{C}^2 + 2(gh-f)\mathrm{BC}$$
$$+ 2(fh-g)\mathrm{AC} + 2(fg-h)\mathrm{AB} \qquad (53).$$

This equation determines K, and then p, λ, μ, ν are known by (51) and (52).

PROP. III. *To find the equations of the normal to a given plane* (50).

The required equations being

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad (54),$$

we have, (18, 52),

$$\left.\begin{aligned} \mathrm{K}\Gamma l &= (1-f^2)\mathrm{A} + (fg-h)\mathrm{B} + (fh-g)\mathrm{C} \\ \mathrm{K}\Gamma m &= (fg-h)\mathrm{A} + (1-g^2)\mathrm{B} + (gh-f)\mathrm{C} \\ \mathrm{K}\Gamma n &= (fh-g)\mathrm{A} + (gh-f)\mathrm{B} + (1-h^2)\mathrm{C} \end{aligned}\right\} \qquad (55);$$

and these equations determine the projecting coefficients, lmn, of the normal.

Hence, also, the equations to the straight line drawn through the point ($\alpha\beta\gamma$) perpendicular to the plane (50), are

$$\frac{x-a}{(1-f^2)\mathrm{A}+(fg-h)\mathrm{B}+(fh-g)\mathrm{C}}=\frac{y-\beta}{(fg-h)\mathrm{A}+(1-g^2)\mathrm{B}+(gh-f)\mathrm{C}}$$
$$=\frac{z-\gamma}{(fh-g)\mathrm{A}+(gh-f)\mathrm{B}+(1-h^2)\mathrm{C}} \quad \ldots\ldots(56).$$

PROP. IV. *Given the equations to a straight line, to find the equation to the plane which is perpendicular to it, and which passes through a point* $(a_1\beta_1\gamma_1)$.

Let (1), that is $\dfrac{x-a}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ denote the straight line. Since the normal to the plane is parallel to this straight line, its directing cosines will (16) be $l+hm+gn$, *etc.*, and the plane passes through the point $(a_1\beta_1\gamma_1)$; hence (49) the equation to the required plane must be

$$(l+hm+gn)(z-a_1)+(hl+m+fn)(y-\beta_1)+(gl+fm+n)(z-\gamma_1)=0 \ldots(57)$$

If the equations to the straight line are given in the form

$$\frac{x-a}{\mathrm{S}}=\frac{y-\beta}{\mathrm{T}}=\frac{z-\gamma}{\mathrm{U}} \ldots\ldots\ldots\ldots\ldots(58),$$

where S, T, U are only *proportional* to the directing cosines, the equation to the required plane would (14, 57) be

$$(\mathrm{S}+h\mathrm{T}+g\mathrm{U})(x-a_1)+(h\mathrm{S}+\mathrm{T}+f\mathrm{U})(y-\beta_1)+(g\mathrm{S}+f\mathrm{T}+\mathrm{U})(z-\gamma_1)=0 \quad \ldots.(59).$$

PROP. V. *To find the angles (cosines* $=\Lambda$, M, N) *which the plane* (50) *makes with the coordinate planes.*

These angles are the same as those which the normal to the plane makes with the normals to the coordinate planes; hence (20, 21, 55) we have

$$\left.\begin{aligned}\Lambda&=\frac{(1-f^2)\mathrm{A}+(fg-h)\mathrm{B}+(fh-g)\mathrm{C}}{\mathrm{K}(1-f^2)^{\frac{1}{2}}}\\ \mathrm{M}&=\frac{(fg-h)\mathrm{A}+(1-g^2)\mathrm{B}+(gh-f)\mathrm{C}}{\mathrm{K}(1-g^2)^{\frac{1}{2}}}\\ \mathrm{N}&=\frac{(fh-g)\mathrm{A}+(gh-f)\mathrm{B}+(1-h^2)\mathrm{C}}{\mathrm{K}(1-h^2)^{\frac{1}{2}}}\end{aligned}\right\} \ldots\ldots\ldots(60).$$

PROP. VI. *To determine the length* (P) *of the perpendicular from a given point* $(a\beta\gamma)$ *on a given plane* (50).

Let r be the distance of any point (xyz) in the plane from the given point; the orthogonal projection of r on the perpendicular is P, and the orthogonal projection of the broken line $(x-a)+(y-\beta)+(z-\gamma)$ is $\lambda(x-a)+\mu(y-\beta)+\nu(z-\gamma)$, $\lambda\mu\nu$ being the directing cosines of the normal to the plane, and hence those of the perpendicular,

$$\therefore\ \mathrm{P}=\lambda(x-a)+\mu(y-\beta)+\nu(z-\gamma)$$
$$=(52),\ \frac{\Gamma}{\mathrm{K}}\left\{\mathrm{A}(x-a)+\mathrm{B}(y-\beta)+\mathrm{C}(z-\gamma)\right\}.$$

Now, (50), $\quad Ax + By + Cz = D,$

$$\therefore \; P = \frac{\Gamma}{K}\Big\{D - (A\alpha + B\beta + C\gamma)\Big\} \ldots\ldots\ldots\ldots (61).$$

Hence if P_1, P_2, P_3 be the perpendiculars from the point $(\alpha\beta\gamma)$ on the planes of yz, xz, and xy respectively, we have

$$P_1 = \frac{\Gamma\alpha}{\sqrt{1-f^2}}, \quad P_2 = \frac{\Gamma\beta}{\sqrt{1-g^2}}, \quad P_3 = \frac{\Gamma\gamma}{\sqrt{1-h^2}} \ldots\ldots\ldots (62),$$

for, since $x = 0$, $y = 0$, and $z = 0$ are the equations to the coordinate planes, the corresponding values of K^2 are (53) $1-f^2$, $1-g^2$, and $1-h^2$ respectively.

Prop. vii. *To find the angle* (ϕ) *between a straight line* (58) *and a plane* (50).

The angle required is the complement of that contained between (58) and the normal to (50). Hence if l', m', n' be the projecting coefficients of (58), and $\lambda\mu\nu$ the directing cosines of the normal to (50), we have (22),

$\sin\phi = \lambda l' + \mu m' + \nu n'$. Now (14, 52), $l' = \frac{S}{W}$, *etc.*, $\lambda = \frac{A\Gamma}{K}$, *etc.*;

$$\therefore \; \sin\phi = \frac{\Gamma}{KW}\Big\{AS + BT + CU\Big\} \ldots\ldots (63),$$

the values of K and W being the same as in (53, 15).

Let ϕ_1, ϕ_2, ϕ_3 be the angles which the straight line (58) makes with the planes of yz, xz, and xy respectively, and ϕ', ϕ'', ϕ''' those which the plane (50) makes with the axes of x, y, and z. Since $x=0$ is the equation to the plane of yz, if we put $A = 1$, $B = C = 0$, in (63), so that $K^2 = 1-f^2$, we shall have $\sin\phi_1 = \frac{\Gamma S}{W\sqrt{1-f^2}}$; also since $y=0$, $z=0$, are the equations of the axis of x, if we put $S = 1$, $T = U = 0$, in (63), so that $W = 1$, we shall have $\sin\phi' = \frac{\Gamma A}{K}$,

$$\therefore \; \sin\phi_1 = \frac{\Gamma S}{W\sqrt{1-f^2}}, \quad \sin\phi_2 = \frac{\Gamma T}{W\sqrt{1-g^2}}, \quad \sin\phi_3 = \frac{\Gamma U}{W\sqrt{1-h^2}} \ldots (64),$$

$$\text{and} \;\; \sin\phi' = \frac{\Gamma A}{K}, \quad \sin\phi'' = \frac{\Gamma B}{K}, \quad \sin\phi''' = \frac{\Gamma C}{K} \ldots\ldots\ldots\ldots\ldots (65).$$

These equations (64) and (65) also immediately follow from (14, 20, 21) and (52).

Prop. viii. *To determine the angle* (Ψ) *between two planes.*

Let $\quad Ax + By + Cz = D \ldots\ldots\ldots\ldots\ldots\ldots\ldots (66),$

and $\quad A_1x + B_1y + C_1z = D_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (67),$

be the equations to the two planes; then $\lambda\mu\nu$ and $\lambda_1\mu_1\nu_1$ being the directing cosines of the normals to the planes, we have (52) $\lambda = \frac{A\Gamma}{K}$, *etc.* and $\lambda_1 = \frac{A_1\Gamma}{K_1}$, *etc.*; now the required angle is the same as that between the normals; hence, (24),

$$KK_1 \cos\Psi = (1-f^2)AA_1 + (1-g^2)BB_1 + (1-h^2)CC_1 + (gh-f)(BC_1+B_1C)$$
$$+(fh-g)(AC_1+A_1C)+(fg-h)(AB_1+A_1B) \ldots\ldots\ldots (68).$$

Also, (26),

$$\frac{K^2K_1^2}{\Gamma^2}\sin^2\Psi = (B_1C-BC_1)^2 + (AC_1-A_1C)^2 + (A_1B-AB_1)^2$$
$$+2f(A_1B-AB_1)(AC_1-A_1C)+2g(B_1C-BC_1)(A_1B-AB_1)$$
$$+2h(B_1C-BC_1)(AC_1-A_1C) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (69).$$

It is almost unnecessary to observe that, in (68) and (69), K_1 is the same function of $A_1B_1C_1$ that K is of ABC.

PROP. IX. *To find the angles* ($a_1a_2a_3$) *which the coordinate planes make with each other.*

Since any two of the coordinate planes make *two* angles (which are supplementary) with each other, $a_1a_2a_3$ will each admit of two values, and this can only be avoided by laying down a conventional rule which shall restrict these symbols to one value each.

Suppose that portion of the plane of xy which contains the positive axis of y to revolve (the shorter way) round the axis of x, until it coincides with that portion of the plane of xz which contains the positive axis of z, then a_1 is the angle through which the plane has revolved. In like manner a_2 is the angle through which that portion of the plane of yz that contains the positive axis of z must revolve round the axis of y to coincide with that portion of the plane of xy which contains the positive axis of x; and a_3 is the angle through which that portion of the plane of xz that contains the positive axis of x must revolve round the axis of z to coincide with that portion of the plane of z which contains the positive axis of y.

It is clear from what has just been said, that if normals be drawn to the planes of xz and xy — the first on the same side of the plane as the *positive* axis of y, and the other on the same side as the *negative* axis of z, then will a_1 be equal to the angle contained between these normals. Now (20), the directing cosines of the first are $\left(0, \frac{\Gamma}{\sqrt{1-g^2}}, 0\right)$, and those of the other are $\left(0, 0, -\frac{\Gamma}{\sqrt{1-h^2}}\right)$; hence (24) $\cos a_1 = \frac{f-gh}{\sqrt{(1-g^2)(1-h^2)}}$;

also, (26) $\sin a_1 = \frac{\Gamma}{\sqrt{(1-g^2)(1-h^2)}}$, (the positive sign being taken in

extracting the square root, because a_1 being less than $180°$, $\sin a_1$ is positive);

$$\therefore \quad \cos a_1 = \frac{f-gh}{\sqrt{(1-g^2)(1-h^2)}}, \quad \cos a_2 = \frac{g-fh}{\sqrt{(1-f^2)(1-h^2)}},$$

$$\cos a_3 = \frac{h-fg}{\sqrt{(1-f^2)(1-g^3)}} \dots\dots (70)^*$$

$$\sin a_1 = \frac{\Gamma}{\sqrt{(1-g^2)(1-h^2)}}, \quad \sin a_2 = \frac{\Gamma}{\sqrt{(1-f^2)(1-h^2)}}$$

$$\sin a_3 = \frac{\Gamma}{\sqrt{(1-f^2)(1-g^3)}} \dots\dots (71).$$

Also,

$$\tan a_1 = \frac{\Gamma}{f-gh}, \quad \tan a_2 = \frac{\Gamma}{g-fh}, \quad \tan a_3 = \frac{\Gamma}{h-fg} \dots (72).$$

The equations (70) and (71) might (as to absolute magnitude) also be easily derived from (68) and (69), and indeed I first deduced them in this way, but I was obliged to call in the aid of spherical trigonometry $\left(\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}\right)$ to determine the *signs* of $\cos a_1$, *etc.* The method given above determines at once both magnitude and sign, and is therefore free from this defect.

Prop. x. *To find the condition that two planes* (66, 67) *may be perpendicular.*

Here $\cos \Psi = 0$, and $\sin \Psi = 0$, hence (68, 69) the condition is

$$(1-f^2)AA_1 + (1-g^2)BB_1 + (1-h^2)CC_1 + (gh-f)(BC_1+B_1C)$$
$$+ (fh-g)(AC_1+A_1C) + (fg-h)(AB_1+A_1B) = 0 \dots (73).$$

Or,

$$(B_1C-BC_1)^2 + (AC_1-A_1C)^2 + (A_1B-AB_1)^2 + 2f(A_1B-AB_1)(AC_1+A_1C)$$
$$+ 2g(B_1C-BC_1)(A_1B-AB_1) + 2h(B_1C-BC_1)(AG_1-A_1C) = \frac{K^2K_1^2}{\Gamma^2} \dots (74).$$

* By means of (70) the expressions (45) for d', d'', d''' the distances of a point $(x_1y_1z_1)$ from the axes may be simplified.

If, as in a previous note, we denote the angles of ordination by $\epsilon_1\ \epsilon_2\ \epsilon_3$, we shall have $1-f^2 = \sin^2 \epsilon_1$, *etc.*; hence (70) $f-gh = \sin \epsilon_2 \sin \epsilon_3 \cos a_1$ *etc.*, and (45) becomes

$$d'^2 = y_1^2 \sin^2\epsilon_3 + z_1^2 \sin^2\epsilon_2 + 2y_1z_1 \sin \epsilon_2 \sin \epsilon_3 \cos a_1,$$
$$d''^2 = x_1^2 \sin^2\epsilon_3 + z_1^2 \sin^2\epsilon_1 + 2x_1z_1 \sin \epsilon_1 \sin \epsilon_3 \cos a_2,$$
$$d'''^2 = x_1^2 \sin^2\epsilon_2 + y_1^2 \sin^2\epsilon_1 + 2x_1y_1 \sin \epsilon_1 \sin \epsilon_2 \cos a_3.$$

Several other formulas may also be modified by means of (70); thus, (60) we have

$$\Lambda = \frac{1}{K}\left\{\sin \epsilon_1 A - \cos a_3 \sin \epsilon_2 B - \cos a_2 \sin \epsilon_3 C\right\}, \textit{ etc.}$$

PROP. XI. *To find the conditions that a straight line* (58) *and a plane* (50) *may be perpendicular.*

The directing cosines of the straight line must coincide with those of the normals to the plane; now the former are (14, 16), proportional to $S+hT+gU$, $hS+T+fU$, and $gS+fT+U$, and (52) the latter to ABC; hence the conditions required are

$$\frac{S+hT+gU}{A}=\frac{hS+T+fU}{B}=\frac{gS+fT+U}{C}\dots\dots(75).$$

PROP. XII. *To determine the conditions that two planes* (66, 67) *may be parallel.*

Since the directing cosines of the normal to one of these planes must be equal to those of the normal to the other, the conditions are (52)

$$\frac{A}{A_1}=\frac{B}{B_1}=\frac{C}{C_1}\dots\dots\dots\dots\dots\dots(76).$$

PROP. XIII. *To find the condition that a straight line* (58) *and a plane* (50) *may be parallel.*

Here ϕ must $=0$, in (63), hence the condition is

$$AS+BT+CU=0\dots\dots\dots\dots\dots\dots(77).$$

If the straight line be not merely parallel to the plane but entirely situated in it, then $(\alpha\beta\gamma)$ being a point in the plane, we must, in addition to (77), have the condition,

$$A\alpha+B\beta+C\gamma=D\dots\dots\dots\dots\dots\dots(78).$$

PROP. XIV. *To find the equation to the plane passing through three given points* $(x_1y_1z_1)$, $(x_2y_2z_2)$ *and* $(x_3y_3z_3)$.

Denote the equation by

$$q_1x+q_2y+q_3z=V\dots\dots\dots\dots\dots\dots(79),$$

we must have

$$q_1x_1+q_2y_1+q_3z_1=V,$$

$$q_1x_2+q_2y_2+q_3z_2=V,$$

and

$$q_1x_3+q_2y_3+q_3z_3=V.$$

The three ratios $q_1\div V$, $q_2\div V$, $q_3\div V$, being obtained from these equations, and $2V$ assumed equal to the common denominator, we shall have

$$\left.\begin{aligned}\pm 2q_1&=y_1z_2-y_2z_1+y_2z_3-y_3z_2+y_3z_1-y_1z_3\\ \pm 2q_2&=x_2z_1-x_1z_2+x_1z_3-x_3z_1+x_3z_2-x_2z_3\\ \pm 2q_3&=x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3\end{aligned}\right\}\dots\dots(80),$$

and, $$\pm 2V=x_1y_2z_3-x_1y_3z_2+x_2y_3z_1-x_2y_1z_3+x_3y_1z_2-x_3y_2z_1\dots(81).$$

We have thus found q_1, q_2, q_3 and V, and hence the equation (79) to the plane required is completely determined. (In the left hand members of (80) and (81) the same sign must be used.)

PROP. XV. *To find the projection* (P) *of a plane area* (Q) *on a plane, the generators of the projecting cylinder being parallel to a given line.*

Let ϕ and Ψ denote the angles which the given straight line makes with the normals to the planes of Q and P. If the projecting cylinder be cut by a plane perpendicular to its generators and R be the area of the section, it is evident that Q and P will both have R for orthogonal projection on this plane; hence ϕ and Ψ being the angles which the planes of Q and P make with that of R, we have $R = Q\cos\phi$, and $R = P\cos\Psi$;

$$\therefore\ P\cos\Psi = Q\cos\phi \dots\dots\dots\dots (82),$$

which determines the projection P.

By the *coordinate projection* of a plane area on one of the coordinate planes, I mean the projection made by a cylinder whose generators are parallel to the coordinate axis out of this plane.

PROP. XVI. *To determine the coordinate projections of any plane area* (Q).

Let Q_1, Q_2, Q_3 be the coordinate projections on the planes of yz, xz, and xy respectively, and $\lambda\mu\nu$ the directing cosines of the normal to the plane of Q. We evidently have by (20 ,82)

$$Q_1 = \frac{\lambda}{e}Q,\quad Q_2 = \frac{\mu}{i}Q,\quad Q_3 = \frac{\nu}{u}Q \dots\dots\dots\dots (82).$$

Hence also,

$$\lambda = \frac{Q_1}{Q}e,\quad \mu = \frac{Q_2}{Q}i,\quad \nu = \frac{Q_3}{Q}u \dots\dots\dots\dots (.84).$$

It will be observed that I here assume Q_1, Q_2, Q_3 to have the same signs as $\lambda\mu\nu$, the directing cosines of the normal to the plane Q, and hence if any one of these cosines, as μ, be negative, the corresponding projection Q_2 must be taken negatively. This must be carefully kept in mind, especially in (85) and (86), which immediately follow.

PROP. XVII. *Required the area of a plane surface in terms of its coordinate projections.*

Substitute the values of $\lambda\mu\nu$ from (84) in (19), and reduce by (20),

$$\therefore\ Q^2 = Q_1^2 + Q_2^2 + Q_3^2 + \frac{2(gh-f)}{\sqrt{(1-g^2)(1-h^2)}}Q_2Q_3 + \frac{2(fh-g)}{\sqrt{(1-f^2)(1-h^2)}}Q_1Q_3$$
$$+ \frac{2(fg-h)}{\sqrt{(1-f^2)(1-g^2)}}Q_1Q_2 \dots\dots\dots\dots (85).$$

By means of (70) this takes the following very simple and elegant form, which so far as I am aware, has not been given before,

$$Q^2=Q_1^2+Q_2^2+Q_3^2-2Q_2Q_3\cos a_1-2Q_1Q_3\cos a_2-2Q_1Q_2\cos a_3 \text{*} \ldots (86).$$

PROP. XVIII. *Required the area* (Q) *of the plane triangle whose angular points are at three given points* $(x_1y_1z_1)$, $(x_2y_2z_2)$ *and* $(x_3y_3z_3)$.

If Q_1, Q_2, Q_3 be the coordinate projections of Q, and q_1, q_2, q_3 have the same values as in (80), it is easily shown that

$$Q_1=q_1\sqrt{1-f^2},\ Q_2=q_2\sqrt{1-g^2},\ Q_3=q_3\sqrt{1-h^2}\ldots\ldots\ldots(87).$$

These substituted in (85) give

$$Q^2=(1-f^2)q_1^2+(1-g^2)q_2^2+(1-h^2)q_3^2+2(gh-f)q_2q_3$$
$$+2(fh-g)q_1q_3+2(fg-h)q_1q_2\ldots\ldots\ldots\ldots\ldots(88),$$

the expression required.

In writing down (87), the signs of Q_1, Q_2, Q_3 and q_1, q_2, q_3 have been taken into account, as well as their absolute magnitudes. For (52, 79) and (83) each of these two sets of quantities is proportional to $\lambda\mu\nu$ the directing cosines of the normal to the plane of Q; hence if in (80), we select that sign which shall make q_1 to have the same sign as Q_1, then shall q_2 and q_3 have the same signs as Q_2 and Q_3, so that we cannot have $Q_2=-q_2\sqrt{1-g^2}$ nor $Q_3=-q_3\sqrt{1-h^2}$. It also appears that since Q_1, Q_2, Q_3 are proportional to q_1, q_2, q_3, we shall determine the signs of the former from those of the latter without knowing the directing cosines of the normal to the plane of Q.

From (20, 84, 87) we have

$$\lambda=\frac{q_1}{Q}\Gamma,\ \mu=\frac{q_2}{Q}\Gamma,\ \nu=\frac{q_3}{Q}\Gamma\ \ldots\ldots\ldots\ldots(89),$$

$\lambda\mu\nu$ being the directing cosines of the normal to the plane of Q.

Substituting the values of q_1, q_2, q_3 as derivable from (89) in (79), we immediately see, by the aid of (49), that if p denote the length of the normal to the plane of Q,

$$p=\frac{V}{Q}\Gamma.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(90).$$

The last result enables us to solve the following problem, with which I shall close this chapter.

* If $r_1r_2r_3$ be the coordinate projections of a terminated straight line r, then $\frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r}$ being the projecting coefficients of r, we shall (12), have

$$r^2=r_1^2+r_2^2+r_3^2+2fr_2r_3+2gr_1r_3+2hr_1r_2,$$

that is, $$r^2=r_1^2+r_2^2+r_3^2+2r_2r_3\cos\epsilon_1+2r_1r_3\cos\epsilon_2+2r_1r_2\cos\epsilon_3,$$

which is remarkably analogous to (86). The analogy would be still closer, were we to measure the inclinations $a_1a_2a_3$ of the coordinate planes to each other, so that these symbols should stand for the supplements of the angles which they now denote.

PROP. XIX. *To find the volume* (V′) *of a tetrahedron whose angular points coincide with the origin, and three given points* $(x_1y_1z_1)$, $(x_2y_2z_2)$ *and* $(x_3y_3z_3)$.

Retaining the preceding notation, we have

$$\mathrm{V}' = \frac{1}{3}\,\mathrm{Q}p =,(90),\frac{1}{3}\Gamma\mathrm{V}.$$

Hence (81) the required volume, is

$$\mathrm{V}' = \pm\frac{1}{6}\Gamma\left\{x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_2y_1z_3 + x_3y_1z_2 - x_3y_2z_1\right\}\ldots(91).$$

If the three points be (X, 0, 0), (0, Y, 0), (0, 0, Z), then

$$\mathrm{V}' = \frac{1}{6}\Gamma\,\mathrm{XYZ}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(92),$$

where X, Y, Z are three contiguous edges of the tetrahedron, and $\Gamma^2 = 1 - f^2 - g^2 - h^2 + 2fgh$, fgh being the cosines of the angles which these edges make with each other.

AN ORGANIZED METHOD OF MAKING THE RESOLUTION REQUIRED IN THE INTEGRATION OF RATIONAL FRACTIONS.

[*Professor De Morgan.*]

In the integration of the rational function $\phi x(x-a)^{-m}(x-b)^{-n}(x-c)^{-p}\ldots$ in which ϕx is rational and integral, the process by which the numerator to $(x-a)^m$ is found, is well known to be the expansion of $\phi x(x-b)^{-n}(x-c)^{-p}\ldots$ in powers of $x-a$, as far as the term involving $(x-a)^{m-1}$; and similarly for the other parts of the resolution; adding the integral quotient, if there be one. A rule may be organized for dividing $\phi a + \phi' a.(x-a) + \ldots$, first by $(a-b+x-a)^n$, then by $(a-c+x-a)^p$, and so on, each division being carried as far as the term in $(x-a)^{m-1}$. The complexity of the examples which any one who learns this rule may venture to undertake, will be the best justification for its publication.

First, to divide $a_0 + a_1x + a_2x^2 + \ldots$ by $(m+x)^n$ as far as the term in x^k. Make a *direct* multiplication of the coefficients by $1, m, m^2, \ldots m^k$, by which I mean, write down

$$a_0,\ a_1m,\ \ldots a_2m^2 \ldots\ldots a_km^k.$$

Write down the first of these, subtract it from the next, *the result* from the next, and so on to the end. Repeat this process n times, the resulting row contains the numerators, and m^n, m^{n+1}, *etc.* are the denominators.

For instance, to divide $3-2x-3x^3+x^4$ by $(2+x)^4$ as far as x^5, we have

$$\begin{array}{rrrrrrr}
 & 3- & 2+ & 0- & 3+ & 1+ & 0 \\
 & 3- & 4+ & 0- & 24+ & 16+ & 0 \\
\hline
1 & 3- & 7+ & 7- & 31+ & 47- & 47 \\
2 & 3- & 10+ & 17- & 48+ & 95- & 142 \\
3 & 3- & 13+ & 30- & 78+ & 173- & 315 \\
4 & 3- & 16+ & 46- & 124+ & 297- & 612
\end{array}$$

Accordingly the result is $\frac{3}{2^4}-\frac{16}{2^5}x+\ldots-\frac{612}{2^9}x^5$.

If we wish to make a further division by $(m'+x)^{n'}$, then, the numerators just given being b_o, b_1, *etc.*, we reduce the preceding fractions to a common denominator by an *inverse* multiplication by powers of m, giving

$$b_o\, m^k,\ b_1 m^{k-1}\ldots\ldots b_{k-1}m,\ b_k\,;$$

and we reserve m^{k+n} for a common denominator factor at the end of the process. We then proceed with powers of m', and repeat the process, noting that nothing beyond x^k in a process can be of use in any one which follows. And it may often happen that a common factor can be taken out of a whole row, and reserved for a numerator factor at the end, or applied in reduction of the reserved (or of any reserved) denominator factor. Of course ϕx should be arranged in powers of $x-a$ by substituting $x+a$ for x, by Horner's method, and then writing $x-a$ for x in the result. I consider the whole of this rule as derived from Horner's suggestions, which are the great abbreviators of operation over the whole working field of algebra.

I shall take as the first instance, $\dfrac{x^4-6x^3}{(x-2)^3(x-1)^2(x-3)(x-5)}$, resolved in my *Differential Calculus* (pp. 272-276) by far less convenient methods. Still that instance is, so far as I know, the most complex which occurs in any elementary work; but it is too trivial to show the power of this rule. Each expansion may be verified by taking the divisors in another order.

First, for $(x-2)^3$

$$\begin{array}{l}
1-6+0\ +0\ +0 \\
1-4-8\ -16-32 \\
1-2-12-40 \\
1-0-12
\end{array}
\qquad
\begin{array}{rrrrl}
 & -32 & -40 & -12 & \\
\text{Num}^{\text{r}}\ 4 & -8 & -10 & -3 & x-1=1+(x-2) \\
\cline{2-4}
 & -8 & -2 & -1 & \\
 & -8 & +6 & -7 & \\
\cline{2-4}
 & -8 & -6 & -7 & x-3=-1+(x-2) \\
 & -8 & +2 & -9 & \\
\cline{2-4}
 & +8 & +2 & +9 & \text{Denom}^{\text{r}}\text{ introduced} \\
 & +8 & -6 & +81 & x-5=-3+(x-2) \\
\cline{2-4}
 & +8 & -14 & +95 &
\end{array}$$

Partial result, $\dfrac{32}{-3}\,\dfrac{1}{(x-2)^3}-\dfrac{56}{(-3)^2}\,\dfrac{1}{(x-2)^2}+\dfrac{380}{(-3)^3}\,\dfrac{1}{x-2}$.

Next, for $(x-1)^2$

$$\begin{array}{l}
1-6+0+0\ \ +0 \\
1-5-5-5\ \ -5 \\
1-4-9-14
\end{array}
\qquad
\begin{array}{rrl}
-5 & -14 & x-2=-1+(x-1) \\
-5 & +14 & \\ \hline
-5 & +19 & \\
-5 & +24 & \\
-5 & +29 & \\ \hline
+5 & +29 & \text{denom}^{\text{r}} \text{ introduced} \\
+5 & -58 & x-3=-2+(x-1) \\ \hline
+5 & -63 & \\
-10 & -63 & \text{denom}^{\text{r}}\ (-2)^2 \\
-10 & +252 & x-5=-4+(x-1). \\ \hline
-10 & +262 &
\end{array}$$

Partial result, $\dfrac{1}{(-2)^2}\left\{\dfrac{-10}{-4}\,\dfrac{1}{(x-1)^2}+\dfrac{262}{(-4)^2}\,\dfrac{1}{x-1}\right\}.$

When only the first power of a factor enters, the factor must simply be expunged, and restored after making x equal to its root. Thus the remaining parts of the resolution are $\dfrac{3^4-6.3^3}{2^2(-2)}\cdot\dfrac{1}{x-3}$ and $\dfrac{5^4-6.5^2}{3^3.4^2.2}\cdot\dfrac{1}{x-5}$.

I have given this example merely that the reader who has examined the case in my *Differential Calculus* may judge of the great amount of labour which is saved. I will now take one of a more complicated character.

Let it be $$\frac{x^7-3x^6+2x^5+x^4-x+1}{(x-1)^4(x+1)^4(x-2)^3(x+2)^2(x-8)}.$$

Take the factors separately, and write down as a heading to each the denominators of the first quotient terms: which will exhibit the multipliers, the numbers of divisions to be made, and the proposed order of the processes.

$$\begin{array}{l}
\qquad (x-1)^4 \\
1-3+2+1+0+0-1+1 \\
1-2+0+1+1+1+0+1 \\
1-1-1+0+1+2+2 \\
1-0-1-1-0+2 \\
1+1+0-1-1
\end{array}
\qquad
\begin{array}{r|rrrrl}
2^4 & (-1)^3 & 3^2 & -7 & & \\
 & 1 & +2 & +2 & -1 & \\
2^4 & 1 & +4 & +8 & -8 & \\ \hline
 & 1 & +3 & +5 & -13 & \\
 & 1 & +2 & +3 & -16 & \\
 & 1 & +1 & +2 & -18 & \\
 & 1 & +0 & +2 & -20 & \\ \hline
\text{num}^{\text{r}}\ 2^2 & 2 & +0 & +1 & -5 & \text{den}^{\text{r}}\ 2^7 \\
(-1)^3 & 2 & -0 & +1 & +5 & \\ \hline
 & 2 & -2 & +3 & +2 & \\
 & 2 & -4 & +7 & -5 & \\
 & 2 & -6 & +13 & -18 & \\ \hline
 & -2 & -6 & -13 & -18 & \text{den}^{\text{r}} \text{ introduced} \\
3^2 & -2 & -18 & -117 & -486 & \\ \hline
 & -2 & -16 & -101 & -385 & \\
 & -2 & -14 & -87 & -298 & \\ \hline
 & -54 & -126 & -261 & -298 & \text{den}^{\text{r}}\ 3^5 \\
-7 & -54 & +882 & -12789 & +102214 & \\ \hline
 & -54 & +936 & -13725 & +115939 &
\end{array}$$

$$\frac{2^2}{2^7.3^5}\left\{\frac{-54}{-7}\cdot\frac{1}{(x-1)^4}+\frac{936}{(-7)^2}\cdot\frac{1}{(x-1)^3}+\frac{-13725}{(-7)^3}\cdot\frac{1}{(x-1)^2}+\frac{115939}{(-7)^4}\cdot\frac{1}{x-1}\right\}.$$

$(x+1)^4 \qquad (-2)^4 \qquad (-3)^3 \qquad 1^2 \qquad -9$

1−3+2 +1 +0 +0 −1 +1		−3	+30	−80	+111	
1−4+6 −5 +5 −5 +4 −3	$(-2)^4$	−3	−60	−320	−888	
1−5+11−16+21−26+30		−3	−57	−263	−625	
1−6+17−33+54−80		−3	−54	−209	−416	
1−7+24−57+111		−3	−51	−158	−258	
		−3	−48	−110	−148	
	num^r (-2^2)	+6	−48	+ 55	− 37	d^r $(-2)^7$
	num^r (-3), $(-3)^3$	−2	−48	−165	−333	
		−2	−46	−119	−214	
		−2	−44	− 75	−139	
		−2	−42	− 33	−106	
	1^2	+54	−378	+ 99	−106	d^r $(-3)^6$
		+54	−432	+531	−637	
		+54	−486	+1017	−1654	
	num^r 27 −9	+2	+162	+3051	+44658	
		+2	+160	+2891	+41767	

$$\frac{(-2)^2(-3)(27)}{(-2)^7\,(-3)^6}\left\{\frac{2}{-9}\cdot\frac{1}{(x+1)^4}+\frac{160}{(-9)^2}\cdot\frac{1}{(x+1)^3}\right.$$
$$\left.+\frac{2891}{(-9)^3}\cdot\frac{1}{(x+1)^2}+\frac{41767}{(-9)^4}\cdot\frac{1}{x+1}\right\}.$$

$(x-2)^3 \qquad 1^4 \qquad 3^4 \qquad 4^2 \qquad -6$

1−3+2+1 +0 +0 −1 +1	1^4	+15	+63	+136	
1−1+0+1 +2 +4 +7 +15		+15	+48	+88	
1+1+2+5 +12+28 +63		+15	+33	+55	
1+3+8+21+54+136		+15	+18	+37	
		+15	+ 3	+34	
	num^r 3 $\quad 3^4$	+5	+ 3	+102	
		+5	− 2	+104	
		+5	− 7	+111	
		+5	−12	+123	
		+5	−17	+140	
		+45	−51	+140	den^r 3^6
	4^2	+45	−204	+2240	
		+45	−249	+2489	
		+45	−294	+2783	
		+720	−1176	+2783	den^r 4^4
	num^r 6^2 −6	20	+196	+2783	
		20	+176	+2607	

$$\frac{3.6^2}{3^6.4^4}\left\{\frac{20}{-6}\cdot\frac{1}{(x-2)^3}+\frac{176}{(-6)^2}\cdot\frac{1}{(x-2)^2}+\frac{2607}{(-6)^3}\cdot\frac{1}{x-2}\right\}.$$

$$\begin{array}{llllllll}
\multicolumn{2}{l}{(x+2)^2} & (-3)^4 & & (-1)^4 & & (-4)^3 & -10 \\
1-3+2 & +1 & +0 & +0 & -1 & +1 & & \\
1-5+12-23 & +46 & -92 & +183 & -365 & & & \\
1-7+26-75 & +196 & -484 & +1151 & & & &
\end{array}$$

$$\begin{array}{r|ll}
 & -365 & +1151 \\
(-3)^4 & -365 & -3453 \\ \hline
 & -365 & -3088 \\
 & -365 & -2723 \\
 & -365 & -2358 \\
 & -365 & -1993 \\ \hline
 & +1095 & -1993 \quad \text{den}^{\text{r}}(-3)^5 \\
(-1)^4 & +1095 & +1993 \\ \hline
 & +1095 & +898 \\
 & +1095 & -197 \\
 & +1095 & -1292 \\
 & +1095 & -2387 \\ \hline
 & -1095 & -2387 \quad \text{den}^{\text{r}}(-1)^5 \\
(-4)^3 & -1095 & +9548 \\ \hline
 & -1095 & +10643 \\
 & -1095 & +11738 \\
 & -1095 & +12833 \\ \hline
 & +4380 & +12833 \quad \text{den}^{\text{r}}(-4)^4 \\
\text{num}^{\text{r}}\ 10\ \ -10 & +438 & -12833 \\ \hline
 & +438 & -13271
\end{array}$$

$$\frac{10}{(-3)^5(-1)^5(-4)^4}\left\{\frac{438}{-10}\cdot\frac{1}{(x+2)^2}-\frac{13271}{(-10)^2}\cdot\frac{1}{x+2}\right\}.$$

$$\begin{array}{lllllllll}
x-8 & 1-3+2 & +1 & +0 & +0 & -1 & +1 & & \\
 & 1+5+42 & +337 & +2696 & +21568 & +172543 & +1380345 & &
\end{array}$$

$$\frac{1380345}{7^4.9^4.9^3.10^2}\cdot\frac{1}{x-8}.$$

All the work of this example is fully written down, so far as an ordinary computer need write, except the multiplications by powers having more than one digit. I have repeated many numbers superfluously, in order that those who are learning the operation may have an indication of every step.

ON THE THEORY OF SYMMETRIC FUNCTIONS.

[*James Cockle, Esq., M.A., Barrister-at-Law.*]

(Continued from page 178.)

The coefficient of P in $-\beta$ may be represented by

$$3(C'Q^2+2D'RQ+, etc.),$$

where

$$C'=n^2 \textstyle\int_n oq^2-(n-1)(n-2)\int_n o\{\int_n q\}^2,$$

and

$$D'=n^2\textstyle\int_n oqr-(n-1)(n-2)\int_n o\int_n q\int_n r\,;$$

but*

$$\textstyle\int_n oq^2=(n-2)\left(\{\int_n q\}^2-\int_n(2q)\right),$$

* Jerrard, *Math. Res.*, p. 5, or p. 57, note.

hence

$$C' = n(n-2)\{f_n q\}^2 - n^2(n-2) f_n(2q),$$
$$= n(n-2)\big(\{f_n q\}^2 - n f_n(2q)\big);$$

again *

$$f_n oqr = (n-2)\big(f_n q\, f_n r - f_n(q+r)\big),$$

consequently

$$D' = n(n-2) f_n q\, f_n r - n^2(n-2) f_n(q+r),$$
$$= n(n-2)\{f_n q\, f_n r - n f_n(q+r)\}:$$

and since neither C' nor D' is necessarily zero we infer that β contains P to one dimension.

Let us next examine the coefficient of P in the expression $p_1 \times a$, which, from what I have already † said, is seen to be free from P^3 and P^2. This coefficient is

$$-f_n o \times (CQ^2 + 2DQR + \ldots.),$$

where

$$C = n f_n q^2 - (n-1)\{f_n q\}^2,$$
$$= -n f_n(2q) + \{f_n q\}^2;$$

also

$$D = n f_n qr - (n-1) f_n q\, f_n r,$$
$$= -n f_n(q+r) + f_n q\, f_n r;$$

let λ be undetermined, then the coefficient of P in the expression

$$\lambda p_1 a - \beta,$$

is equal to

$$-(\lambda n C + 3C')Q^2 - 2(\lambda n D + 3D')QR + \ldots.;$$

now, if

$$\lambda n C + 3C' = 0,$$

we have

$$\lambda = -\frac{3C'}{nC} = 3(n-2);$$

but the value of $\lambda = 3(n-2)$ satisfies the relation

$$\lambda n D + 3D' = 0,$$

hence the expression

$$3(n-2)\, p_1 a - \beta,$$

is altogether free from P; changing its sign, substituting for a and β, and dividing by 2, it becomes

$$3n^2 p_3 - 3n(n-2) p_2 p_1 + (n-1)(n-2) p_1^3,$$

which I call a critical function of the third degree, and which I shall represent by β'. I ought perhaps to have added to this last discussion the remark that, if we had considered any other term of the expression for y (for instance $Q' x q'$), the result would have been the same as for Rx^r, which I have supposed to form part of the right hand side of (2) ‡. In what follows, I shall carry the argument so far only as to overcome the algebraical diffi-

* Ibid, p. 57, note. † *Supra*, p. 177. ‡ *Supra*, p. 176.

culty of the question, and shall leave the reader to supply for himself the necessary generalization of the process of reasoning.

We shall next proceed to the function

$$24n^3p_4-(n-1)(n-2)(n-3)p_1^4,$$

which I shall represent by γ, and in which the coefficient of P^4 is

$$\begin{aligned}&\textstyle\int_n o^4-(n-1)(n-2)(n-3)\{-\int_n o\}^4\\ &=n^4(n-1)(n-2)(n-3)-(n-1)(n-2)(n-3)n^4=0\,;\end{aligned}$$

hence γ is free from P^4. Again, the coefficient of P^3 in γ is

$$\begin{aligned}&4(n^3\textstyle\int_n o^3q-(n-1)(n-2)(n-3)\{\int_n o\}^3\int_n q)Q+\ldots.\\ &=4\{n^3(n-1)(n-2)(n-3)-(n-1)(n-2)(n-3)n^3\}Q\textstyle\int_n q+\ldots.\\ &=0Q+\ldots\ldots\end{aligned}$$

whence we may infer that P^3 vanishes from γ. So, the coefficient of P^2 in γ is

$$\begin{aligned}&10n^3\{Q^2\textstyle\int_n o^2q^2+2QR\int_n o^2qr+\ldots\}\\ &=10(n-1)(n-2)(n-3)\left\{Q^2\{\textstyle\int_n o\}^2\{\int_n q\}^2+2QR\{\int_n o\}^2\int_n q\int_n r\right\}\,;\end{aligned}$$

in which last expression rejecting the factor 10, and taking the coefficient Q^2, we obtain

$$\begin{aligned}&n^3\textstyle\int_n o^2q^2-(n-1)(n-2)(n-3)\{\int_n o\}^2\{\int_n q\}^2\\ &=n^3(n-2)(n-3)\textstyle\int_n q^2-(n-1)(n-2)(n-3)\,n^2\{\int_n q\}^2\\ &=n^2(n-2)(n-3)\left\{\{\textstyle\int_n q\}^2-n\int_n(2q)\right\}\\ &=A'',\text{ suppose.}\end{aligned}$$

Also let $B''=n^3\int_n o^2qr-(n-1)(n-2)(n-3)\,n^2.\int_n q\int_n r$;

then,

$$\begin{aligned}\textstyle\int_n o^2qr&=n^2\textstyle\int_n q\int_n r-\{5n\int_n q\int_n r+n^2\int_n(q+r)\}\\ &\quad+2\{2\textstyle\int_n q\int_n r+2n\int_n(q+r)\}+\{2\int_n q\int_n r+n\int_n(q+r)\}-6\int_n(q+r)\\ &=(n-2)(n-3)\{\textstyle\int_n q\int_n r-\int_n(q+r)\}.\end{aligned}$$

I may observe, in passing, that the last process not only illustrates a formula given at pages 58-59 (note) of Mr. Jerrard's *Researches*, but its result shows that

$$\textstyle\int_n o^2qr=(n-2)(n-3)\int_n qr,$$

as in the case * when q and r are equal. We now see that

$$B''=n^2(n-2)(n-3)\{\textstyle\int_n q\int_n r-n\int_n(q+r)\},$$

and that

$$\gamma=10(A''Q^2+2B''QR+\ldots.)P^2+\ldots\ldots,$$

and, consequently, that γ involves P^2.

Let λ, as before, be indeterminate, and let us inquire whether such a value may not be given to λ as to cause P^2 to vanish from

$$\gamma+\lambda\beta p_1,$$

an expression which, as attention to what precedes will convince the reader, is free from P^4 and P^3, and in which the coefficient of P^2 is

$$(10A''+3\lambda nC')Q^2+2(10B''+3\lambda nD')QR+\ldots.\,;$$

* As to this case see Jerrard's *Math. Res.*, p. 5.

hence, since

$$\frac{A''}{C'} = \frac{B''}{D'} = -(n-3),$$

we see that if

$$\lambda = -\frac{10}{3}(n-3),$$

the coefficient of P^2 in $\gamma + \lambda\beta p_1$ will vanish. Hence into the function

$$72p_4 - 60n^2(n-3)p_3p_1 + 7(n-1)(n-2)(n-3)p^4_1,$$

we see that P^2 does not enter.

The above investigations might have been somewhat shortened, thus, from the equation

$$\lambda n D + 3D' = 0,$$

we might have inferred that

$$\lambda n C + 3C' = 0,$$

the latter equation being what the former becomes when $q = r$, and this method I shall employ in future. To the reader of Mr. Jerrard's *Researches* the two cases will however be somewhat different, for in determining the value of such a function as $f_n o q^2$ we can at once refer to the formula of page 5 of Mr. Jerrard's work, while for such a function as $f_n o q r$ we are obliged to have recourse to the note at page 57 of that work. The following generalization of Mr. Jerrard's formula will be found useful, and may be proved thus :—

Supposing the quantities $p, q, r \ldots$ to be μ in number, the quantity

$$f_n o^{m-\mu} pqr \ldots$$

will consist of

$$n(n-1)(n-2).(n-m+1)$$

terms, of which the

$$n(n-1)(n-2).(n-\mu+1)$$

comprised in the expression

$$f_n pqr \ldots$$

will recur. Hence we see that

$$\frac{f_n o^{m-\mu} pqr \ldots}{f_n pqr \ldots} = \frac{n(n-1)\ldots(n-m+1)}{n(n-1)\ldots(n-\mu+1)},$$

as in the case of $p = q = r =$, *etc.*

Postscript.—In a note at page 179 of this volume, I observe that in alluding to my " Account" of the Method of Vanishing Groups, a reference to the *volume* of the *Philosophical Magazine* is omitted. The " Account" is in the 32d volume of that work. I take this opportunity of referring the reader to pp. 257-258 of the second volume of this work, where I gave a partial discussion of the subject treated of by Professor Hearn in the last number of the *Mathematician (supra* pp. 195-200). This discussion I have since completed in the *Mechanics' Magazine.* Professor Hearn has founded his paper on the *analytical* as I have my discussion on the *geometrical* properties of surfaces of the second degree. The reader ought to examine both the discussions.

2, Church-yard Court, Temple, 27th October, 1848.

NOTES ON TRIGONOMETRY.

[*Mr. Fenwick.*]

The following notes contain improvements it is hoped on some of the processes in elementary trigonometry.

I. *Formulæ involving an arc less than a quadrant and multiples of a quadrant.*

Let A be an arc, less than a quadrant, of the circle whose radius is the linear unit, and π, as usual, the semi-circumference of the same circle; then

$$\sin A = \sin(\pi - A) = -\sin(\pi + A) = -\sin(-A) \ldots\ldots\ldots(1),$$

$$\cos A = -\cos(\pi - A) = -\cos(\pi + A) = \cos(-A) \ldots\ldots\ldots(2).$$

These relations will also hold if we add to the arc A or take from it, any number of circumferences; for the resulting arcs will have the *same extremities* as at present. Whence, adding n circumferences, or $2n\pi$, to A in (1), we get

$$\sin A = \sin(2n\pi + A) = \sin\{(2n+1)\pi - A\} = -\sin\{(2n+1)\pi + A\}$$

$$= -(2n\pi - A) = \pm\sin(2n\pi \pm A) = \pm\sin\{(2n+1)\pi \mp A\} \ldots.(3).$$

By deducting n circumferences from the arc A in (1), we get in a similar way

$$\sin A = \pm\sin(-2n\pi \pm A) = \pm\sin\{-(2n-1)\pi \mp A\} \ldots\ldots(4).$$

Hence, if p be a positive or negative integer of the series 0, 2, 4, 6...., and q a positive or negative integer of the series 1, 3, 5...., we have by (3) and (4),

$$\sin(p\pi + A) = \sin A \ldots.(5), \qquad \sin(p\pi - A) = -\sin A \ldots\ldots(6),$$

$$\sin(q\pi + A) = -\sin A \ldots.(7), \qquad \sin(q\pi - A) = \sin A \ldots\ldots(8).$$

In a precisely similar way we get by means of (2),

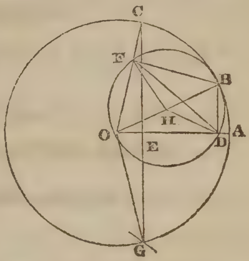
$$\cos(p\pi + A) = \cos A \ldots.(9), \qquad \cos(p\pi - A) = \cos A \ldots.(10),$$

$$\cos(q\pi + A) = -\cos A \ldots.(11), \qquad \cos(q\pi - A) = -\cos A \ldots.(12).$$

These important formulæ are well known. The use which I propose to make of them will be seen as I proceed.

II. *New and simple demonstrations of the formulæ for the sine and cosine of the sum and difference of two arcs.*

Let AB, BC be two arcs of the circle ACG, whose radius is the linear unit. From B draw BD, BF perpendicular respectively to the radii OA, OC, and from C draw CE perpendicular to OA, meeting the circle again in G. Bisect OB in H; join FH, HD, OG; and denote the arcs AB, BC, by A and B.



Then because each of the angles ODB, OFB is a right angle, a circle described from H, with $\frac{1}{2}$OB as radius, will pass through the points O, D, B, F. Hence the angle FHD = 2COA = COG; the triangles FHD, COG, are consequently similar, and therefore FD = $\frac{1}{2}$GC = EC, since FH = $\frac{1}{2}$CO. Also the perpendicular from H on FD = $\frac{1}{2}$OE; whence the difference of the perpendiculars from O, B, on FD, is equal to OE, as each of these quantities is double of the perpendicular from H on DF.

Hence (*Euc. vi.* D. C.)

$$OB.EC = BD.OF + BF.OD, \text{ and } OB.OE = OD.\,OF - BD.BF;$$

or
$$\sin(A + B) = \sin A \cos B + \sin B \cos A \ldots\ldots\ldots\ldots(13),$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B \ldots\ldots\ldots\ldots(14).$$

If we interchange the perpendiculars BD, CE, so that CE may represent the *difference* of A and B, we shall get in a similar way,

$$\sin(A - B) = \sin A \cos B - \sin B \cos A \ldots\ldots\ldots\ldots(15),$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B \ldots\ldots\ldots\ldots(16).$$

In this case, OE is equal to the *sum* of the perpendiculars from O and B on FD.

We may remind the student that (14, 15, 16) can be deduced from (13). For instance

$$\cos(A + B) = \sin\{\tfrac{1}{2}\pi - (A + B)\} = \sin\{(\tfrac{1}{2}\pi - A) + (-B)\}$$
$$= \sin(\tfrac{1}{2}\pi - A)\cos(-B) + \sin(-B)\cos(\tfrac{1}{2}\pi - A) = \cos A \cos B - \sin A \sin B;$$

since $\sin(\frac{1}{2}\pi - A) = \cos A$, $\cos(-B) = \cos B$, $\sin(-B) = -\sin B$,

and $\cos(\frac{1}{2}\pi - A) = \sin A$. And similarly for the others.

To generalize the formulæ (13, 14, 15, 16), we will now endeavour to prove that they hold for all values of A and B, whether positive or negative.

Put $A = m\pi - x$, $B = n\pi - y$; and let $A + B = (m+n)\pi - (x+y)$, be positive and less than $\frac{1}{2}\pi$; m and n being any positive or negative integers. It will hence be obvious that x and y are any positive or negative arcs. Now by (13),

$$\sin\{(m+n)\pi - (x+y)\} = \sin(m\pi - x)\cos(n\pi - y) + \sin(n\pi - y)\cos(m\pi - x) \quad \ldots (17).$$

Hence if m be even and n odd, and consequently $m+n$ odd, then by (8),

$$\sin\{(m+n)\pi-(x+y)\}=\sin(x+y).$$

Also by (6, 8, 10, 12),

$\sin(m\pi-x)=-\sin x$, $\cos(n\pi-y)=-\cos y$, $\sin(n\pi-y)=\sin y$,

$\cos(m\pi-x)=\cos x$. Whence

$$\sin(m\pi-x)\cos(n\pi-y)+\sin(n\pi-y)\cos(m\pi-x)$$
$$=\sin x\cos y+\sin y\cos x.$$

Hence by (17),

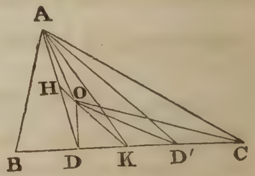
$$\sin(x+y)=\sin x\cos y+\sin y\cos x.$$

The same also follows when m and n are both even or both odd, and therefore the formula (13) holds universally. And so for the formulæ (14, 15, 16).

III. *Investigations of three important problems in Trigonometrical Surveying.*

I. *To find the height of an object from angles of elevation taken on a horizontal plane, at three given stations in a straight line, not in the direction of the object.*

Let AO be the object perpendicular to the plane DOC in which are the given stations D, K, C, in the same straight line. Denote the angles of elevation at C, K, D, that is, the angles ACO, AKO, ADO, by α, β, γ; and put $CK=a$, $KD=b$, and $AO=x$. Then



$$AC=x\operatorname{cosec}\alpha,\quad AK=x\operatorname{cosec}\beta,$$
$$AD=x\operatorname{cosec}\gamma.$$

Hence

$$\cos ACD=\frac{a^2+x^2\operatorname{cosec}^2\alpha-x^2\operatorname{cosec}^2\beta}{2ax\operatorname{cosec}\alpha}=\frac{(a+b)^2+x^2\operatorname{cosec}^2\alpha-x^2\operatorname{cosec}^2\gamma}{2(a+b)x\operatorname{cosec}\alpha},$$

or $$\{(a+b)(\operatorname{cosec}^2\alpha-\operatorname{cosec}^2\beta)+a(\operatorname{cosec}^2\gamma-\operatorname{cosec}^2\alpha)\}x^2=ab(a+b).$$

This equation by means of the general relation

$$\sin(A+B)\sin(A-B)=\sin^2A\sin^2B(\operatorname{cosec}^2B-\operatorname{cosec}^2A),$$

reduces to the following :—

$$\{(a+b)\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma+a\sin(\alpha+\gamma)\sin(\alpha-\gamma)\sin^2\beta\}x^2$$
$$=ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma,$$

or $$(1+\lambda)x^2=\mu\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

where $\lambda = \dfrac{a\sin(\alpha+\gamma)\sin(\alpha-\gamma)\sin^2\beta}{(a+b)\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma}$, $\mu = \dfrac{ab\sin^2\alpha\sin^2\beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}$.

Now *if* α *be greater than* γ *and less than* β; λ and μ will both be positive, and hence we may assume

$$\lambda = \tan^2\phi;\ \text{whence by (18), } x^2\sec^2\phi = \mu, \text{ or } x^2 = \mu\cos^2\phi.$$

Hence to determine ϕ and x, we have the logarithmic equations,

$$2\log\tan\phi = \log a + \log\sin(\alpha+\gamma) + \log\sin(\alpha-\gamma) + 2\log\sin\beta + \log\operatorname{cosec}(\beta+\alpha)$$
$$+\log\operatorname{cosec}(\beta-\alpha) + 2\log\operatorname{cosec}\gamma - \log(a+b) - 60 \ldots\ldots (19),$$
$$2\log x = \log a + \log b + 2\log\sin\alpha + 2\log\sin\beta + 2\log\cos\phi$$
$$+\log\operatorname{cosec}(\beta+\alpha) + \log\operatorname{cosec}(\beta-\alpha) - 80 \ldots\ldots\ldots (20).$$

If α *be greater than either* β *or* γ (β being either greater or less than γ), then λ and μ will both be negative, and hence we must assume

$$\lambda = \sec^2\phi;\ \text{whence } x^2\tan^2\phi = \frac{ab\sin^2\alpha\sin^2\beta}{\sin(\alpha+\beta)\sin(\alpha-\beta)},$$

or
$$x^2 = \frac{ab\sin^2\alpha\sin^2\beta\cot^2\phi}{\sin(\alpha+\beta)\sin(\alpha-\beta)}.$$

Consequently the equations of solution in this case, are

$$2\log\sec\phi = \log a + \log\sin(\alpha+\gamma) + \log\sin(\alpha-\gamma) + 2\log\sin\beta$$
$$+\log\operatorname{cosec}(\alpha+\beta) + \log\operatorname{cosec}(\alpha-\beta) + 2\log\operatorname{cosec}\gamma - \log(a+b) - 60 \ldots (21),$$
$$2\log x = \log a + \log b + 2\log\sin\alpha + 2\log\sin\beta + 2\log\cot\phi$$
$$+\log\operatorname{cosec}(\alpha+\beta) + \log\operatorname{cosec}(\alpha-\beta) - 80 \ldots\ldots\ldots (22).$$

Lastly, *if* α *be less than either* β *or* γ (β being either greater or less than γ), then λ will be negative and μ positive.

Hence we must assume

$$\lambda = \sin^2\phi;\ \text{whence } x^2\cos^2\phi = \mu, \text{ or } x^2 = \mu\sec^2\phi.$$

The equations therefore to determine ϕ and x, are the following :—

$$2\log\sin\phi = \log a + \log\sin(\gamma+\alpha) + \log\sin(\gamma-\alpha) + 2\log\sin\beta$$
$$+\log\operatorname{cosec}(\beta+\alpha) + \log\operatorname{cosec}(\beta-\alpha) + 2\log\operatorname{cosec}\gamma - \log(a+b) - 60 \ldots (23),$$
$$2\log x = \log a + \log b + 2\log\sin\alpha + 2\log\sin\beta + 2\log\sec\phi$$
$$+\log\operatorname{cosec}(\beta+\alpha) + \log\operatorname{cosec}(\beta-\alpha) - 80 \ldots\ldots\ldots (24)$$

I am not aware that the formulæ (23) and (24) have been noticed by any writer on this subject, though they are essential to the complete solution of the problem.

EXAMPLE.

Let $a=50$, $b=60$, $\alpha=30^\circ\ 40'$, $\beta=40^\circ\ 33'$, $\gamma=50^\circ\ 23'$.

The formulæ of solution in this case are (23) and (24), as α is less than either β or γ.

Now

$$a+b=110, \qquad \gamma+\alpha=81^\circ\ 3', \qquad \beta+\alpha=71^\circ\ 13',$$
$$\gamma-\alpha=19^\circ\ 43', \qquad \beta-\alpha=\ 9^\circ\ 53'.$$

To find ϕ by (23) :—

$$\begin{aligned}
\log 50 &= 1.6989700\\
\sin 81^\circ\ 3' &= 9.9946798\\
\sin 19^\circ\ 43' &= 9.5281053\\
2\sin 40^\circ\ 33' &= 19.6259756\\
\text{cosec } 71^\circ\ 13' &= 10.0237679\\
\text{cosec } \ 9^\circ\ 53' &= 10.7653751\\
2 \text{ cosec } 50^\circ\ 23' &= \underline{20.2266488}\\
&\ 21.8635225\\
\log 110 &= \underline{2.0413927}\\
2\sin\phi &= 19.8221298\\
\sin\phi &= 9.9110649
\end{aligned}$$

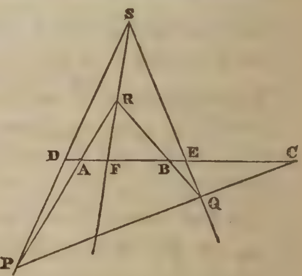
Hence $\phi=54^\circ\ 34'\ 12''.61$.

To find x by (24) :—

$$\begin{aligned}
\log 50 &= 1.6989700\\
\log 60 &= 1.7781513\\
2\sin 30^\circ\ 40' &= 19.4152128\\
2\sin 40^\circ\ 33' &= 19.6259756\\
2\sec 54^\circ\ 34'\ 12''.61 &= 20.4735852\\
\text{cosec } 71^\circ\ 13' &= 10.0237679\\
\text{cosec } \ 9^\circ\ 53' &= \underline{10.7653751}\\
2\log x &= 3.7810379\\
\log x &= 1.8905189
\end{aligned}$$

or $x=77.7175$, the required height.

2. *Given the distances of three remote objects* P, Q, R, *from each other, and the angles they subtend at a station* S, *in the same horizontal plane ; to find the distance of the station* S *from each of the objects.*



Put $RQ=a$, $PQ=b$, $PR=c$; and denote the angles P,R,Q, by A,B,C. Also put the angle $RSP=\alpha$, $RSQ=\beta$, $RPS+RQS=B-(\alpha+\beta)=2m$, $RPS-RQS=2\theta$; then $RPS=m+\theta$, $RQS=m-\theta$.

Now

$$\frac{RP}{RS}=\frac{\sin\alpha}{\sin(m+\theta)}, \quad \frac{RS}{RQ}=\frac{\sin(m-\theta)}{\sin\beta}, \quad \frac{RQ}{RP}=\frac{\sin A}{\sin C};$$

hence by multiplication,

$$1=\frac{\sin\alpha\sin(m-\theta)\sin A}{\sin(m+\theta)\sin\beta\sin C}, \quad \text{or} \quad \frac{\sin(m-\theta)}{\sin(m+\theta)}=\frac{\sin\beta\sin C}{\sin\alpha\sin A}.$$

But $$\frac{\sin(m-\theta)}{\sin(m+\theta)}=\frac{\tan m-\tan\theta}{\tan m+\tan\theta}; \text{ whence}$$

$$\frac{\tan m-\tan\theta}{\tan m+\tan\theta}=\frac{\sin\beta\sin C}{\sin\alpha\ \sin A}=h, \text{ suppose.}$$

From this we readily get

$$\tan\theta = \frac{1-h}{1+h}\tan m.$$

Hence to find h and θ, we have the equations

$$\log h = \log\sin\beta + \log\sin C + \log\operatorname{cosec}\alpha + \log\operatorname{cosec}A - 40 \ldots(25),$$

$$\log\tan\theta = \log(1-h) + \log\tan m - \log(1+h) \ldots\ldots\ldots\ldots(26).$$

If h be greater than unity and $\tan m$ positive, then $\tan\theta$ will be negative, and hence θ will be negative. (See the example for illustration.)

Whence finally,

$$PS = c\,\frac{\sin(m+\theta+\alpha)}{\sin\alpha},\ RS = c\,\frac{\sin(m+\theta)}{\sin\alpha},\ QS = a\,\frac{\sin(m-\theta+\beta)}{\sin\beta};$$

or $$\log PS = \log c + \log\sin(m+\theta+\alpha) + \log\operatorname{cosec}\alpha - 20 \ldots\ldots\ldots(27),$$

$$\log RS = \log c + \log\sin(m+\theta) + \log\operatorname{cosec}\alpha - 20 \ldots\ldots\ldots(28),$$

$$\log QS = \log a + \log\sin(m-\theta+\beta) + \log\operatorname{cosec}\beta - 20 \ldots\ldots\ldots(29).$$

Cor. 1. When S is on PQ, then $2m = A + C$.

Cor. 2. When S is any where in the angular space formed by RP, RQ, or these produced (in the direction of RP, RQ), then $2m = 360° - (\alpha+\beta+B)$.

The student will have little difficulty in modifying the formulæ for other positions of S *.

EXAMPLE.

Let $a = 263$, $b = 405$, $c = 214$, $\alpha = 13° 30' 20''$, $\beta = 29° 50' 12''$.

The angles A, B, C, are found by the usual methods to be

$$A = 35° 45' 53'',\ B = 115° 50' 20'',\ C = 28° 23' 47''.$$

Hence $m = \frac{1}{2}\{B - (\alpha+\beta)\} = 36° 14' 54''$.

To find h by (25):—

$\sin 29° 50' 12''$ =	9.6968185
$\sin 28° 23' 47''$ =	9.6772133
$\operatorname{cosec} 13° 30' 20''$ =	10.6316394
$\operatorname{cosec} 35° 45' 53''$ =	10.2332466
$\log h$ =	0.2389178
or h =	1.733476
Hence $1 - h$ = —	.733476
$1 + h$ =	2.733476

To find θ by (26):—

(As $1-h$ is *negative*, θ is negative.)

$\log .733476$ =	1.8653859
$\tan 36° 14' 54''$ =	9.7305832
	9.5959691
$\log 2.733476$ =	0.4367153
$\tan\theta$ =	9.1592538

Hence $\theta = -(11° 7' 48'')$.

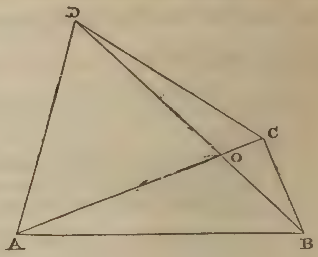
Whence $m+\theta+\alpha = 38° 37' 26''$, $m+\theta = 25° 7' 6''$, $m-\theta+\beta = 77° 12' 54''$.

* In all the solutions that I have seen of this and the following problem, the resulting expressions are either left in a form not adapted to logarithms, or they are reduced to this form, by introducing 45° into the formulæ, in addition to a subsidiary angle.

To find PS by (27):—

$$\begin{aligned} \log 214 &= 2.3304138 \\ \sin 38^\circ\ 37'\ 26'' &= 9.7953275 \\ \operatorname{cosec} 13^\circ\ 30'\ 20'' &= \underline{10.6316394} \\ \log PS &= 2.7573807, \text{ or } PS = 571.98. \end{aligned}$$

3. *To find the distance between two inaccessible objects* C *and* D, *situated in the same plane with the observer.*



Let $AB = a$, be a given line in the plane of the observer.

Put

$$\begin{aligned} \text{angle } DAB &= \alpha, & CBA &= \gamma, \\ CAB &= \beta, & DBA &= \delta, \end{aligned}$$

$$BDC + ACD = \beta + \delta = 2m,$$

$$BDC - ACD = 2\theta;$$

then $BDC = m + \theta, \quad ACD = m - \theta.$

Now

$$\frac{DO}{CO} = \frac{\sin(m-\theta)}{\sin(m+\theta)}, \quad \frac{AO}{DO} = \frac{\sin(\alpha+\delta)}{\sin(\alpha-\beta)}, \quad \frac{BO}{AO} = \frac{\sin\beta}{\sin\delta}, \quad \frac{OC}{BO} = \frac{\sin(\gamma-\delta)}{\sin(\beta+\gamma)}.$$

Proceeding with these as in the former problem, we get

$$\frac{\tan m - \tan\theta}{\tan m + \tan\theta} = \frac{\sin(\alpha-\beta)\sin\delta\,\sin(\beta+\gamma)}{\sin(\alpha+\delta)\sin\beta\,\sin(\gamma-\delta)} = h,$$

and

$$\tan\theta = \frac{1-h}{1+h}\tan m.$$

Also,

$$DC = DA.\frac{\sin(\alpha-\beta)}{\sin(m-\theta)} = a.\frac{\sin\delta}{\sin(\alpha+\delta)}.\frac{\sin(\alpha-\beta)}{\sin(m-\theta)}.$$

Hence to find CD, the required distance, we have the equations,

$$\log h = \log\sin(\alpha-\beta) + \log\sin\delta + \log\sin(\beta+\gamma) + \log\operatorname{cosec}(\alpha+\delta)$$
$$+ \log\operatorname{cosec}\beta + \log\operatorname{cosec}(\gamma-\delta) - 60 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

$$\log\tan\theta = \log(1-h) + \log\tan m - \log(1+h) \quad \ldots\ldots\ldots\ldots\ldots (31),$$

$$\log CD = \log a + \log\sin\delta + \log\sin(\alpha-\beta) + \log\operatorname{cosec}(\alpha+\delta)$$
$$+ \log\operatorname{cosec}(m-\theta) - 40 \quad \ldots\ldots\ldots\ldots (32).$$

Royal Military Academy,
Feb. 1, 1849.

NEW AND SIMPLE PROCESS FOR DETERMINING ALL THE THREE ROOTS OF A CUBIC EQUATION.

[*Dr. Rutherford.*]

The cubic equation whose roots are $a+\sqrt{\beta}$, $a-\sqrt{\beta}$ and r is

$$x^3-(2a+r)x^2+(a^2+2ra-\beta)x-r(a^2-\beta)=0.$$

Now if the roots of this equation be diminished by r, the coefficients of the transformed equation will readily furnish the values of a and β without resolving the resulting quadratic equation. Thus we have

$$\begin{array}{rrl}
1-(2a+r) & +(a^2+2ra-\beta) & -r(a^2-\beta)\ (r \\
\underline{\qquad\qquad r} & \underline{-2ra\qquad} & \underline{+r(a^2-\beta)} \\
-2a & a^2-\beta & \\
\underline{\qquad\qquad r} & \underline{-2ra+r^2} & \\
-2a+r & a^2-2ra+r^2-\beta & \\
\underline{\qquad\qquad r} & \underline{3a^2+6ra\quad-3\beta} & =\text{3 times coefficient of } x. \\
\underline{2)-2a+2r} & 4a^2+4ra+r^2-4\beta & \\
-a+r & \underline{-(4a^2+4ra+r^2)} & =\text{square of coefficient of } x^2. \\
\underline{\qquad -r} & \underline{4)-4\beta} & \\
-a & -\beta &
\end{array}$$

Hence if $x^3+ax^2+bx+c=0$ be the proposed cubic equation, the process developed above furnishes the following type of solution.

$$\begin{array}{rrl}
1+a & +b & +c\ (r \\
\underline{r} & \underline{ra'} & \underline{rb'} \\
a' & b' & 0 \\
\underline{r} & \underline{ra''} & \\
a'' & b'' & \\
\underline{r} & \underline{3b}\ \text{add} & \\
\underline{2)\ a'''} & \cdots & \\
\cdots & \underline{a^2}\ \text{subtract} & \\
\underline{-r} & 4)\ \underline{-4\beta} & \\
-a & -\beta &
\end{array}$$

Consequently the values of r, a and β are all found, and thus the three roots are completely determined, whether the two roots $a\pm\sqrt{\beta}$ be real or unreal. If the roots of the cubic equation be denoted by r_1, r_2 and r_3, and the same process of reduction be adopted, r_1 denoting the real root, the values of r_2+r_3 and $(r_2-r_3)^2$ would be found in the same manner. If the value of $(r_2-r_3)^2$ be positive, the two roots r_2 and r_3 are real, and if negative, they are unreal.

When the value of r has been obtained to the required degree of accuracy the values of a and β may be found very simply from the coefficients of the first and second powers of the unknown quantity in the given equation, by the following process.

From the coefficient of x^2 we have

$$-2a = -(2a+r)+r \quad \text{or} \quad a = \tfrac{1}{2}\{(2a+r)-r\}.$$

And from the coefficient of x we get

$$-\beta = (a^2+2ra-\beta) - a(2r+a).$$

EXAMPLE.

Find all the roots of the cubic equation $x^3+8x^2+6x-75{\cdot}9=0$.

This equation has one positive root, and it lies between 2 and 3. Developing this root by Horner's process, and carrying out the preceding operation for finding a and β, we have the subjoined computation.

1 + 8	+6	−75·9 (2·4257122884
10	26	52
12	50	−23·900
144	5576	22·304
148	6168	−1·596
1522	619844	1·239688
1424	622892	−356312
15265	62365525	311827625
15270	62441875	−44484375
152757	6245256799	43716798
152764	6246326147	−767577
1527711	624634142411	624634
1527712	624635670123	−142943
15277132	624635975666	124927
15277134	624636281209	−18016
152771362	624636311763	12493
152771364	624636342317	−5523
1527713668	624636354539	4997
1527713676	624636366760	−526
15277136848	624636367982	500
15277136856	624636369204	−26
152771368644	624636369265	25
152771368648	62·4636369326	−1
2)15·2771368652	18	
7·6385684326	80.4636369326	
−2·4257122884	−64	
5·2128561442 = −a	4)16·4636369326	
	4·1159092332 = −β	

Hence the three roots of the given cubic equation are

$$2{\cdot}4257122884 \text{ and } -5{\cdot}2128561442 \pm \sqrt{-4{\cdot}1159092332}.$$

If the value of β be positive, then the roots of the form $a \pm \sqrt{\beta}$ are real, and their values may be found by extracting the square root of β, and adding that square root to, and subtracting it from, the value of a determined in the first column.

In a small tract which I have recently published, entitled "*The Complete Solution of Numerical Equations, in which, by one uniform process, the imaginary as well as the real roots are easily determined,*" two other methods of finding all the three roots of a cubic equation are given; but the process here developed is neater than either of those referred to, since the values of a and β are deduced very simply from the coefficients of the given and transformed equations, and the root which has been found by Horner's process of developement. By this method, the resolution of the resulting quadratic equation is completely avoided, and the remaining roots are obtained without any difficulty.

Royal Military Academy,
Feb. 2, 1849.

APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

[*Mr. Robert Finlay, Professor of Mathematics and Natural Philosophy, Manchester New College.*]

(Continued from page 138.)

APPLICATION OF THE THEORY OF RECIPROCAL POLARS TO SURFACES OF THE SECOND DEGREE.

XLI.

To find the equation of the surface reciprocal to any surface of the second degree, the director being a sphere.

Let the equation to the director be

$$x^2 + y^2 + z^2 = 1,$$

and that of the given surface

$$Ax^2 + A'y^2 + A''z^2 + 2(Byz + B'zx + B''xy) + 2(Cx + C'y + C''z) + E = 0 \,..\, (1);$$

also, let $x'y'z'$ denote any point in the given surface (1), and $x_1y_1z_1$ the corresponding point of the required surface: then (XV) the equation of the polar plane of the point $x_1y_1z_1$ is

$$x_1x + y_1y + z_1z = 1 \ldots\ldots\ldots\ldots\ldots (a),$$

and (XVIII) the equation of the tangent plane applied to the given surface (1) at the point $x'y'z'$ is

$$(Ax' + B''y' + B'z' + C)x + (B''x' + A'y' + Bz' + C')y + (B'x' + By' + A''z' + C'')z$$
$$+ Cx' + C'y' + C''z' + E = 0;$$

and, since these two planes are identical (XXIV), we have the following conditions:—

$$A\,x'+B''y'+B'z'+C\;+x_1(Cx'+C'y'+C''z'+E)=0,$$
$$B''x'+A'y'+B\,z'+C'\;+y_1(Cx'+C'y'+C''z'+E)=0,$$
$$B'x'+B\,y'+A''z'+C''+z_1(Cx'+C'y'+C''z'+E)=0.$$

By comparing these with equations (a), No. XXII, we see that they may be deduced from them by changing a, b, c into $x_1 y_1 z_1$ respectively; hence by adopting the notation and results of the number referred to, which were given mainly with prospective reference to the present problem, we shall have

$$-x'=\frac{A_1x_1+B_3y_1+B_2z_1+C_1}{C_1x_1+C_2y_1+C_3z_1+E_1},$$

$$-y'=\frac{B_3x_1+A_2y_1+B_1z_1+C_2}{C_1x_1+C_2y_1+C_3z_1+E_1},$$

$$-z'=\frac{B_2x_1+B_1y_1+A_3z_1+C_3}{C_1x_1+C_2y_1+C_3z_1+E_1}.$$

Now since the plane (a) passes through the point $x'y'z'$ (XXIV), we have

$$x_1x'+y_1y'+z_1z'=1,$$

and by substituting the preceding values of x', y', z' in this equation, we get

$$A_1x_1^2+A_2y_1^2+A_3z_1^2+2(B_1y_1z_1+B_2z_1x_1+B_3x_1y_1)+2(C_1x_1+C_2y_1+C_3z_1)$$
$$+E_1=0\ldots(2),$$

which is the equation of the reciprocal polar of the surface (1), in relation to the given sphere.

By substituting for A_1, A_2, A_3, B_1, *etc.* their values given by equations (12), (13), (14), (15), No. XXII, this equation becomes

$$(A'C''^2+A''C'^2+EB^2-EA'A''-2BC'C'')x^2$$
$$+(A''C^2+A\,C''^2+EB'^2-EA''A-2B'C''C)y^2$$
$$+(A\,C'^2+A'C^2+EB''^2-EA\,A'-2B''C\,C')z^2$$

$$+2\{E(AB-B'B'')+C''(B''C-AC')+C(B'C'-B\,C)\}yz$$
$$+2\{E(A'B'-B''B)+C(BC'-A'C'')+C'(B''C''-B'C')\}zx$$
$$+2\{E(A''B''-B\,B')+C'(B'C''-A''C)+C''(B\,C-B''C'')\}xy$$
$$+2\{C\,(B^2-A'A'')+C'(A''B''-BB')+C''(A'B'-B''B)\}x$$
$$+2\{C'(B'^2-A''A)+C''(A\,B-B'B'')+C\,(A''B''-B\,B')\}y$$
$$+2\{C''(B''^2-A\,A')+C(A'B'-B''B)+C'(A\,B-B'B'')\}z$$
$$+AB^2+A'B'^2+A''B''^2-AA'A''-2BB'B''=0\ldots\ldots\ldots\ldots(2');$$

in which the coefficients are expressed immediately in terms of those of the reciprocal surface (1).

XLII.

By comparing the equations (1) and (2) of the sphero-polar-reciprocal surfaces, a great variety of properties may be immediately deduced, of which the following may be noticed here.

1st. When the condition $A_1=0$ is satisfied, the axis of x is parallel to a side of the asymptotic cone of the surface (2). Again, by equations (18), No. XXII, the condition $A_1=0$ shows that the pole of the plane of $y\,z$, taken with respect to the surface (1), lies in that plane; and therefore (XVIII) that the plane of $y\,z$ touches the surface (1); hence *if a straight line be drawn through the centre of the director parallel to any side of the asymptotic cone of one of the reciprocal surfaces, the plane drawn through the centre of the director perpendicular to this line will touch the other surface;* and conversely, *if a plane be drawn through the centre of the director touching one of the surfaces, the straight line drawn through the centre of the director perpendicular to that plane, is parallel to a side of the asymptotic cone of the other surface.*

2nd. When $B_1=0$ the axes of y and z are parallel to two of the principal axes of the surface (2); but by equations (16) and (17), No. XXII, the condition $B_1=0$ gives $h'=0$, $g''=0$, from which it appears that the pole of the plane of xy is in the plane of yz, and that the pole of the plane of yz is in the plane of xy; hence *if two straight lines be drawn through the centre of the director parallel to two of the principal axes of one of the reciprocal surfaces, and if two planes be drawn through the same point perpendicular to these lines, the pole of each of these planes taken with respect to the other surface, will lie on the other plane.* Conversely, *if two perpendicular planes be so related to a surface of the second degree that each passes through the pole of the other, and if the reciprocal polar of the surface be taken in relation to any sphere having its centre on the intersection of the two planes, two of the principal axes of the polar surface will be respectively perpendicular to those planes.*

3rd. When $C_1=0$ it is evident from equations (19), No. XXII, that the centre of the surface (1) is on the plane of yz; and the same condition implies that the diametral plane of the surface (2) which bisects chords parallel to the axis of x passes through the origin; hence *if any plane be drawn through the centres of the director and of one of the reciprocal surfaces, the diametral plane of the other surface which bisects chords perpendicular to it will pass through the centre of the director.* Reciprocally, *if a diametral plane of one of the reciprocal surfaces be drawn through the centre of the director, the plane drawn through that centre, perpendicular to the chords bisected by the former plane, will pass through the centre of the other surface.*

4th. When $E_1=0$ it is evident from equations (19) No. XXII, that the centre of the surface (1) is at infinity; and in this case the reciprocal surface (2) passes through the origin; hence, *when the centre of the director is on one of the reciprocal surfaces, the other surface is a paraboloid,*

its centre being at infinity: and conversely, *when one of the surfaces is a paraboloid the reciprocal surface passes through the centre of the director.*

XLIII.

In virtue of equations (19), No. XXII, it is evident that the linear part of equation (2) may be written in the form

$$-2E_1(kx+k'y+k''z)+E_1$$

and therefore (XV) the equation of the polar plane of the origin, with respect to the surface (2), is

$$kx+k'y+k''z-1=0;$$

but this equation also represents the polar plane of the point $k\ k'\ k''$ in relation to the director; hence, *the polar plane of the centre of the director taken with respect to one of the reciprocal surfaces, coincides with the polar plane of the centre of the other surface taken in relation to the director.*

XLIV.

The equation (1) of the proposed surface may evidently be written in the form

$$(AE-C^2)x^2+(A'E-C'^2)y^2+(A''E-C''^2)z^2$$
$$+2(BE-C'C'')yz+2(B'E-C''C)zx+2(B''E-CC')xy$$
$$+(Cx+C'y+C''z+E)^2=0 \ldots\ldots\ldots\ldots\ldots\ldots (1');$$

hence, in order that it may represent a surface of revolution having a focus at the origin, the conditions are

$$AE-C^2=A'E-C'^2=A''E-C''^2 \ldots\ldots\ldots\ldots\ldots\ldots (b),$$
$$BE-C'C''=0,\ B'E-C''C=0,\ B''E-CC''=0 \ldots\ldots (c);$$

and the latter system of equations gives

$$E=\frac{C'C''}{B}=\frac{C''C}{B'}=\frac{CC'}{B''} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d),$$

$$BC=B'C'=B''C'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e).$$

Now in virtue of equations (d) we have

$$C^2-AE=\frac{C}{B''}(B''C-AC')=\frac{E}{C'}(B''C-AC'),$$

$$C'^2-A'E=\frac{C'}{B}(BC'-A'C'')=\frac{E}{C''}(BC'-A'C''),$$

$$C''^2-A''E=\frac{C''}{B'}(B'C''-A''C)=\frac{E}{C}(B'C''-A''C);$$

by means of which equations (12), No. XXII, are easily reduced to the form

$$-A_1E=(C'^2-A'E)(C''^2-A''E),$$
$$-A_2E=(C''^2-A''E)(C^2-AE),$$
$$-A_3E=(C^2-AE)(C'^2-A'E);$$

hence, when the conditions (b) are satisfied, we shall have $A_1=A_2=A_3$. Again, in virtue of the conditions (c) and (e), equations (13), No. XXII, give $B_1=0$, $B_2=0$, $B_3=0$: hence, when the conditions (a) and (b) are satisfied simultaneously, the equation (2) of the reciprocal surface becomes

$$A_1(x^2+y^2+z^2)-E_1(kx+k'y+k''z-1)=0;$$

and therefore that surface is a sphere, the coordinates of whose centre are proportional to k, k', k''; hence, *the surface reciprocal to any surface of revolution of the second degree, with respect to a sphere having its centre at a focus of the surface, is a sphere; and the centres of the three surfaces are in the same straight line.*

XLV.

If equation (2) be written in the form

$$(A_1E_1-C_1^2)x^2+(A_2E_1-C_2^2)y^2+(A_3E_1-C_3^2)z^2$$
$$+2(B_1E_1-C_2C_3)yz+2(B_2E_1-C_3C_1)zx+2(B_3E_1-C_1C_2)xy$$
$$+(C_1x+C_2y+C_3z+E_1)^2=0;$$

we obtain by substituting for A_1, A_2, A_3, B_1, *etc.*, their values (XXII)

$$(B^2-A'A'')x^2+(B'^2-A''A)y^2+(B''^2-AA')z^2$$
$$+2(AB-B'B'')yz+2(A'B'-B''B)zx+2(A''B''-BB')xy$$
$$+\frac{E_1^2}{M}(kx+k'y+k''z-1)=0\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where, for the sake of brevity, we put

$$M=(A'A''-B^2)C^2+(A''A+B'^2)C'^2+(AA'-B''^2)C''^2$$
$$+2(B'B''-BA)C'C''+2(B''B-B'A')C''C+2(BB'-B''A'')CC'+E_1E.$$

Again, if we assume

$$\mu=(a'a''-\beta^2)\gamma^2+(a''a-\beta'^2)\gamma'^2+(aa'-\beta''^2)\gamma''^2$$
$$+2(\beta'\beta''-\beta a)\gamma'\gamma''+2(\beta''\beta-\beta'a')\gamma''\gamma+2(\beta\beta'-\beta''a'')\gamma\gamma'+\epsilon_1\epsilon,$$

it is evident that the surfaces whose equations are

$$ax^2+a'y^2+a''z^2+2(\beta yz+\beta'zx+\beta''xy)+2(\gamma x+\gamma'y+\gamma''z)+\epsilon=0\,..\,(4),$$

and

$$(\beta^2-a'a'')x^2+(\beta'^2-a''a)y^2+(\beta''^2-aa')z^2$$
$$+2(a\beta-\beta'\beta'')yz+2(a'\beta'-\beta''\beta)zx+2(a''\beta''-\beta\beta')xy$$
$$+\frac{\epsilon_1^2}{\mu}(\lambda x+\lambda'y+\lambda''z-1)^2=0\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

will also be reciprocal polars, $\lambda\lambda'\lambda''$ being the centre of the surface (4), and ϵ_1 denoting the function $a\beta^2+a'\beta'^2+a''\beta''^2-aa'a''-2\beta\beta'\beta''$. Now if the surfaces (1) and (4) be similar, we shall have

$$\frac{A}{a}=\frac{A'}{a'}=\frac{A''}{a''}=\frac{B}{\beta}=\frac{B'}{\beta'}=\frac{B''}{\beta''}=\sigma \text{ suppose};$$

hence, in this case, if we multiply equation (5) by σ^2 and subtract the result from (3), we shall find

$$\frac{E_1^2}{M}(kx+k'y+k''z-1)^2-\frac{\epsilon_1^2\sigma^2}{\mu}(\lambda x+\lambda' y+\lambda'' z-1)^2=0,$$

from which it is evident that the surfaces (3) and (5) intersect on two planes which pass through the line of intersection of the planes

$$kx+k'y+k''z-1=0 \text{ and } \lambda x+\lambda' y+\lambda'' z-1=0.$$

Hence we see that *if two surfaces of the second degree be similar, their polar reciprocal surfaces, taken with respect to any sphere, intersect each other in two planes, which pass through the intersection of the polar planes of the centres of the two given surfaces taken with respect to the director.*

XLVI.

Hitherto the equations of the reciprocal surfaces have been exhibited in their most general form, but they may be considerably simplified by taking the axes of co-ordinates parallel to the principal axes of the given surface, the origin being still at the centre of the director. In this case we have $B=0$, $B'=0$, $B''=0$; therefore the equation of the given surface becomes

$$Ax^2+A'y^2+A''z^2+2(Cx+C'y+C''z)+E=0 \;\ldots\ldots (6),$$

and the equation of its sphero-polar-reciprocal surface is

$$\begin{aligned}(A'C''^2+A''C'^2-EA'A)''x^2+(A''C^2+AC''^2-EA''A)y^2\\ +(AC'^2+A'C^2-EAA')z^2\\ -2(AC'C''yz+A'C''Czx+A''CC'xy)\\ -2(CA'A''x+C'A''Ay+C''AA'z)-AA'A''=0\ldots\ldots(7).\end{aligned}$$

When $C=0$ the plane of yz is a principal plane of the surface (6); and the axis of x is a principal axis of the surface (7), or the plane of yz is parallel to one of its principal sections; hence, *when the centre of the director is on a principal section of one of the reciprocal surfaces the other surface has one of its principal sections parallel to it.*

When $C=0$ and $C'=0$ the axis of z coincides with a principal axis of the given surface (6), and the principal axes of the surface (8) are parallel to the axes of coordinates; hence, *if the centre of the director be on a principal axis of the given surface, the principal axes of the reciprocal surfaces are parallel.*

When $C=0$, $C'=0$, and $C''=0$, the equations of the reciprocal surfaces become

$$Ax^2+A'y^2+A''z^2+E=0 \text{ and } \frac{x^2}{A}+\frac{y^2}{A'}+\frac{z^2}{A''}+\frac{1}{E}=0;$$

consequently, *if the centre of the director be at the centre of the given surface, the principal axes of the reciprocal surfaces will coincide in direction, and their lengths will be reciprocally proportional.*

XLVII.

Returning to the general equation (1), let $C=0$, $C'=0$, $E=0$; then it is evident that $A_3=0$, or that the surface (1) touches the plane of xy. But, in this case, the equation (2′) of the reciprocal surface becomes

$$C''^2(A'x^2+Ay^2-B''xy)$$
$$+2C''\{(A'B'-B''B)x+(AB-B'B'')y+(B''^2-AA')z\}+E_1=0,$$

which evidently represents a paraboloid, having its axis parallel to the axis of z; hence, *when the centre of the director is on the given surface, the reciprocal surface is a paraboloid, whose axis is parallel to the normal to the given surface drawn through the centre of the director.*

It is evident from this property, that if the centre of the director be upon one of the principal sections of either of the reciprocal surfaces, the axis of the other surface will be parallel to the plane of that section.

XLVIII.

To find the polar reciprocal surface of a given surface of the second degree, in relation to any given surface of the second degree taken as director.

Let the equation of the director be

$$Px^2+P'y^2+P''z^2+2(Qyz+Q'zx+Q''xy)+2(Rx+R'y+R''z)+T=0,$$

and that of the given surface

$$Ax^2+A'y^2+A''z^2+2(Byz+B'zx+B''xy)+2(Cx+C'y+C''z)+E=0 \quad \ldots.(6);$$

then if $x'y'z'$ be any point on the latter surface, and xyz the corresponding point on the required surface, the equation of the tangent plane applied to the given surface at the former point is

$$(Ax'+B''y'+B'z'+C)x+(B''x'+A'y'+Bz'+C')y$$
$$+(B'x'+By'+A''z'+C'')z+Cx'+C'y'+C''z'+E=0,$$

and the equation of the polar plane of the latter point taken with respect to the director is

$$(Px_1+Q''y_1+Q'z_1+R)x+(Q''x_1+P'y_1+Qz_1+R')y$$
$$+(Q'x_1+Qy_1+P''z_1+R'')z+Rx_1+R'y_1+R''z_1+T=0;$$

and since these planes are identical (XXIV.) we shall have

$$\frac{Ax'+B''y'+B'z'+C}{Cx'+C'y'+C''z'+E}=\frac{Px_1+Q''y_1+Q'z_1+R}{Rx_1+R'y_1+R''z_1+T}=-X,$$

$$\frac{B''x'+A'y'+Bz'+C'}{Cx'+C'y'+C''z'+E}=\frac{Q''x_1+P'y_1+Qz_1+R'}{Rx_1+R'y_1+R''z_1+T}=-Y,$$

$$\frac{B'x'+By'+A''z'+C''}{Cx'+C'y'+C''z'+E}=\frac{Q'x_1+Qy_1+P''z_1+R''}{Rx_1+R'y_1+R''z_1+T}=-Z.$$

From these we obtain, by the process in No. XLI.,

$$A_1X^2+A_2Y^2+A_3Z^2+2(B_1YZ+B_2ZX+B_3XY)$$
$$+2(C_1X+C_2Y+C_3Z)+E_1=0;$$

or by restoring the values of X, Y, Z,

$$
\begin{aligned}
&A_1(Px_1+Q''y_1+Q'z_1+R)^2+A_2(Q''x_1+P'y_1+Qz_1+R')^2\\
&\qquad\qquad\qquad\qquad+A_3(Q'x_1+Qy_1+P''z_1+R'')^2\\
&+2B_1(Q''x_1+P'y_1+Qz_1+R')(Q'x_1+Qy_1+P''z_1+R'')\\
&+2B_2(Q'x_1+Qy_1+P''z_1+R'')(Px_1+Q''y_1+Q'z_1+R)\\
&+2B_3(Px_1+Q''y_1+Q'z_1+R)(Q''x_1+P'y_1+Qz_1+R')\\
&-2C_1(Px_1+Q''y_1+Q'z_1+R)(Rx_1+R'y_1+R''z_1+T)\\
&-2C_2(Q''x_1+P'y_1+Qz_1+R')(Rx_1+R'y_1+R''z_1+T)\\
&-2C_3(Q'x_1+Qy_1+P''z_1+R'')(Rx_1+R'y_1+R''z_1+T)\\
&+\ E_1(Rx_1+R'y_1+R''z_1+T)^2=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),
\end{aligned}
$$

which is the required equation.

XLIX.

When the director is a central surface, this equation may be greatly simplified by taking a system of its conjugate diameters as axes of coordinates; for in this case we have

$$Q=0,\ Q'=0,\ Q''=0;\ R=0,\ R'=0,\ R''=0;\ T=-1,$$

and the equation becomes

$$
\begin{aligned}
&A_1P^2x^2+A_2P'^2y^2+A_3P''^2z^2+2(B_1P'P''yz+B_2P''Pzx+B_3PP'xy)\\
&\qquad\qquad+2(C_1Px+C_2P'y+C_3P''z)+E_1=0\ \ldots\ldots\ldots(7').
\end{aligned}
$$

Now in virtue of equations (19), No. XXII, the linear part of this equation may be written in the form

$$-2\,E_1(P\,kx+P'k'y+P''k''z)+E_1$$

hence, (XV) the equation of the polar plane of the origin is

$$Pkx+P'k'y+P''k''z=1,$$

from which it is evident that the theorem of No. XLIII is true when the director is any central surface of the second degree.

L.

Equation (7′), being analagous to equation (2) of No. XLI, suggests properties similar to those announced in No. XLII. Thus, by supposing one of the coefficients A_1, A_2, A_3 to vanish, we see that if a straight line be drawn parallel to any side of the asymptotic cone of one of the reciprocal surfaces (7′), the diametral plane of the director which bisects chords parallel to it will touch the other surface (6); and the converse.

Again, by supposing one of the coefficients B_1, B_2, B_3, to vanish, it may be demonstrated, that if two conjugate diameters of the director be drawn parallel to two conjugate diameters of the given surface, and if the two diametral planes of the director conjugate to these be drawn, the pole of each of these planes taken with regard to the other surface will lie on the other plane; and the converse.

If we now suppose one of the coefficients C_1, C_2, C_3 to vanish, we see that if any plane be drawn through the centres of the director and of one of the reciprocal surfaces, the diametral plane of the other surface, which bisects chords parallel to the diameter of the director conjugate to the former plane, will pass through the centre of the director.

Lastly, the condition $E_1=0$ shows that when the centre of the director is on one of the surfaces, the centre of the other surface is at infinity, and therefore the latter surface is a paraboloid.

LI.

If the principal axes of the director be taken as axes of coordinates, it is evident (XLIV) that the conditions in order that the surface (6) may be a surface of revolution, having a focus at the origin, are

$$C^2-AE=C'^2-A'E=C''^2-A''E,$$

$$BE-C'C''=B'E-C''C=B''E-CC'=0:$$

but in virtue of these conditions we have seen that

$$A_1=A_2=A_3 \text{ and } B_1=B_2=B_3=0;$$

and therefore the equation (7′) of the reciprocal surface becomes

$$A_1(P^2x^2+P'^2y^2+P''^2z^2)+2(C_1Px+C_2P'y+C_3P''z)+E_1=0,$$

which denotes a surface of the second degree having its axes parallel to those of the director; hence, *if one of the reciprocal surfaces have a focus at the centre of the director, the principal axes of the other surface will be parallel to those of the director, and proportional in magnitude to their squares.*

It is evident from this property, that when the axes of the director are equal (whether real or imaginary), the reciprocal surface (7′) will be a sphere.

This theorem is evidently a generalization of the one given in No. XLIV, and by repeating the process in No. XLV, it is easy to show that the theorem there announced is true when the director is any central surface of the second degree.

If the system of conjugate diameters of the director parallel to a system of conjugate diameters of the given surface, be taken as axes of coordinates, the equations of the reciprocal surfaces will have nearly the same forms as those exhibited in No. XLVI, from which the student will readily deduce results similar to the ones there enunciated.

Again, it may be proved as in No. XLVII, that when the centre of the director is on the given surface, the reciprocal surface is a paraboloid, whose axis is parallel to the diameter conjugate to the diametral plane of the director which touches the given surface.

LII.

When the director is a paraboloid, if any diameter be taken as the axis of x, and the tangent plane applied at its vertex as the plane of yz, the axes of y and z being parallel to any system of conjugate diameters of a section parallel to the plane of yz, the equation of the director becomes

$$P'y^2+P''z^2+2Rx=0;$$

hence, in this case, the equation of the reciprocal surface will be (XLVIII)

$$E_1R^2x^2+A_2P'^2y^2+A_3P''^2z^2+2(B_1P'P''yz-C_3P''Rzx-C_2P'Rxy)$$
$$-2(C_1R^2x-B_3P'Ry-B_2P''Rz)+A_1R^2=0\ldots.(7'').$$

The equation of the diametral plane which bisects chords of this surface parallel to the axis of x is

$$E_1Rx - C_3P''z - C_2P'y - C_1R = 0,$$

or by No XXII, $$R(k+x) + P'k'y + P''k''z = 0;$$

and since this is evidently the equation of the polar plane of the point $kk'k''$, taken with respect to the director, we have the following theorem :—

The diametral plane of one of the reciprocal surfaces, which bisects chords parallel to the axis of the director, coincides with the polar plane of the centre of the other surface taken in relation to the director.

LIII.

By comparing the equations (6) and (7″), the following properties of the reciprocal surfaces which they represent may readily be deduced.

1st. When $A_1 = 0$, the surface (7″) passes through the origin, and the reciprocal surface (6) touches the plane of yz; hence *the tangent plane applied to the director at any point of the curve in which it is cut by one of the reciprocal surfaces, touches the other surface;* and conversely, *if a common tangent plane be applied to the director and one of the surfaces, the other surface will pass through the point in which it touches the director.*

It is evident from this theorem that if a developable surface be circumscribed to the director and one of the reciprocal surfaces, the other surface will pass through the curve in which it touches the director.

2nd. When $A_2 = 0$, the axis of y is parallel to a side of the asymptotic cone of the surface (7″), and the plane of zx touches the reciprocal surface (6); hence *if a straight line be drawn parallel to any side of the asymptotic cone of one of the reciprocal surfaces, the diametral plane of the director which bisects chords parallel to it will touch the other surface;* and conversely, *if any diametral plane of the director touch one of the surfaces, the chords which it bisects will be parallel to a side of the asymptotic cone of the other surface.*—The condition $A_3 = 0$ leads to the same result.

3rd. When $E_1 = 0$, the surface (6) is a paraboloid, and the axis of x is parallel to a side of the asymptotic cone of the surface (7″); hence *if one of the reciprocal surfaces be a paraboloid, the asymptotic cone of the other surface will have a side parallel to the axis of the director;* and reciprocally, *if the asymptotic cone of one of the surfaces have a side parallel to the axis of the director, the other surface will be a paraboloid.*

4th. The condition $B_1 = 0$ implies that the axes of y and z are parallel to two conjugate diameters of the surface (7″), and also that in relation to the surface (6) each of the planes of xy and zx passes through the pole of the other. Now it is well known that if the director and one of the reciprocal surfaces be cut by any plane, the curves of section have in general a system of conjugate diameters of the one parallel to a system of conjugate diameters of the other; hence *if the diametral planes of the director be drawn, which bisect chords parallel to these, the pole of each of these planes, taken with regard to the other surface, will be on the other plane.*

Reciprocally, *if two diametral planes of the director be drawn, such that the pole of each in relation to one of the reciprocal surfaces lies on the other, the chords of the director bisected by these planes will be parallel to two conjugate diameters of the other surface.*

5th. When $B_3 = 0$ the diametral plane of the surface ($7''$) which bisects chords parallel to the axis of y passes through the origin, and each of the planes of yz and zx passes through the pole of the other, taken in relation to the surface (6); hence *if the diametral planes of the director and of one of the reciprocal surfaces be constructed, which bisect chords parallel to a given straight line, and a tangent plane be applied to the director at a point of intersection of these two planes; this tangent plane and the diametral plane of the director will be such, that the pole of each, taken in relation to the other surface, will lie on the other plane.* The condition $B_2 = 0$ gives the same result.

6th. When $C_1 = 0$ the diametral plane of the surface ($7''$) which bisects chords parallel to the axis of the director, passes through the origin, and the centre of the surface (6) lies on the plane of yz; hence *if the diametral plane of one of the reciprocal surfaces, which bisects chords parallel to the axis of the director, cut the director, the tangent plane applied to the director at any point of the curve of intersection will pass through the centre of the other surface.*

7th. When $C_3 = 0$ the axes of z and x are parallel to two conjugate diameters of the surface ($7''$), and the centre of the surface (6) is on the plane of xy; hence, *if a diameter of the director and a tangent at its vertex be drawn parallel to two conjugate diameters of one of the surfaces, the diametral plane of the director which bisects chords parallel to the tangent will pass through the centre of the other surface.* The condition $C_2 = 0$ gives a similar result.

LIV.

Taking $R = 1$, in virtue of equations (19), No. XXII., equation ($7''$) may be written in the form

$$E_1 x(x + 2P'k'y + 2P''k''z + 2k)$$
$$+ A_2 P'^2 y^2 + A_3 P''^2 z^2 + 2B_1 P'P''yz + 2B_3 P'y + 2B_2 P''z + A_1 = 0,$$

from which it is evident that the surface ($7''$) passes through the curves in which the cylinder whose equation is

$$A_2 P'^2 y^2 + A_3 P''^2 z^2 + 2B_1 P'P''yz + 2B_3 P'y + 2B_2 P''z + A_1 = 0$$

is cut by the planes whose equations are

$$x = 0 \text{ and } x + 2P'k'y + 2P''k''z + 2k = 0.$$

The latter of these evidently denotes the polar plane of the point $2k$, $2k'$, $2k''$, taken in relation to the director; hence *the tangent plane applied to the director at any point, and the polar plane of the point obtained by drawing a line from the point of contact to the centre of one of the reciprocal surfaces and producing it till the produced part becomes*

equal to the line itself, intersect the other surface in two curves which lie on a cylinder having its axis parallel to the axis of the director.

LV.

When the coordinate axes are rectangular, or the origin is at the principal vertex of the director, we have seen in No. XLIV. that the conditions in order that the given surface (6) may be a surface of revolution having a focus at the origin of coordinates, give $A_1 = A_2 = A_3$, $B_1 = 0$, $B_2 = 0$, $B_3 = 0$; hence in this case the equation obtained in the last No. becomes

$$A_1(P'^2y^2 + P''^2z^2 + 1) + E_1x(x + 2P'k'y + 2P''k''z + 2k) = 0.$$

Similarly, the equation to the reciprocal polar of any other surface of revolution of the second degree, having one of its foci at the principal vertex of the director, will be of the form

$$A'_1(P'^2y^2 + P''^2z^2 + 1) + E'_1x(x + 2P'k_2 + 2P''k_3 + 2k_1) = 0,$$

where k_1, k_2, k_3 denote the coordinates of the centre of the given surface. Now by eliminating the function $P'^2y^2 + P''^2z^2 + 1$ from the last two equations we obtain

$$x\left\{\frac{E_1}{A_1}(x + 2P'k'y + 2P''k''z + 2k) - \frac{E'_1}{A'_1}(x + 2P'k_2y + 2P''k_3z + 2k_1)\right\} = 0,$$

which evidently represents two planes; hence, *if any two surfaces of revolution of the second degree have the principal vertex of a paraboloid for a common focus, their polar reciprocal surfaces, taken with respect to the paraboloid, will cut each other in two plane curves; one of which lies in the plane which touches the paraboloid at the principal vertex, and the other in a plane passing through the intersection of the polar planes of the other two foci of the given surfaces taken in relation to the director.*

Scholium. The examples now given may be sufficient to indicate the method of applying the general theory given in Nos. XXIV., XXV., *etc.* to the investigation of the reciprocal polar of any given surface—a problem which, as far as I am aware, had not previously received adequate attention, and to which I have chiefly confined myself in this paper. The geometrical mode of investigation, which is almost exclusively adopted by writers on this subject, although admirably suited to prove the general properties of reciprocal surfaces, is not competent to find the equation of one of the surfaces when that of the other is given. This problem being solved, the theory of reciprocal surfaces is complete, and nothing remains but to apply and develope it.

The student who is not acquainted with the geometrical method of exhibiting the theory of reciprocal polars, is referred to PONCELET'S "*Mémoire sur la théorie générale des polaires reciproques,*" in the fourth volume of the "*Journal von Crelle;*" and to CHASLES' "*Memoir on spheropolar reciprocals,*" in the fifth volume of the "*Memoires de l'Académie de Bruxelles.*"

SOLUTIONS OF MATHEMATICAL EXERCISES.

CLI. *By Simon Lhuilier.*

Let a sphere and plane which cut each other (not in a great circle) be given, and likewise any number of points in space; it is required to inscribe in the circle of the sphere made by the plane, a spherical polygon (composed of arcs of great circles) so that the planes of these great circles shall pass through the given points.

[SOLUTION. *After M. Lhuilier, Geneva; Mem. del' Acad., Berlin,* 1796.]

Let the given points on the surface of the sphere be P_1, P_2,......P_n, and O the centre of the sphere. Draw lines P_1O, P_2O,......P_nO to the centre O, meeting the given plane which cuts the sphere in $p_1, p_2, \ldots\ldots p_n$. Through these points describe a polygon, which shall have its angular points in the circumference of the circle in which the sphere is intersected by the plane. Then planes through these lines and the centre of the sphere will cut the spherical surface in the great circle polygon required.

The conclusion is too obvious to require a formal demonstration.

It is also obvious that a corresponding construction will be applicable to surfaces of the second order, the given plane not being diametral, and the sides of the polygon being diametral sections of the surface. Lhuilier unnecessarily limits the solution to surfaces of revolution.

It is, moreover, equally obvious that instead of diametral sections being prescribed for the sides of the required polygon, that the same construction would apply to sections through any one given point whatever, and to any given plane cutting the surface. The only limitation necessary is that the given point shall not be in the given plane.

CLX. *By Lexell.*

ABCD is a quadrilateral inscribed in a circle (whose centre is O); and the opposite sides are produced to meet, viz. AB and CD in E, and BC and AD in F; join EO, FO, meeting the circle in R and S; and take

$$OP : OR :: OR : OE, \text{ and } OQ : OS :: OS : OF.$$

Then if EQ, FP be drawn, they will be perpendicular to FO, EO.

[FIRST SOLUTION. *Lexell, Nov. Acta Petrop.* 1780.]

Draw lines from O and P to the angular points A, B, C, D of the quadrilateral; then OE : OA : : OA : OP, and the triangle OAP is similar to OEA. In a similar manner the triangles OBP, OCP, ODP are respectively similar to OEB, OEC, OED.

Hence we have OAP=OBP=OEB, and the four points A, B, P, O, are in the circumference of a circle; hence AOB=APB, and APO = ABO = BPE. In a similar manner it is shown that DPO = DCO = CPE; hence APD = BPC.

But sin ADP : sin DPA :: AP : AD, and (since DPA = BPC)

sin DPA : sin PCB :: BC : BP; whence

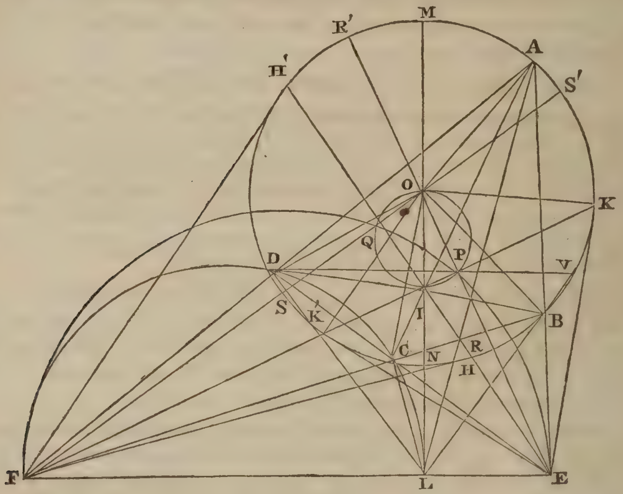
sin ADP : sin PCB :: AP.BC : AD.BP :: AP.CE : AE.BP,

by reason of the similar triangles CBE and ADE.

Again, $AP : AO :: AE : EO$, and $PB : AO(=OB) : BE : EO$;
whence $AP : PB :: AE : BE$, and consequently

$$\sin ADP : \sin PCB :: AE.EC : AE.EB :: CE : BE.$$

But $CE : BE :: \sin CBE : \sin BCE :: \sin CDF : \sin DCF :: CF : FD$;
therefore we have $\sin ADP : \sin PCB :: CF : FD$.



Also, $\sin FPD : \sin ADP :: FD : FP$ (since $\sin FDP = \sin ADP$),
and $\sin PCB : \sin FPC :: FP : FC$ (since $\sin FCP = \sin BCP$),
therefore we get

$$FC : FD :: \sin ADP : \sin PCB :: \sin ADP \sin FPC : \sin FPD \sin PCB.$$

From this we obtain $\sin FPC = \sin FPD$, or $FPC = FPD$; but $OPD = CPE$; hence $FPO = FPE =$ a right angle.

Cor. $EO.OF \cos EOF = (\text{radius})^2$.

[SECOND SOLUTION. *Mr. J. W. Elliott, Greatham.*]

From the point F draw the tangents FH, FH′, and join HH′ meeting OF in Q′ (suppose). Also draw the diagonals AC, BD intersecting in I, and join OH.

Now by known properties of the inscribed quadrilateral, the points H, I, Q′, H′ are in a straight line, so also are the points E, H, I; therefore the line EHI coincides with HIQ′. But HIQ′ is perpendicular to OF (by construction); therefore also EIQ′ is perpendicular to OF.

Moreover FHO is a right angle; therefore (Euc. vi 8. Cor.)

$$OF : OH :: OH : OQ',$$

that is, $$OF : OS :: OS : OQ',$$

but, $$OF : OS :: OS : OQ.$$

Therefore $OQ' = OQ$; hence the point Q′ coincides with Q, and the line EQ with EQ′; wherefore EQ is perpendicular to FO.

In like manner, it may be shown that FP is perpendicular to EO.

I shall subjoin a few theorems, which may not perhaps be without interest.

(*A*). If EO, FO be produced to meet the circle in R′ and S′; the lines ER′, FS′ are harmonically divided in R, P and S, Q.

For by the exercise, OE : OR :: OR : OP, therefore

$$OE-OR : OE+OR :: OR-OP : OR+OP;$$

that is, $ER : ER' :: PR : PR'$.

Similarly, $FS : FS' :: QS : QS'$.

(*B*). The line joining the centre of the circle and the point of intersection of the diagonals, cuts at right angles the line joining the points of concourse.

For FP and EQ each pass through I; therefore the point of intersection of the diagonals is also the point of intersection of the perpendiculars drawn from the angles F, E to the opposite sides of the triangle FOE: hence OIL cuts EF at right angles.

(*C*). The square on the line joining the points of concourse, is equal to the sum of the squares on the *two* tangents drawn from these points.

For let the tangent from E touch the circle in K: then

$$EF^2 = FO^2 + EO^2 - 2FO.OQ, \textit{ by Euc. ii. } 13,$$
$$= FO^2 - OH^2 + EO^2 - OK^2, \textit{ by the exercise},$$
$$= FH^2 + EK^2.$$

Cor. i. $EF^2 = FH^2 + EK^2 = BF.CF + CE.DE$. This is at once obvious from Euc. iii. 36.

Cor. ii. Since $FO^2 - FO.OQ = OF.FQ$, and $EO^2 - FO.OQ = EO^2 - EO.OP = OE.EP$; therefore

$$EF^2 = OE.EP + OF.FQ.$$

(*D*). The difference of the squares on the tangents FH, EK is equal to the difference of the squares of the segments FL, EL.

For, $OL^2 + FL^2 = OF^2 = FH^2 + OH^2$,
$OL^2 + EL^2 = OE^2 = EK^2 + OK^2$.

But $OH = OK$; therefore $FL^2 - EL^2 = FH^2 - EK^2$.

There are many other properties connected with this exercise, a few of which I shall enumerate.

(*E*). Tangents to the circle at B and C, cut the line EQ in the same point.

(*F*). The points B, O, Q, C are in the circumference of a circle; so also are the points F, E, P, Q.

(*H*). The rectangle $BQ.CQ = QH^2$, *etc.*

[THIRD SOLUTION. *Mr. Thomas Wilkinson, Burnley.*]

The proportions reduced to equalities are

$$OP.OE = OR^2 = OS^2 = OQ.OF,$$

and hence a circle will pass through the four points E, P, Q, F, and therefore FQE = FPE. From E draw EK to touch the circle at K, and join OK, PK; then since OP : OK :: OK : OE, and the angle at O common to the two triangles POK, EOK, they are similar, and hence the angle OPK is equal to OKE, that is, to a right angle.

N N

Draw OL perpendicular to EF meeting the circle in M and N, then by a known property *(Lawson's Prop. xi)*, we have

$$FD.FA = FL^2 + ML.LN, \text{ and } ED.EC = EL^2 + ML.LN;$$

whence $\quad FD.FA + ED.EC = FL^2 + EL^2 + 2ML.LN;$

but $\quad FD.FA + ED.EC = FE^2$ *(Pott's App., p. 52, Pr. 123)*

$$= FL^2 + EL^2 + 2FL.LE;$$

wherefore $\; ML.LN = EL.LF$, and hence also we have

$$EL^2 + FL.LE = EL^2 + ML.LN, \text{ or}$$

$$EL.EF = ED.EC = EK^2 = EO.EP.$$

Whence a circle will pass through the points F, L, P, O, and the angle FPO is equal to FLO, that is, to a right angle, as was to be proved.

Cor. 1. FE is the diameter of the circle through the points F,Q,P,E.

Cor. 2. The perpendicular from O upon FE will pass through the intersection of FP and EQ.

Solutions of great merit have been received from Dr. Burns, and Messrs. Ashton, P. Elliott, Greenup, Levy, Maver, Pownall, Roy, Smyth, and C. A. Weddle.

CLXI. *By* ——.

Give a proof of Euler's construction of the triangle inscribed in a circle, whose sides pass through three given points (see p. 147).

[Solution. *Mr. David Maver, Bon Accord School, Aberdeen.*]

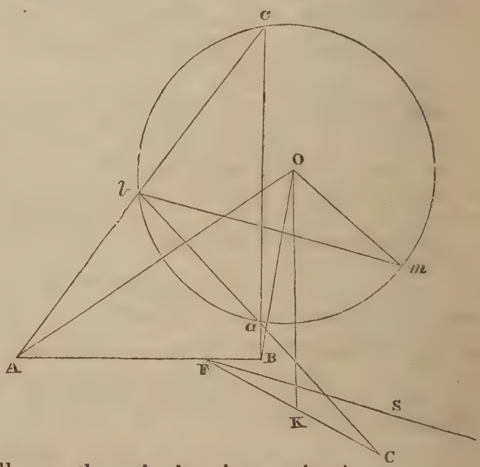
(The reader will readily supply the lines and references in the figure.)

Let O be the centre of the given circle, and A, B, C the three given points. Let the radius be unity, and in the line BA take $BF = \dfrac{BO^2-1}{AB}$.

Join FC, and in it take $FK = \dfrac{FO^2-1}{FC}$. Then from the centre O draw the radius Om so that

$$\cos KOm = \frac{\cos BFC}{KO};$$

bisect the angle BFC by the line FS, and draw mb parallel to FS meeting the circle again in b. Join Cb cutting the circle in a; join Ba and produce it to c; then if cb be joined and produced it will pass through the given point A.



From the point K draw KTG perpendicular to Om cutting Om in T and the circle in G; make TOI = BFC; then

$$\frac{OT}{OK} = \cos KOm = \frac{\cos BFC}{KO} = \frac{\cos TOI}{KO} = \frac{TO}{OK.OI};$$

whence OI = 1 = radius of the circle bmc. Join Ib, bG, then the angle Ibm = ½ angle IOm = SFC, and SF being parallel to bm, therefore bI is

parallel to FC. In a similar manner bG is parallel to AB. Join Fa and produce it till it cut the circle, if possible, in a point n different from G; then since from the above $FC.FK = FO^2 - 1 = Fa.Fn$, consequently $FC : Fa :: Fn : FK$, and hence (*Euc.* vi. 6) the angle FnK or anK=FCa $= ab$I $= a$GK. The point n coincides therefore with G. Also since bG is parallel to AB, the angle aFB $= b$Ga $= bca$. Produce cb to cut AB in a point p, then by similar triangles $pB.BF = cB.Ba = BO^2 - 1 = AB.BF$, wherefore finally p coincides with A.

[It will be seen by the following solution, that Exercises 162 and 164 are particular cases of 175; the last exercise, therefore, is given in this place. It would seem, moreover, that exercises 162 and 164 have already been discussed in the *Cambridge Mathematical Journal.* Of this, doubtless, the proposer (the same correspondent sent both exercises) was not aware, nor indeed were the Editors.]

CLXXV. *Mr. John Walker, Trinity College, Dublin.*

Find the locus of the intersection of three planes at right angles to each other, each of which touches one of three confocal ellipsoids.

[SOLUTION. *Mr. Thomas Weddle; and similarly by Messrs. P. Elliott, Lamesley, Durham; N. Smyth, Galway; and C. A. Weddle, Durham Training School.*]

Since the ellipsoids are confocal, their equations may be denoted as follows:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ ....}(1),$$

$$\frac{x^2}{a^2+h} + \frac{y^2}{b^2+h} + \frac{z^2}{c^2+h} = 1 \text{ ....}(2),$$

and

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1 \text{ ....}(3).$$

Let (lmn), $(l_1m_1n_1)$, and $(l_2m_2n_2)$ be the directing cosines of the normals to tangent planes (at right angles to each other) to (1), (2) and (3), respectively. By *Gregory's Solid Geometry*, p. 140, the equations to the tangent planes to (1) and (2) are

$$lx+my+nz = \sqrt{(a^2l^2+b^2m^2+c^2n^2)} \text{ ....}(4),$$

and, $$l_1x+m_1+n_1z = \sqrt{\{(a^2+h)l_1^2+(b^2+h)m_1^2+(c^2+h)n_1^2\}};$$

or, since $l_1^2+m_1^2y+n_1^2 = 1$, the latter becomes,

$$l_1x+m_1y+n_1z = \sqrt{(a^2l_1^2+b^2m_1^2+c^2n_1^2+h)} \text{ ....}(5).$$

Similarly the equation to the tangent plane to (3) is

$$l_2x+m_2y+n_2z = \sqrt{(a^2l_2^2+b^2m_2^2+c^2n_2^2+k)} \text{ ....}(6).$$

Now it is evident that the locus required will be obtained by eliminating $l\,m\,n\,l_1$, *etc.*, from (4, 5, 6). This is easily effected, for we have only to add the squares of these equations, and recollect that since the tangent planes are at right angles to one another, we have, $l^2+l_1^2+l_2^2=1$, $m^2+m_1^2+m_2^2=1$, $n^2+n_1^2+n_2^2=1$, $lm+l_1m_1+l_2m_2=0$, $ln+l_1n_1+l_2n_2=0$, and $mn+m_1n_1+m_2n_2=0$. The equation to the locus is thus found to be

$$x^2+y^2+z^2 = a^2+b^2+c^2+h+k \text{ ....}(7),$$

and it is therefore a sphere concentric with the ellipsoids. If a denote the *greatest* axis of any of these ellipsoids, b_1 the *mean* axis of a second, and c_2, the *least* axis of the third, we see that the square of the radius of this sphere $= a^2 + b_1^2 + c_2^2$; for $b_1^2 = b^2 + h$, and $c_2^2 = c^2 + k$.

The preceding investigation is a slight modification* of a solution given in a paper in the *Cambridge Mathematical Journal*, first series, vol. i. p. 119, (p. 130, 2nd edition)—the same paper also containing solutions of Exercises 162 and 164. Indeed the problem discussed above includes both these problems as particular cases; it becomes the former when h and k both $= 0$, and the latter when $a = b = c$.

* This consists almost entirely in the introduction of h and k, which somewhat simplifies the investigation.

It appears from a remark of the author (whose name is not given) of the paper mentioned above, that there is a solution of this problem in the 19th volume of the *Annales de Mathématiques*.

Mr. A. Roy, Dundee, sent also a good solution.

In addition to the above, the following solution of Exercise 164 may interest some of our readers.

CLXIV. *By δ.*

The radii of three concentric spheres are R, R′, R″; it is required to find the equation of the surface which is the locus of the point of intersection of three planes at right angles to each other, one of which constantly touches one of the three given spheres.

[SOLUTION. *Messrs. P. Elliott, Lamesley; W. Mortimer, Seedley, Manchester; John Stafford, Newcastle; Thomas Weddle, Wimbledon (second solution), and C. A. Weddle, Training School, Durham.*]

Construct a rectangular parallelopiped having its three dimensions equal to R, R′ and R″; place one of its angular points at the common centre O, of the spheres, and let P be the angular point opposite to O. Let this parallelopiped be turned round the point O; then the faces that intersect in P will evidently always be tangent planes to the three spheres, each to each, and hence P will describe the locus required. But $OP = \sqrt{(R^2 + R'^2 + R''^2)}$, is constant, and hence the locus is a concentric sphere, the square of whose radius is equal to the sum of the squares of the radii of the three given spheres, and whose equation is

$$X^2 + Y^2 + Z^2 = R^2 + R'^2 + R''^2.$$

Good solutions were also received from Mr. J. W. Elliott, Greatham (two solutions); A. D. B., and Mr. Nicholas Smyth, Galway.

To Exercise 162, good solutions were received from the following correspondents:—

Messrs. Peter Elliott, Lamesley; J. W. Elliott, Greatham, Stockton (two solutions); T. Greenup; the Rev. J. C. Bruce's Academy, Newcastle; W. Mortimer, Seedley, Manchester; A. Roy, Dundee; N. Smyth, Galway; C. A. Weddle, Training School, Durham; and Stephen Watson, Shotley Bridge.

CLXIII. *Mr. Thomas Weddle.*

Join the angular points of a tetrahedron, and the points in which an inscribed surface of the second degree touches the faces of the tetrahedron; if three of these lines pass through a point, the fourth will pass through the same point.

[SOLUTION. *Mr. Weddle, the proposer.*]

Let $t=0 \ldots (1)$, $u=0 \ldots (2)$, $v=0 \ldots (3)$, $w=0 \ldots (4)$, be the equations to the faces of the tetrahedron. Supposing t, u, v and w to have been multiplied by arbitrary constants, it follows from what has been shown by me at p. 261, vol. ii. of the *Mathematician*, that the equation to the inscribed surface of the second degree is

$$t^2+u^2+v^2+w^2+2(tu+vw)\cos\theta+2(tv+uw)\cos\phi$$
$$+2(tw+uv)\cos\Psi=0 \ldots (5);$$

where the constants θ, ϕ and Ψ are subject to the condition

$$\theta+\phi+\Psi=2\pi \ldots\ldots\ldots\ldots (6).$$

The equation (5) may by aid of (6) be put under the form

$$\{t+u\cos\theta+v\cos\phi+w\cos\Psi\}^2+\{-u\sin\theta+v\sin\phi+w\sin\Psi\}^2$$
$$-4vw\sin\phi.\sin\Psi=0 \ldots\ldots (7).$$

To find the point of contact of the surface with the face (4) of the tetrahedron, put $w=0$ in (7), and we must then have $t+u\cos\theta+v\cos\phi=0$, and $-u\sin\theta+v\sin\phi=0$, whence, (6),

$$\frac{t}{\sin\Psi}=\frac{u}{\sin\phi}=\frac{v}{\sin\theta} \ldots\ldots\ldots\ldots (8),$$

and (8) and (4) determine the point of contact of (4) and (5); but the equations (8) are those of a straight line passing through the intersection of (1, 2, 3); hence the equations (8) are those of the straight line joining the point of contact of (4, 5) and the opposite angle of the tetrahedron. In a similar way we shall find that the equations to the other three lines mentioned in the enunciation are,

$$\frac{t}{\sin\phi}=\frac{u}{\sin\Psi}=\frac{w}{\sin\theta} \ldots\ldots\ldots\ldots (9),$$

$$\frac{t}{\sin\theta}=\frac{v}{\sin\Psi}=\frac{w}{\sin\phi} \ldots\ldots\ldots\ldots (10),$$

and, $$\frac{u}{\sin\theta}=\frac{v}{\sin\phi}=\frac{w}{\sin\Psi} \ldots\ldots\ldots\ldots (11);$$

and we have to show that if three of the lines (8, 9, 10, 11) pass through a point, the fourth will pass through the same point.

Now in order that (9, 10, 11) may pass through the same point we must have

$$\sin^2\theta=\sin^2\phi=\sin^2\Psi \ldots\ldots\ldots\ldots (12);$$

hence either $\sin\theta=\sin\phi=\sin\Psi$, or one of the quantities $\sin\theta$, $\sin\phi$, $\sin\Psi$ must have a different sign from the other two, such as, $\sin\theta=\sin\phi=-\sin\Psi$. In the former case (8, 9, 10, 11) become,

$$\left.\begin{array}{l} t=u=v \\ t=u=w \\ t=v=w \\ u=v=w \end{array}\right\} \ldots\ldots\ldots\ldots (13),$$

and these lines all pass through the point $t=u=v=w$.

But if $\sin\theta = \sin\phi = -\sin\Psi$, (8, 9, 10, 11) will become,

$$t = -u = -v$$
$$t = -u = \quad w$$
$$t = -v = \quad w$$
$$-u = -v = \quad w,$$

and these lines pass through the point $t = -u = -v = w$.

Hence, in every case, if three of the lines pass through a point, the fourth will pass through the same point. The truth of the theorem is consequently established.

By (12) we have $\cos^2\theta = \cos^2\phi = \cos^2\Psi$; hence $\cos\theta = \cos\phi = \cos\Psi$, or $-\cos\theta = \cos\phi = \cos\Psi$, or $\cos\theta = -\cos\phi = \cos\Psi$, or $\cos\theta = \cos\phi = -\cos\Psi$; but it is unnecessary to consider any of these except the first; for if we take $-\cos\theta = \cos\phi = \cos\Psi$ in (5), we shall, after changing the signs of v and w, get the same result as if we had taken $\cos\theta = \cos\phi = \cos\Psi$.

From (6) we have, $1 - \cos^2\theta - \cos^2\phi - \cos^2\Psi + 2\cos\theta\cos\phi\cos\Psi = 0$, and if $\cos\theta = \cos\phi = \cos\Psi$, this equation becomes, $1 - 3\cos^2\theta + 2\cos^3\theta = 0$, the roots of which are 1, 1, and $-\frac{1}{2}$, so that we must have either $\cos\theta = \cos\phi = \cos\Psi = 1$, or $\cos\theta = \cos\phi = \cos\Psi = -\frac{1}{2}$; but the former is inadmissible, for then (5) would denote a plane. Substitute, therefore, $\cos\theta = \cos\phi = \cos\Psi = -\frac{1}{2}$ in (5), and we have,

$$t^2+u^2+v^2+w^2 - tu - tv - tw - uv - uw - vw = 0 \ldots\ldots (14);$$

and this is the general equation to surfaces of the second degree which are inscribed in the tetrahedron, and which are such that the four lines drawn from the points of contact to the opposite angular points of the tetrahedron pass through the same point. The equations to these lines are given in (13), and all the four lines pass through the point $t=u=v=w$.

Since the surface (14) evidently meets the tangent plane $t = 0$ in one point only,* it is evident that (14) must denote an *umbilical* surface, that is, either an ellipsoid, an hyperboloid of two sheets, or an elliptic paraboloid.

I take this opportunity of observing, that I have succeeded in obtaining the general equation to surfaces touching the faces of a tetrahedron in a form different from (5), and as this equation may in some cases be of easier application than (5), I shall give it here.

It is not difficult to show, (by aid of (6),) that (5) may be written in the form,

$$\tan\tfrac{1}{2}\phi\tan\tfrac{1}{2}\Psi\,(t+u-v-w)^2 + \tan\tfrac{1}{2}\theta\,.\tan\tfrac{1}{2}\Psi\,(t-u+v-w)^2$$
$$+ \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi\,(t-u-v+w)^2 = (t+u+v+w)^2.$$

Also, since $\frac{1}{2}\theta + \frac{1}{2}\phi + \frac{1}{2}\Psi = \pi$, we have,

$$\cot\tfrac{1}{2}\phi.\cot\tfrac{1}{2}\Psi + \cot\tfrac{1}{2}\theta\,.\cot\tfrac{1}{2}\Psi + \cot\tfrac{1}{2}\theta\,.\cot\tfrac{1}{2}\phi = 1;$$

hence, if we put $l = \tan\frac{1}{2}\phi\,.\tan\frac{1}{2}\Psi$, $m = \tan\frac{1}{2}\theta\,.\tan\frac{1}{2}\Psi$, and $n = \tan\frac{1}{2}\theta\,.\tan\frac{1}{2}\phi$, we see that the general equation to surfaces of the second degree touching the faces (1, 2, 3, 4) of a tetrahedron may be written

$$l(t+u-v-w)^2+m(t-u+v-w)^2+n(t-u-v+w)^2=(t+u+v+w)^2 \ldots (15),$$

* This is not necessarily true of the general surface (5), for θ, ϕ, and Ψ may not be real quantities. This will be evident on looking over the investigations at pp. 259—261, Vol. II. of the *Mathematician*.

where the arbitrary constants l, m, n are subject to the condition

$$l^{-1}+m^{-1}+n^{-1}=1 \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

It is worthy of observation, that if we suppose the constants l, m, n to be subject to the condition $l+m+n=1$, instead of the condition (16), then (15) will denote surfaces of the second degree circumscribed about the tetrahedron. This remark is contained in a letter which I received from Mr. Hearn—the student will find little difficulty in establishing the truth of it.

There are yet other forms of the equation to surfaces of the second degree touching the faces of a tetrahedron, which I have recently found to be useful. In the paper previously quoted I have shown that this equation may be written

$$t^2+u^2+v^2+w^2+(tu+vw)(\alpha+\alpha^{-1})+(tv+uw)(\beta+\beta^{-1})$$
$$+(tw+uv)(\gamma+\gamma^{-1})=0,$$

where $\alpha\beta\gamma=1$.

Now if we assume $\alpha=\dfrac{q}{r}$, $\beta=\dfrac{r}{p}$, and therefore $\gamma=\dfrac{p}{q}$, the equation will become

$$t^2+u^2+v^2+w^2+\left(\frac{q}{r}+\frac{r}{q}\right)(tu+vw)+\left(\frac{p}{r}+\frac{r}{p}\right)(tv+uw)$$
$$+\left(\frac{p}{q}+\frac{q}{p}\right)(tw+uv)=0 \ldots (17).$$

This equation has the advantage of being symmetrical without requiring any relation among the arbitrary constants. These advantages may likewise be gained as follows:—Put $\theta=\epsilon-\mu$, and $\phi=\mu-\delta$, then, (6), $\Psi=2\pi+(\delta-\epsilon)$, and these values substituted in (5), give

$$t^2+u^2+v^2+w^2+2(tu+vw)\cos(\epsilon-\mu)+2(tv+uw)\cos(\mu-\delta)$$
$$+2(tw+uv)\cos(\delta-\epsilon)=0 \ldots\ldots\ldots\ldots (18),$$

in which the constants δ, ϵ, μ, are not subject to any restriction.

The reader must be careful to recollect that, in all the preceding equations, t, u, v, and w are supposed to be multiplied by arbitrary constants; so that if $t'=0$, $u'=0$, $v'=0$, and $w'=0$, be the *given* equations to the faces of the tetrahedron, and A, B, C, and D be arbitrary constants, then will $t=\text{A}t'$, $u=\text{B}u'$, $v=\text{C}v'$, and $w=\text{D}w'$.

CLXV. *Mr. R. H. Wright, M.A., London.*

Two equal cones of equal densities float in a fluid whose density is likewise given, and unequal weights are placed on their horizontal bases over their centres of gravity; find how far the one will sink below the other. Also, if d be the original draft of the fluid for each, w the weight of each cone, m_1 and m_2 very small weights applied as above, δ the difference in draft of fluid; show that

$$\delta=\frac{d}{3w}\left(m_2-m_1\right).$$

[SOLUTION. *Dr. Burns, Rochester; and similarly by Messrs. P. Elliott; A. Roy; C. A. Weddle; and R. H. Wright, the proposer.*]

Let the specific gravity of the cones be to that of the fluid as s to 1; then $1 : s^{\frac{1}{3}} :: a : d$, where a is the altitude of each cone. Also let x_1 and x_2 be the drafts when the weights m_1 and m_2 are placed on the cones; then

$$d^3 : x_1^3 :: w : w + m_1, \text{ and } x_1 = d\left(1 + \frac{m_1}{w}\right)^{\frac{1}{3}},$$

$$d^3 : x_2^3 :: w : w + m_2, \text{ and } x_2 = d\left(1 + \frac{m_2}{w}\right)^{\frac{1}{3}};$$

hence $x_2 - x_1 = d\left\{\left(1 + \frac{m_2}{w}\right)^{\frac{1}{3}} - \left(1 + \frac{m_1}{w}\right)^{\frac{1}{3}}\right\}$

$$= d\left(1 + \frac{1}{3}\cdot\frac{m_2}{w} - \frac{1}{9}\cdot\frac{m_2^2}{w^2} + etc. - 1 - \frac{1}{3}\cdot\frac{m_1}{w} + \frac{1}{9}\cdot\frac{m_1^2}{w^2} - etc.\right)$$

$$= \frac{d}{3w}\left(m_2 - m_1\right) - \frac{d}{9w^2}\left(m_2^2 - m_1^2\right) + etc.$$

When m_1 and m_2 are very small when compared with w, all the powers, except the first, may be neglected, and therefore in this case

$$\delta = \frac{d}{3w}\left(m_2 - m_1\right).$$

Scholium A right cone of uniform specific gravity cannot *practically* float as stated in the above exercise, and it may be shewn that a force applied at the vertex, not less than that denoted by $(s^{\frac{3}{4}} - s)\times$ solidity, is requisite to cause the cone to float with the vertex downward, and the axis vertical.

Let b be the radius of the cone's base and a the altitude; then $\frac{1}{3}\pi b^2 a$ = solidity and $\frac{1}{3}\pi b^2 a\, s$ = weight. Also let x = the part of the axis immersed; then

$$a : b :: x : \frac{bx}{a} = \text{the radius of the section at depth } x,$$

and $\quad \dfrac{\pi b^2 x^3}{3a^2}$ = buoyant force; hence we have

$$\frac{\pi b^2 x^3}{3a^2} - \frac{\pi b^2 a\, s}{3} = \frac{\pi b^2}{3a^2}\left(x^3 - a^3 s\right) = \text{force at the vertex.}$$

Since $\quad \dfrac{\pi b^2 x^2}{3a^2} : \dfrac{\pi b^2 a\, s}{3} :: \dfrac{3a}{4} : \dfrac{3a^4 s}{4x^3}$ = distance from the vertex of the common centre of gravity of the cone and the force or weight applied at the vertex; consequently $\dfrac{3x}{4} = \dfrac{3a^4 s}{4x^3}$, and therefore $x^4 = a^4 s$, or $x = \pm a s^{\frac{1}{4}}$; hence substituting for x^3 we get

$$\frac{\pi b^2}{3a^2}\left(a^3 s^{\frac{3}{4}} - a^3 s\right) = \frac{\pi b^2 a}{3}\left(s^{\frac{3}{4}} - s\right).$$

Now $s^{\frac{2}{3}} - s$ being a positive quantity when s is a proper fraction, and $\frac{\pi b^2 a}{3}\left(s^{\frac{2}{3}} - s\right)$ applied at the vertex is requisite to cause the cone to float with indifference when the vertex is downward and the axis vertical, a still greater force is necessary to make the cone float with stability in that position.

If however a certain part from the vertex of each cone be formed of a substance of greater specific gravity than the fluid, the cones will float with stability, as supposed in this elegant exercise.

Good solutions on precisely similar principles were sent by Messrs. Bolton, P. Elliott, Greenup, Smyth, and Septimus Tebay, Gas Works, Preston.

CLXVI. *Dr. Burns, Rochester.*

Given $\tan^{-1} x^2 + \tan^{-1} x = \tan^{-1} \frac{1}{3}$, to find x.

[First Solution. *Messrs. Thomas Bolton, King's College, London; William Mortimer, Manchester; Stephen Watson, Castleside; and Thomas Wilkinson, Burnley.*]

Since by trigonometry, $\tan^{-1}x^2 + \tan^{-1}x = \tan^{-1}\frac{x^2+x}{1-x^3}$;

$$\therefore \quad \tan^{-1}\frac{x^2+x}{1-x^3} = \tan^{-1}\frac{1}{3}.$$

Inversely, $\frac{x^2+x}{1-x^3} = \frac{1}{3}$; therefore by clearing of fractions, and transposing

$$x^3 + 3x^2 + 3x = 1.$$

Add 1 to both sides, and we get

$$(x+1)^3 = 2 : \quad \therefore \; x = \sqrt[3]{2} - 1 = \cdot 259921.$$

[Second Solution. *Messrs. Peter Elliott, Lamesley; William Sherret, Cuminestone; and John Stafford, St. Ann's School, Newcastle.*]

By the general formula, $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}$,

we have $$\tan^{-1}x^2 + \tan^{-1}x = \tan^{-1}\frac{x^2+x}{1-x^3}.$$

Comparing this with the given equation, we evidently have

$$\frac{x^2+x}{1-x^3} = \frac{1}{3}, \text{ or } x^3 + 3x^2 + 3x + 1 = 2;$$

$$\therefore \quad x = \sqrt[3]{2} - 1 = \cdot 259921 = \tan 14^\circ\, 34'\, 11''.$$

Nearly according to one or other of the above were the solutions of Messrs. W. S. Ashton, Darwen; A. D. B., Hammersmith; W. H. Gibb, Manchester; Thos. Greenup, Mathematical Master at Mr. Bruce's Academy, Newcastle-on-Tyne; Thomas Johnson, St. Helens; Patrick Kincella, Leighlin Bridge; A. Roy, Dundee; Nicholas Smyth, Galway; Mortimer Dover Tait, at Mr. Hoole's Academy, Blackburn; C. A. Weddle, Training School, Durham; and Function.

CLXVII. *James Lockhart, Esq., Fasnacloich, Argyleshire.*

In cubic equations of the general form

$$x^3 + ax^2 + bx + c = 0:$$

prove that if $a^2b^2 + 18abc$ is greater than $4ca^3 + 4b^3 + 27c^2$, the roots are real; if less, two roots are imaginary; and if equal, two roots are equal to each other.

[FIRST SOLUTION. *Messrs. A. Roy, and C. A. Weddle.*]

Transform the given equation in the ordinary way into another wanting the second power of the unknown quantity, and it becomes

$$y^3 - \frac{a^2 - 3b}{3}y + \frac{2a^3}{27} - \frac{ab}{3} + c = 0 \ldots\ldots\ldots\ldots (1).$$

Now the roots of the given equation differ from those of (1) by a constant quantity only; hence the criterion for determining the nature of the roots of (1) will apply equally to the proposed equation. If for (1) we write $y^3 + Ay + B = 0$, the value of y by Cardan's rule is

$$y = \left\{-\frac{B}{2} + \sqrt{\left(\frac{B^2}{4} + \frac{A^3}{27}\right)}\right\}^{\frac{1}{3}} + \left\{-\frac{B}{2} - \sqrt{\left(\frac{B^2}{4} + \frac{A^3}{27}\right)}\right\}^{\frac{1}{3}};$$

hence when $\frac{B^2}{4} + \frac{A^3}{27}$ is negative, two values of y are imaginary. Now replacing the values of A and B we have

$$\frac{B^2}{4} + \frac{A^3}{27} = \frac{4ca^3 + 4b^3 + 27c^2 - a^2b^2 - 18abc}{108},$$

which is negative, or positive, or nothing, according as $18a^2b^2 + 18abc$ is greater or less than $4ca^3 + 4b^3 + 27c^2$, or equal to it. Now it is well known that, according as $\frac{B^2}{4} + \frac{A^3}{27}$ is negative, nothing or positive, the roots are all real and unequal, two of them are equal, or two imaginary; hence applying the criterion to (1) we have according as

$$4a^2b^2 + 18abc >, =, \text{ or } < 4ca^3 + 4b^3 + 27c^2,$$

the roots of the given equation are all real and unequal, or two roots equal, or two roots imaginary.

The solutions of Messrs. Elliott, Mortimer, Stafford, Tait, and Watson were similar to the above.

[SECOND SOLUTION. *By A. D. B., Hammersmith; and similarly by Dr. Burns, Rochester; Mr. W. S. Ashton, Darwen; Mr. W. H. Gibb, Manchester; and Mr. Thomas Johnson, St. Helen's.*]

Sturm's theorem applied to the general cubic equation gives the following functions,

$$\begin{aligned} X &= x^3 + ax^2 + bx + c \\ X_1 &= 3x^2 + 2ax + b \\ X_2 &= 2(a^2 - 3b)x + ab - 9c = Ax + B \\ X_3 &= 2a\,AB - A^2b - 3B^2. \end{aligned}$$

Let A be negative, then X_3 must also be negative; for $A < 0$ involves the condition $a^2 < 3b$, and b must be positive. Now $3B^2$ is essentially a positive

quantity, and since b is positive, the quantity A^2b must likewise be positive. If therefore X_3 be positive, then will

$$2aAB \gtrdot A^2b + 3B^2.$$

But $2aAB - 3B^2$ is a maximum when $B = \frac{1}{3}aA$, in which case the remainder is $\frac{1}{3}a^2A^2$, a quantity which is always $\lessdot A^2b$, if $a^2 \lessdot 3b$, or $A \lessdot 0$. Consequently $A \lessdot 0$ necessarily makes $2aAB - A^2b - 3B^2 \lessdot 0$, and $2aAB - A^2b - 3B^2 \gtrdot 0$ includes the condition $A \gtrdot 0$. The function X_3 therefore contains all that is requisite to determine the nature of the roots of the proposed equation according to Sturm's theorem. Substituting for A and B we find X_3 to be

$$a^2b^2 + 18abc - 4ca^3 - 4b^3 - 27c^2;$$

and according as this quantity is *positive*, *negative*, or *zero*, so are the roots of the equation all real, two impossible, or two of them equal.

[THIRD SOLUTION. *Mr. J. Greenup, Mathematical Master, the Rev. J. C. Bruce's Academy, Newcastle.*]

The given equation when the roots are diminished by a quantity d becomes

$$(x+d)^3+a(x+d)^2+b(x+d)+c=0,$$

or $$x^3+(3d+a)x^2+(3d^2+2ad+b)x+d^3+ad^2+bd+c=0.$$

Now as d is quite arbitrary, we may assume $d=\frac{1}{3}\{-a\pm\sqrt{(a^2-3b)}\}$,

or $$3d^2+2ad+b=0.$$

Therefore the equation becomes, by putting

$$m=-a\pm\sqrt{(a^2-3b)},$$

$$x^3\pm(a^2-3b)^{\frac{1}{2}}x^2+\tfrac{1}{27}\{m^3+3am^2+9bm\}+c=0.$$

Hence if $c \gtrdot \frac{1}{27}(m^3+3am^2+9bm)$, the equation has two imaginary roots, clearing of fractions, expanding, transposing, *etc.*, it appears that if $a^2b^2+18abc \gtrdot 4a^3c+4b^3+27c^2$, the roots are real, if less, two are imaginary, and if equal, two roots are equal.

Cor. When $a=0$ the case corresponds with question 156, No. 3, Mathematician.

The solutions of Mr. N. Smyth, Mr. T. Wilkinson, and Mr. P. Kincella were omitted with much regret.

MATHEMATICAL EXERCISES—*(continued.)*

179. *Mr. W. H. Levy, Shalbourne.*

If the radii of the escribed circles of any plane triangle be in harmonical progression, then will the sides of the triangle be in arithmetical progression.

180. *Mr. Septimus Tebay, Gas Works, Preston.*

Three circles, radii r_1, r_2, r_3 touch each other on a plane. Let three other circles, radii ρ_1, ρ_2, ρ_3 be described each touching two of the former circles and a common tangent; then will

$$\frac{1}{\sqrt{\rho_1}}+\frac{1}{\sqrt{\rho_2}}+\frac{1}{\sqrt{\rho_3}}=\frac{2}{\sqrt{r_1}}+\frac{2}{\sqrt{r_2}}+\frac{2}{\sqrt{r_3}}.$$

181. *Lieut. A. R. Clarke, R.E.*

If the angular points of a triangle be joined with the opposite points of contact of the escribed parabolas, the three lines will meet in a point, and the sum of its distance from the sides of the triangle is

$$\frac{1}{\mathrm{D}} \cdot \frac{a^2b^2 + a^2c^2 + b^2c^2}{ab + ac + bc},$$

where D is the diameter of the circumscribed circle.

182. *Dr. Burns, Rochester.*

Given $\sin\theta = \sin\theta^2 = -\sin\theta^3 = \sin\theta^4 = -\sin\theta^5 = \sin\theta^6 = -$ *etc.*; find θ a whole number of degrees greater than $1°$ and less than $90°$.

183. *James Lockhart, Esq., Fasnacloich, Argyleshire.*

Find the first two least values of n that are not prime numbers, when $\frac{2^n - 2}{n}$ is a whole number.

184 *Dr. Rutherford.*

If the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ has two roots of the form $+\sqrt{-\beta}$, and $-\sqrt{-\beta}$, there exists among the coefficients a, b, c, d, e the relation

$$\frac{ab - c}{ad - e} = \frac{ad - e}{cd - be}, \quad \text{or} \quad \frac{abc - a^2d - c^2}{ab^2 - 2ad - bc - e} = \frac{e}{d}.$$

185. *Mr. Robert Harley, Blackburn, Lancashire.*

Required the value of the expectation of a person who is to receive a number of crowns, of which the digits only are known.

186. *Dr. Burns.*

Find a multiplier which will make $\sqrt{11} - \sqrt{7} + \sqrt{5} - \sqrt{3}$ a rational quantity, and show how a multiplier may be found which will make $\sqrt{a} \pm \sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3} \pm \dots \pm \sqrt{a_n}$, a rational quantity.

187. *Dr. Rutherford.*

Find the values of the unreal roots of the equation

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$$

188. *Mr. Thomas Weddle, Wimbledon, Surrey.*

If two triangles be inscribed in, or circumscribed about a conic section, there exists another conic to which the angles of each of the triangles are the poles of the opposite sides.

189. *Mr. Septimus Tebay, Preston.*

Three planes xy, xz, yz intersect at right angles. Let six straight lines be drawn through their common point of intersection; two at right angles in the plane xy; two at right angles in the plane xz; and two at right angles in the plane yz; prove that the sum of the moments of inertia of any body (M) about these six lines as axes, is

$$4\Sigma(x^2m) + 4\Sigma(y^2m) + 4\Sigma(z^2m);$$

x, y, z being the coordinates of any particle (m) of the solid.

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A NEW AND EASY METHOD OF APPROXIMATING TO THE UNREAL ROOTS OF TRINOMIAL EQUATIONS.

[*Mr. Thomas Weddle.*]

The discovery of a *practical* method of approximating to the *un*real roots of numerical equations is a problem which I have made many unavailing attempts to solve. Among various artifices which have at different times flitted through my mind is the assumption of x (the unknown) $= r(\cos\theta \pm \sqrt{-1}\sin\theta)$; and since $x^n = r^n(\cos n\theta \pm \sqrt{-1}\sin n\theta)$, it is easy to get two equations connecting r and θ, but I have never been able to obtain a general solution of these equations in a form suited to practice. The recent publication of Dr. Rutherford's ingenious tract* (which those interested in the solution of equations will do well to consult) has recalled this subject to my mind, and it has occurred to me that the preceding form may, at any rate, be advantageously employed in the solution of equations *having three terms only*. The method which I have devised seems to effect the discovery of the unreal roots of trinomial equations at as small an expenditure of labour as could, I think, be reasonably expected; the number of figures that can be obtained by our common logarithmic and trigonometric tables (which I make use of) is indeed limited, but the extent to which the approximation can be carried by means of these will, I doubt not, be found amply sufficient in almost every case.

Let $$ax^m + bx^n + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

denote any trinomial equation, and first let m and n be prime to each other.

Assume $$x = r(\cos\theta \pm \sqrt{-1}\sin\theta) \ldots\ldots\ldots\ldots\ldots(2),$$

in which we may suppose r to be a real number, positive or negative, and θ to be comprised between the limits 0° and 90°.

* The Complete Solution of Numerical Equations, in which, by one Uniform Process, the Imaginary as well as the Real Roots are easily determined. By William Rutherford, LL.D., F.R.A.S. G. Bell, 186, Fleet Street.

Substitute (2) in (1), and equate the real and unreal parts of the resulting equation separately to zero,

$$\therefore \quad ar^m \cos m\theta + br^n \cos n\theta + c = 0,$$

and

$$ar^m \sin m\theta + br^n \sin n\theta \qquad = 0.$$

Solve these two equations, considering r^m and r^n as the unknowns;

$$\therefore \quad r^m = \frac{c}{a} \frac{\sin n\theta}{\sin(m-n)\theta} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and,

$$r^n = -\frac{c}{b} \frac{\sin m\theta}{\sin(m-n)\theta} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now

$$(r^m)^n = (r^n)^m,$$

$$\therefore \quad \frac{c^n}{a^n} \cdot \frac{(\sin n\theta)^n}{\{\sin(m-n)\theta\}^n} = \frac{c^m}{(-b)^m} \cdot \frac{(\sin m\theta)^m}{\{\sin(m-n)\theta\}^m}:$$

or, reducing,

$$(-b)^m . \{\sin n\theta\}^n . \{\sin(m-n)\theta\}^{m-n} = a^n c^{m-n} \{\sin m\theta\}^m \ldots\ldots (5).$$

Having found θ from this equation, we shall then know r from (3) or (4). Should m be even, we must, to avoid ambiguity in sign, use (4), but if n be even, then (3) must be used. If both m and n be odd, then of course either (3) or (4) may be employed.

On the whole, I am inclined to think that the rule of Double Position is best adapted to find the values of θ, and the method as applied to the preceding equation (5), will be best seen in an example as under. I may here, once for all, observe that, since I take the logarithm of each side of (5), it makes very little matter whether the coefficients a, b, c be large or small numbers, also the only multipliers m, n, and $m-n$ will always be small. It will also be seen that whatever be the degree of the equation, the actual labour required to determine *a single pair* of unreal roots is nearly the same, and thus the work necessary to determine *all* the roots will be nearly proportional to the degree of the equation.

Example. Let $x^{10} + 2x^3 - 5 = 0$, be proposed for solution.

Equation (5) here becomes

$$2^{10} (\sin 3\theta)^3 . (\sin 7\theta)^7 = -5^7 (\sin 10\theta)^{10} \ldots\ldots\ldots\ldots (6).$$

Since the right hand member of this equation is essentially negative, one and one only of the two, $\sin 3\theta$, $\sin 7\theta$ must be negative. When the latter is negative, write $-\sin(7\theta - 180°)$ for $\sin 7\theta$; but when the former is negative, write $-\sin(3\theta - 180°)$ for $\sin 3\theta$; take the logarithms of the resulting equations, and we have

$$3 \log \sin 3\theta + 7 \log \sin(7\theta - 180°) - 10 \log \sin 10\theta - 1{\cdot}882490 = 0 \ldots (7),$$

and

$$3 \log \sin(3\theta - 180°) + 7 \log \sin 7\theta - 10 \log \sin 10\theta - 1{\cdot}882490 = 0 \,..(8).$$

On referring to (6), it will be seen that θ must be comprised between $\frac{180°}{7}\left(=25°\frac{5}{7}\right)$ and $\frac{360°}{7}\left(=51°\frac{3}{7}\right)$, or between $60°$ and $\frac{540°}{7}\left(=77°\frac{1}{7}\right)$.

To examine the first interval we must make use of (7), while to examine the latter, we must employ (8); recollecting that, since $\sin 10\theta$ enters (6), in an even power, we may, in every case, treat $\sin 10\theta$ as a positive quantity, whether it be so or not.

By actual trial, we find one value of θ between 32° and 33°, another between 39° and 40°, a third between 69° and 70°, and the fourth between 73° and 74°.

Substitute 32° and 33° successively for θ in the left hand member of (7), and the results are $-1{\cdot}078$ and $\cdot345$;

$$\therefore \quad 1{\cdot}078 + {\cdot}345 : 1{\cdot}078 :: 1^\circ = 60' : 45', \text{ nearly.}$$

Put $\theta = 32^\circ\ 45'$ in (7), and we get $-{\cdot}042269$, combine this with the previously found result (·345) for $\theta = 33^\circ$, and we have

$${\cdot}042 + {\cdot}345 : {\cdot}042 :: 15' : 1'{\cdot}6, \text{ nearly.}$$

$$\therefore \quad \theta = 32^\circ\ 46'{\cdot}6, \text{ nearly.}$$

Write 32° 46′·6 for θ in (7), and we have $-{\cdot}002136$; put therefore $\theta = 32^\circ\ 47'$, and we have $-{\cdot}007939$,

$$\therefore \quad {\cdot}002136 + {\cdot}007939 : {\cdot}002136 :: {\cdot}4 : {\cdot}0848;$$

$$\therefore \quad \theta = 32^\circ\ 46'{\cdot}6848, \text{ very nearly.}$$

Since $m(=10)$ is here even, and $n(=3)$ odd, we must take (4) from which to compute the value of r,

$$\therefore \quad r^3 = \frac{5}{2} \cdot \frac{\sin 10\theta}{\sin 7\theta}.$$

For the preceding value of θ this becomes,

$$r^3 = 2{\cdot}5\, \frac{\sin(360^\circ - 10\theta)}{\sin(7\theta - 180^\circ(} = 2{\cdot}5\, \frac{\sin 32^\circ\ 13'{\cdot}152}{\sin 49^\circ\ 26'{\cdot}7936}.$$

Hence r is positive, and throwing (2) into the form $r \cos\theta \pm \sqrt{-1}\, r\cos\theta . \tan\theta$, we may finish the computation as under:

log 2·5	=	·3979400
log sin 32° 13′·152	=	9·7268573
		10·1247973
log sin 49° 26′·7936	=	9·8806992
	3 |	·2440981
log r	=	·0813660
log cos 32° 46′·6848	=	9·9246792
$\therefore r\cos\theta = 1{\cdot}014017$		·0060452
log tan 32° 46′·6848	=	9·8088283
$\therefore r\sin\theta = {\cdot}652940$		$\bar{1}{\cdot}8148735$

Hence one pair of unreal roots is $1{\cdot}014017 \pm {\cdot}652940\sqrt{-1}$.

Since $\{x - r(\cos\theta + \sqrt{-1}\sin\theta)\}.\{x - r(\cos\theta - \sqrt{-1}\sin\theta)\}$
$$= x^2 - 2r\cos\theta\, x + r^2,$$

if we wished to find the real quadratic factors of the equation, we may find $r\cos\theta$ as above; also having found

$$\log r = \cdot 0813660,$$

we have $$\log r^2 = \cdot 1627320, \qquad \therefore r^2 = 1\cdot 454561.$$

Hence the quadratic factor is

$$x^2 - 2\cdot 028034\,x + 1\cdot 454561.$$

In a similar manner we shall find the other values of θ, and the corresponding values of x, and, collecting the whole, we have as under.

The values of θ are

$$32^\circ\ 46'\cdot 6848, \qquad 39^\circ\ 48'\cdot 5693, \qquad 69^\circ\ 20'\cdot 3093, \text{ and } 73^\circ\ 45'\cdot 5432.$$

The corresponding values of x are

$$1\cdot 014017 \pm \cdot 652940\sqrt{-1},$$
$$-\cdot 890959 \pm \cdot 742569\sqrt{-1},$$
$$-\cdot 392217 \pm 1\cdot 040085\sqrt{-1},$$
and, $$\cdot 345219 \pm 1\cdot 185091\sqrt{-1}.$$

The two *real* roots of the equation (which cannot of course be found by the above method) will be most easily obtained by the process which constitutes my "New Method of Solving Numerical Equations of all Orders;"* and they are thus found to be

$$1\cdot 091469 \text{ and } -1\cdot 243589.$$

Also the real quadratic factors of the equation are

$$x^2 - 2\cdot 028034\,x + 1\cdot 454561,$$
$$x^2 + 1\cdot 781918\,x + 1\cdot 345216,$$
$$x^2 + \cdot 784434\,x + 1\cdot 235611,$$
$$x^2 - \cdot 690438\,x + 1\cdot 523617,$$
and, $$x^2 + \cdot 152120\,x - 1\cdot 357339,$$

the last factor arising from the two real roots.

It only remains to dispose of the case in which m and n have a common measure.

In this case (1) may be written,

$$ax^{hv} + bx^{kv} + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

h and k being prime to each other.

Let $z = x^v$, then (9) becomes

$$az^h + bz^k + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Assume $z(=x^v) = r(\cos\theta \pm \sqrt{-1}\sin\theta)$, and proceed precisely as before to ascertain the values of r and θ.

First, let v be odd; then r_1 and θ_1 being corresponding values of r and θ, we have

* It is not an unimportant feature of this method, that it admits of use being made of a table of logarithms, thus frequently greatly facilitating the calculations.

$$x^v = r_1\{\cos\theta_1 \pm \sqrt{-1}\sin\theta_1\} = r_1\{\cos(2l\pi+\theta_1) \pm \sqrt{-1}\sin(2l\pi+\theta_1)\};$$

$$\therefore\ x = \sqrt[v]{r_1}.\left\{\cos\frac{2l\pi+\theta_1}{v} \pm \sqrt{-1}\sin\frac{2l\pi+\theta_1}{v}\right\}\ldots\ldots\ldots(11),$$

and we shall obtain $2v$ values of x by making l successively $= 0, 1, 2 \ldots.,$ $v-1$.

Secondly, let v be even, then r_1 being a *positive* value of r, and θ_1 the corresponding value of θ,

$$x = \pm\sqrt[v]{r_1}.\left\{\cos\frac{2l\pi+\theta_1}{v} \pm \sqrt{-1}\sin\frac{2l\pi+\theta_1}{v}\right\}\ldots\ldots\ldots\ldots(12),$$

and we shall obtain $2v$ values of x by making l successively $= 0, 1, 2 \ldots.,$ $\frac{v-2}{2}$.

But if $-r_1$ be a *negative* value of r (v still being even) and θ_1 the corresponding value of θ,

$$\therefore\ x^v = -r_1\{\cos\theta_1 \pm \sqrt{-1}\sin\theta_1\}$$
$$= r_1[\cos\{(2l+1)\pi-\theta_1\} \pm \sqrt{-1}\sin\{(2l+1)\pi-\theta_1\}];$$

$$\therefore\ x = \pm\sqrt[v]{r_1}\left\{\cos\frac{(2l+1)\pi-\theta_1}{v} \pm \sqrt{-1}\sin\frac{(2l+1)\pi-\theta_1}{v}\right\}\ldots(13),$$

and we shall obtain $2v$ values of x by making l successively $= 0, 1, 2 \ldots.$ $\frac{v-2}{2}$.

If r_1 or $-r_1$ (as the case may be), be a real root of (10), we have only to suppose $\theta_1 = 0$ in (11), (12), or (13), and take only v values.

The computation of each of the preceding expressions may be conducted precisely as before, but all this is too plain to require any illustration.

Wimbledon, Feb. 8th, 1849.

REMARK ON HORNER'S METHOD OF SOLVING EQUATIONS.

[*Professor De Morgan.*]

The superiority of Horner's * method for the numerical solution of equations is, I think, now admitted in this country even by those who do not yet see that it is the natural and proper extension of those particular cases of itself known by the name of *division* and *extraction of the square root*. Every advance upon preceding solutions, in the number of figures obtained, has a tendency to set this superiority in a clearer light: for which reason I think it desirable to publish the following account.

Wallis happened to choose as his example of the Newtonian method the equation $x^3 - 2x = 5$; from which it has arisen that almost every succeeding writer, who has had anything new to produce in numerical solution, has taken this as one, often as the first, of his instances. Fourier, by his

* Horner (*Leybourn's Repository*, vol. v., part ii., page 52) describes his having met with his own method, so far as the cube root is concerned (but long after he had published) in Ingram's edition of Melrose's Arithmetic (see my *Arithmetical Books*, page 88). This book was published in 1816. But Alexander Ingram had previously published this method (which is his, and not Melrose's) in his edition of Hutton's Arithmetic, Edinburgh, 1807, duodecimo. The Appendix is his own, and contains the method.

posthumous work on equations (page 209), appears to have carried the solution by his method to 32 decimal places, or 33 places. The labour of this method, as compared with that of Horner, is very great.

A few years ago, I proposed to one of my classes at University College to carry the solution beyond 32 decimals. Four or five of the members* of that class carried the work, independently, to 51 decimal places, or 52 places, the last decimal place being three units in excess, as I now know. The result was published in the article *Involution and Evolution* in the *Supplement* to the *Penny Cyclopædia*, and in the last edition of my *Arithmetic*.

Previously to the last Christmas vacation, I proposed the same exercise to one of my classes, desiring that the solution might be carried beyond that of their predecessors. Some answers which I received may be described (noting only the *correct* places) as of 75, 65, 63, 58, 57, and 52 decimal places. One answer, however, carried the work to 101 decimal places, or 102 places altogether. The producer of this answer is Mr. William Harris Johnston, of Dundalk, and of the Excise Office. On receiving this answer, the extent of which went much beyond any thing I had expected, I requested Mr. Johnston, by way of verification, to undertake the equation $y^3 - 90\,y^2 + 2500\,y - 16000 = 0$, exacting a promise that he would not look at his former work until the second work was finished. This promise was given, but was needless as to the last half of the solution (unpublished), since I afterwards found that I had his only copy of his first solution in my own possession all the time. The relation between the roots of these equations is $y = 30 - 10\,x$, x being the root of the first, and y that of the second. Accordingly, each place of y is the defect † from 9 of the following place of x. Mr. Johnston's value of y agrees with what it ought to be, from x, to the very last place. Both are as follows :—

If $x^3 - 2x = 5$, then $x = 2{\cdot}0945514815423265914823865405793029638573$

$\frac{1}{10}\,y = {\cdot}9054485184576734085176134594206970361426$

06105628239180304128529045312189983483667146267281777157775786

93894371760819695871470954687810016516332853732718222842242 14

In truth, it is carried one place further, as appeared on comparing the work of the two. Mr. Johnston has stopped short of the extreme end of his own work to give the last figure with certainty : but by the eye alone it can be seen that the last 6 in x might be replaced by $59 +$, and the last 4 in $\frac{1}{10}\,y$ by $41 -$.

My reason for publishing this account is, to fix a limit which any future proposed method of solving equations to great extent (I do not speak of special methods for a few figures) must pass before it has established itself against Horner's. Fourier, who certainly lent some new power to Newton's method, went to 33 figures, but more than ten youths, in their elementary studies, have carried the result by Horner's process beyond fifty figures; and one to 103 figures. I may add that the complete method—invented (or at least extended from the ordinary evolution of the square root) in some of its main parts two centuries and a half ago by Vieta—thrown by for at least a century and a half, as superseded by shorter methods—suddenly restored by

* In verification of their solution, I may state that the same thing was done at the same time, at my request, by a connexion of my own, who was then a student of King's College, London, who has since gained a senior wrangler's place.

† The class in question had not read anything in the theory of equations, and had not then even commenced algebra. Horner's method was given to them as a means of mechanical exercise in computation.

a then obscure country schoolmaster, who invented the part which had always been wanting—and not only restored to algebra, but made to take a place in arithmetic, from which it will not be driven as long as the common rule of division remains—possesses an historical interest which, as sufficiently appears above, is sufficient to induce beginners to exercise themselves in computation of a more extensive character than it is usually thought reasonable to require of them.

ON THE ELEMENTARY DEVELOPMENTS OF ALGEBRA.

[*Mr. Finlay, Professor of Mathematics and Natural Philosophy, Manchester New College.*]

1. If two variable quantities be so related that the value of the one depends on that of the other, the one is said to be a *function* of the other. Thus, if $y=3x^2$ when $x=1$ we have $y=3$, when $x=2$ we have $y=12$, when $x=3$ we have $y=27$, and so on; hence the value of y depends on the value assigned to x, and therefore by the definition y is a function of x. So also the logarithm of a number is a function of the number, the sine of an angle a function of the angle, and so on.

It is evident from the preceding definition that *no function of a variable can vanish for all values that can possibly be assigned to the variable.* For, if it could, the value of the function would always be zero, and therefore would not depend on that of the variable.

2. If the equation

$$A+Bx+Cx^2+etc.=A'+B'x+C'x^2+etc.\dots\dots\dots(a)$$

be identical, or if it be true for every value that can possibly be assigned to x, the coefficients of the same powers of x will be equal to one another.

To prove this important theorem, we observe that if the conditions

$$A'=A,\ B'=B,\ C'=C,\ etc.\dots\dots\dots\dots\dots\dots(b),$$

be satisfied, the two members of equation (a) will evidently be equal for all values of x. Besides it is only in this way that the equation can be made identically true. For, by transposing all the terms to the same side, we obtain

$$0=A-A'+(B-B')x+(C-C')x^2+etc.\dots\dots\dots(a');$$

and if any of the coefficients, $C-C'$ for instance, did not vanish, the corresponding term $(C-C')x^2$ would be a function of x, and therefore could not vanish for *all* values of x(1). If two or more of the coefficients of equation (a') be finite, it may be shown in a similar manner that equation (a') cannot be true except for particular values of x. Hence, in order that the proposed equation may be identically true, the conditions (b) are necessary and sufficient.

The proposition is evidently true when the indices of the powers of x are negative or fractional. It also holds when the development proceeds according to a series of sines or cosines of multiples of the variable, or generally according to a series of functions of the variable which are subject to any given law; the same proof being applicable in all cases.

3. The method of *indeterminate* or *conditional coefficients* is a process by which the development of a function may be effected, by assuming for it a

series of terms arranged according to the powers * of the variable on which it depends, with coefficients which are to be found from the condition that the assumed equation may be identical. As an example, let it be required to divide $a^n - x^n$ by $a - x$, when n is positive and entire. By the nature of division the result is of the form

$$A_0 + A_1 x + A_2 x^2 + \textit{etc.}$$

and therefore we may assume

$$\frac{a^n - x^n}{a - x} = A_0 + A_1 x + A_2 x^2 \ldots + A_i x^i + \textit{etc.},$$

where A_0, A_1, A_2, *etc.*, are conditional coefficients, the values of which are to be determined from the condition that the assumed equation may be identically true. Clearing this equation of fractions, we get

$$\begin{aligned} a^n - x^n = A_0 a &+ A_1 a x + A_2 a x^2 \ldots\ldots + A_i a x^i + \textit{etc.} \\ &- A_0 x - A_1 x^2 \ldots\ldots - A_{i-1} x^i - \textit{etc.} \end{aligned}$$

and since this equation is true for all values of x, the following conditions must be satisfied:

$$aA_0 - a^n = 0, \quad aA_1 - A_0 = 0, \quad aA_2 - A_1 = 0 \ldots\ldots$$
$$aA_n - A_{n-1} = -1, \quad aA_{n+1} - A_n = 0, \textit{ etc.}$$

From these equations we easily obtain

$$A_0 = a^{n-1}, \quad A_1 = a^{n-2}, \quad A_2 = a^{n-3}, \ldots A_{n-1} = a^0 = 1,$$
$$A_n = 0, \qquad A_{n+1} = 0, \quad A_{n+2} = 0, \textit{ etc.}$$

and by introducing these values of A_0, A_1, A_2, *etc.* in the assumed equation we get for the required quotient

$$a^{n-1} + a^{n-2} x + a^{n-3} x^2 \ldots + a x^{n-2} + x^{n-1},$$

the same result that would be found by division.

4. To find the value of $\frac{a^n - x^n}{a - x}$ when $x = a$.

First, when n is a whole positive number, we have (3)

$$\frac{a^n - x^n}{a - x} = a^{n-1} + a^{n-2} x + a^{n-3} x^2 \ldots + a x^{n-2} + x^{n-1}.$$

Now when $x = a$ each term in the second member of this equation becomes a^{n-1}; consequently in this case we have

$$\frac{a^n - a^n}{a - a} = n a^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots (c).$$

Secondly, let $n = -p$, where p is positive and entire; then since $a^{-p} = \frac{1}{a^p}$, $x^{-p} = \frac{1}{x^p}$, we shall have

$$a^{-p} - x^{-p} = \frac{x^p - a^p}{a^p x^p};$$

$$\therefore \quad \frac{a^{-p} - x^{-p}}{a - x} = -\frac{1}{a^p x^p} \cdot \frac{a^p - x^p}{a - x},$$

* The assumed development may proceed according to any regular series of known functions of the variable. Thus, if fx denote any function of x, we may in general assume

$$fx = A + B\cos x + C\cos 2x + D\cos 3x + \textit{etc.},$$

but in this case the values of the coefficients A, B, C, D, *etc.*, cannot in general be found in a finite form, except by means of definite integrals. If however we assume

$$fx = A + Bx + Cx^2 + Dx^3 + \textit{etc.},$$

the values of the coefficients A, B, C, *etc.*, are in general readily determined by the ordinary processes of elementary algebra.

and when $x = a$, we obtain by the first case,

$$\frac{a^{-p} - a^{-p}}{a - a} = \frac{-1}{a^{2p}}.p.a^{p-1} = -p.a^{-p-1} \text{ ....} (c').$$

Since this equation is the same that would be obtained by substituting $-p$ for n in equation (c), it follows that equation (c) holds good when n is a negative integer.

Thirdly, let $n = \frac{p}{q}$, where $\frac{p}{q}$ is a fraction in its lowest terms. If we assume

$$\sqrt[q]{a} = a', \ \sqrt[q]{x} = x';$$

we obtain by raising both sides of each equation to the power p and also the power q,

$$a^{\frac{p}{q}} = a'^p, \ x^{\frac{p}{q}} = x'^p, \ a = a'^q, \ x = x'^q;$$

$$\therefore \ \frac{a^{\frac{p}{q}} - x^{\frac{p}{q}}}{a - x} = \frac{a'^p - x'^p}{a'^q - x'^q} = \frac{a'^p - x'^p}{a' - x'} \div \frac{a'^q - x'^q}{a' - x'};$$

hence when $x = a$, or $x' = a'$, we obtain by the first and second cases,

$$\frac{a^{\frac{p}{q}} - a^{\frac{p}{q}}}{a - a} = \frac{pa'^{p-1}}{qa'^{q-1}} = \frac{p}{q} a'^{p-q} = \frac{p}{q} a^{\frac{p}{q}-1} \text{ ....} (c''),$$

from which it appears that equation (c) is true when n is a positive or negative fraction.

5. To develop $(1 + z)^n$ in a series of ascending powers of z, when the index n is positive or negative, fractional or entire.

Let the required development be denoted by $a + a_1 z + a_2 z^2 + a_3 z^3 +$ *etc.*, so that

$$(1 + z)^n = a + a_1 z + a_2 z^2 + a_3 z^3 + etc. \text{ ....} (d);$$

then, since this equation is true for all values of z, we shall have

$$(1 + x)^n = a + a_1 x + a_2 x^2 + a_3 x^3 + etc.;$$

and by subtracting this equation from the former one, we get

$$(1 + z)^n - (1 + x)^n = a_1(z - x) + a_2(z^2 - x^2) + a_3(z^3 - x^3) + etc.$$

By dividing the first member of this equation by $1 + z - (1 + x)$, and the second member by its equal $z - x$, we obtain

$$\frac{(1 + z)^n - (1 + x)^n}{1 + z - (1 + x)} = a_1 + a_2(z + x) + a_3(z^2 + zx + x^2) + etc. ;$$

and when $x = z$ this becomes (4)

$$n(1 + z)^{n-1} = a_1 + 2a_2 z + 3a_2 z^2 + etc.$$

Multiplying both sides of this equation by $1 + z$, we get

$$\begin{aligned} n(1 + z)^n = a_1 &+ 2a_2 z + 3a_3 z^2 + etc., \\ &+ a_1 z + 2a_2 z^2 + etc.; \end{aligned}$$

and by substituting for $(1+z)^n$ its development (f), this becomes

$$na + na_1z + na_2z^2 + na_3z^3 + na_4z^4 + etc.$$
$$= a_1 + (2a_2 + a_1)z + (3a_3 + 2a_2)z^2 + (4a_4 + 3a_3)z^3 + etc.$$

Since this equation is identical, we obtain (2)

$$a_1 = na,$$

$$2a_2 + a_1 = na_1 \text{ or } a_2 = \frac{n-1}{2} a_1 = \frac{n(n-1)}{2} a,$$

$$3a_3 + 2a_2 = na_2 \text{ or } a_3 = \frac{n-2}{3} a_2 = \frac{n(n-1)(n-2)}{1.2.3} a,$$

$$4a_4 + 3a_3 = na_3 \text{ or } a_4 = \frac{n-3}{4} a_3 = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} a, \ etc.$$

and by substituting these values of a_1, a_2, a_3, *etc.*, in the development (d), we get

$$(1+z)^n = a\left\{1 + nz + \frac{n(n-1)}{1.2} z^2 + \frac{n(n-1)(n-2)}{1.2.3} z^3 + etc.\right\}.$$

When $z = 0$ this equation gives $a = (1)^n$, and therefore, if we admit that any power or root of 1 is equal to 1, the required development will be

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{1.2} z^2 + \frac{n(n-1)(n-2)}{1.2.3} z^3 + etc. \ldots . (1).$$

By substituting $\frac{h}{x}$ for z in this equation, and multiplying both sides by x^n, we shall have

$$(x+h)^n = x^n + \frac{n}{1} x^{n-1}h + \frac{n(n-1)}{1.2} x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1.2.3} x^{n-3}h^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} . x^{n-4}h^4 + etc. \ldots . (1')$$

which is Newton's binomial theorem.

6. If t_i denote the i^{th} term of the development which constitutes the second member of equation (1), we shall have

$$t_i = \frac{n(n-1)(n-2) \ldots . (n-i+2)}{1.2.3 \ldots (i-1)} . z^{i-1} \ldots\ldots\ldots\ldots . (2);$$

hence when $n - i + 2 = 0$, or $i = n + 2$, the i^{th} term, and all those which follow it, vanish. Consequently, when n is a positive integer, the development of $(1+z)^n$ is finite, and contains only $n+1$ terms. But when n is fractional or negative, the preceding value of i is impossible, and therefore the development contains an infinite number of terms.

From equation (2) we obtain

$$t_{i+1} = \frac{n(n-1)(n-2) \ldots . (n-i+2)(n-i+1)}{1.2.3 \ldots . (i-1)i} z^i;$$

$$\therefore \quad \frac{t_{i+1}}{t_i} = \frac{n-i+1}{i} z = \left(\frac{n+1}{i} - 1\right) z.$$

In this equation n is invariable, but i increases by unity from term to term; and, when the series is infinite, i may become greater than any given number. Consequently the ratio of two consecutive terms of the development approaches the limit z as i increases, and therefore the series is divergent when $z \triangleright 1$. Hence it is evident that the series (1′) is divergent when $h \triangleright x$.

7. To develop a^x in a series of ascending powers of x.

Let $c_\circ + cx + c_2x^2 + c_3x^3 + c_4x^4$ *etc.*, denote the required development; then since $a^\circ = 1$, we have $c_\circ = 1$.

$$\therefore \quad a^x - 1 = cx + c_2x^2 + c_3x^3 + c_4x^4 + etc. \quad \ldots\ldots\ldots (e),$$

$$a^z - 1 = cz + c_2z^2 + c_3z^3 + c_4z^4 + etc.,$$

and by subtraction we obtain

$$a^x - a^z = c(x - z) + c_2(x^2 - z^2) + c_3(x^3 - z^3) + etc.$$

Again, since $a^x - a^z = a^z(a^{x-z} - 1)$, we get, in virtue of equation (e),

$$a^x - a^z = a^z \{c(x-z) + c_2(x-z)^2 + c_3(x-z)^3 + etc.\};$$

and by equating this to the former value of $a^x - a^z$, we obtain

$$a^z \{c(x-z) + c_2(x-z)^2 + etc.\} = c(x-z) + c_2(x^2 - z^2) + etc.$$

Dividing by $x-z$, and taking $x = z$ in the result, we get

$$c.a^x = c + 2c_2x + 3c_3x^2 + etc.;$$

hence by substituting for a^x its development (e) we obtain

$$c(1 + cx + c_2x^2 + c_3x^3 + etc.) = c + 2c_2x + 3c_3x^2 + etc.$$

Since this equation is identical, we shall have (2),

$$2c_2 = c^2 \quad \text{or} \quad c_2 = \frac{c^2}{2},$$

$$3c_3 = cc_2 \quad \text{or} \quad c_3 = \frac{c^3}{1.2.3},$$

$$4c_4 = cc_3 \quad \text{or} \quad c_4 = \frac{c^4}{1.2.3.4}, \; etc.;$$

and by substituting these values of c_2, c_3, c_4, *etc.* in equation (e) we obtain

$$a^x = 1 + cx + \frac{c^2x^2}{1.2} + \frac{c^3x^3}{1.2.3} + \frac{c^4x^4}{1.2.3.4} + etc. \ldots\ldots (3).$$

8. To determine the coefficient c, put $x = \frac{1}{c}$ in equation (3); then

$$a^{\frac{1}{c}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + etc.;$$

hence if ϵ denote the second member of this equation, so that

$$\epsilon = 2{\cdot}718281828459, \; etc. \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

we shall have $a^{\frac{1}{c}} = \epsilon$, or $a = \epsilon^c$; from which we obtain by taking logarithms,

$$c = \frac{\log a}{\log \epsilon} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If l' denote the logarithm of any number in the system whose base is ϵ, we have $l'\epsilon = 1$,

$$\therefore \; c = l'a \ldots\ldots\ldots\ldots\ldots\ldots (5'),$$

and by substituting this in equation (3) we get

$$a^x = 1 + x.l'a + \frac{x^2}{1.2}(l'a)^2 + \frac{x^3}{1.2.3}(l'a)^3 + etc, \ldots\ldots (3'),$$

which is a more definite form of the required development.

When $a = \epsilon$ we have $c = 1$,

$$\therefore \; \epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + etc. \ldots (3''),$$

which is the simplest form of the exponential theorem.

9. The value of the coefficient c may also be obtained, by means of the binomial theorem, as follows. Let $a = 1 + b$, then (5)

$$a^x = 1 + \frac{x}{1}.b + \frac{x(x-1)}{1.2}.b^2 + \frac{x(x-1)(x-2)}{1.2.3}.b^3 + etc.;$$

hence by substituting for a^x its development (e) we shall have

$$cx + c_2x^2 + c_3x^3 + etc. = \frac{x}{1}b + \frac{x(x-1)}{1.2}b^2 + \frac{x(x-1)(x-2)}{1.2.3}b^3 + etc.$$

$$\therefore \; c + c_2x + c_3x^2 + etc. = b + \frac{x-1}{1.2}b^2 + \frac{(x-1)(x-2)}{2.3}b^3 + etc.$$

and when $x = 0$ this becomes

$$c = b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \tfrac{1}{4}b^4 + etc. \ldots\ldots\ldots (6).$$

By equating this with the value of c obtained in the last number, we have

$$\log(1+b) = \log \epsilon.\{b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \tfrac{1}{4}b^4 + etc.\} \ldots\ldots (7),$$

from which we also deduce

$$l'(1+b) = b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3 - \tfrac{1}{4}b^4 + etc. \ldots\ldots (7'),$$

by passing to the system of logarithms whose base is ϵ.

10. To develop $l(1+x)$ in a series of terms arranged according to the ascending powers of x, l denoting the logarithm in the system whose base is a.

Let the required development be denoted by $m_0 + mx + m_2x^2 + m_3x^3 + m_4x^4 + etc.$; then since $l(1) = 0$, we have $m_0 = 0$.

$$\therefore \; l(1+x) = mx + m_2x^2 + m_3x^3 + m_4x^4 + etc. \ldots\ldots (f),$$

$$l(1+z) = mz + m_2z^2 + m_3z^3 + m_4z^4 + etc.;$$

and by subtraction we obtain

$$l(1+x) - l(1+z) = m(x-z) + m_2(x^2 - z^2) + m_3(x^3 - z^3) + etc.$$

But, by the nature of logarithms,

$$l(1+x) - l(1+z) = l\left(\frac{1+x}{1+z}\right) = l\left(1 + \frac{x-z}{1+z}\right);$$

and by equation (f)

$$l\left(1+\frac{x-z}{1+z}\right)=m\,\frac{x-z}{1+z}+m_2\left(\frac{x-z}{1+z}\right)^2+m_3\left(\frac{x-z}{1+z}\right)^3+etc.;$$

hence by equating these two values of $l(1+x)-l(1+z)$ we obtain

$$m\left(\frac{x-z}{1+z}\right)+m_2\left(\frac{x-z}{1+z}\right)^2+etc.=m(x-z)+m_2(x^2-z^2)+m_3(x^3-z^3)+etc.$$

Dividing both sides of this equation by $x-z$, and taking $x=z$ in the result, we have

$$\frac{m}{1+x}=m+2m_2x+3m_3x^2+4m_4x^3+etc.,$$

or, by developing the first member by division,

$$m-mx+mx^2-mx^3+etc.=m+2m_2x+3m_3x^2+4m_4x^3+etc.;$$

and, since this equation is identical, we shall have (2)

$$m_2=-\frac{m}{2},\ m_3=\tfrac{1}{3}m,\ m_4=-\tfrac{1}{4}m,\ etc.;$$

consequently the required development is

$$l(1+x)=m(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+etc.)\ldots\ldots\ldots\ldots(8).$$

11. In this equation m may have any value whatever. When $m=1$, the logarithms are called NAPERIAN, and are usually denoted by the characteristic l'; hence

$$l'(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+etc.\ldots\ldots\ldots\ldots(8'),$$

and by comparing this with the general equation (8), we shall have

$$l(1+x)=m.l'(1+x)\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

This equation shows that the logarithm of any number in the system whose base is a, may be found by multiplying the Naperian logarithm of the number by a constant quantity m. On account of this property the constant m is called the modulus of the system of logarithms whose base is a. Since $la=1$, if we take $1+x=a$, equation (9) gives

$$m=\frac{1}{l'a};$$

and therefore the modulus of any system of logarithms is the reciprocal of the Naperian logarithm of its base.

By comparing equations (7) and (8), we easily infer that $m=l'e$, and therefore $l'e=1$; consequently the number ϵ, which represents the infinite series

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+etc.,$$

is the base of the Naperian system of logarithms.

12. By writing $\dfrac{h}{x}$ for x in equation (8′) we get

$$l'\left(1+\frac{h}{x}\right)=\frac{h}{x}-\frac{1}{2}\frac{h^2}{x^2}+\frac{1}{3}\frac{h^3}{x^3}-\frac{1}{4}\frac{h^4}{x^4}+etc.,$$

and when h is negative this becomes

$$l'\left(1-\frac{h}{x}\right)=-\frac{h}{x}-\tfrac{1}{2}\frac{h^2}{x^2}-\tfrac{1}{3}\frac{h^3}{x^3}-\tfrac{1}{4}\frac{h^4}{x^4}-etc.;$$

hence by subtraction

$$l'\left(\frac{x+h}{x-h}\right)=2\left(\frac{h}{x}+\tfrac{1}{3}\frac{h^3}{x^3}+\tfrac{1}{5}\frac{h^5}{x^5}+etc.\right)\ldots\ldots\ldots\ldots(10).$$

If we assume

$$\frac{x+h}{x-h}=\frac{p+q}{p},\text{ or }\frac{h}{x}=\frac{q}{2p+q},$$

equation (10) becomes

$$l'(p+q)-l'p=2\left\{\frac{q}{2p+q}+\tfrac{1}{3}\left(\frac{q}{2p+q}\right)^3+\tfrac{1}{5}\left(\frac{q}{2p+q}\right)^5+etc.\right\}\ldots(11),$$

and when $q=1$, we have

$$l'(p+1)=l'p+2\left\{\frac{1}{2p+1}+\tfrac{1}{3}\left(\frac{1}{2p+1}\right)^3+\tfrac{1}{5}\left(\frac{1}{2p+1}\right)^5+etc.\right\}\ldots(11').$$

The last equation enables us to compute the Naperian logarithm of any of the natural numbers when the logarithm of the number immediately preceding it is known. Thus, since $l'(1)=0$, we have

$$l'2=2\left\{\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\frac{1}{7}\left(\frac{1}{3}\right)^7+etc.\right\},$$

$$l'3=2\left\{\frac{1}{5}+\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\frac{1}{7}\left(\frac{1}{3}\right)^7+etc.\right\}+l'2,\ etc.$$

13. To develop the angle whose tangent is x, in a series of ascending powers of x.

Let $ax+a_2x^2+a_3x^3+a_4x^4+etc.$ denote the required development.

$$\therefore\ \tan^{-1}x=ax+a_2x^2+a_3x^3+a_4x^4+etc.\ldots\ldots\ldots\ldots(g),$$

$$\tan^{-1}z=az+a_2z^2+a_3z^3+a_4z^4+etc.,$$

and by subtraction we obtain

$$\tan^{-1}x-\tan^{-1}z=a(x-z)+a_2(x^2-z^2)+a_3(x^3-z^3)+etc.$$

Now by trigonometry $\tan^{-1}x-\tan^{-1}z=\tan^{-1}\left(\dfrac{x-z}{1+xz}\right)$;

$$\therefore a(x-z)+a_2(x^2-z^2)+a_3(x^3-z^3)+etc.=a\left(\frac{x-z}{1+xz}\right)+a_2\left(\frac{x-z}{1+xz}\right)^2+etc.$$

Dividing both sides by $x-z$ and taking $x=z$ in the result, we obtain

$$a+2a_2x+3a_3x^2+etc.=\frac{a}{1+x^2}=a-ax^2+ax^4-etc.;$$

and since this equation is identical we shall have

$$a_2=0,\ a_3=-\frac{1}{3}a,\ a_4=0,\ a_5=\frac{1}{5}a,\ a_6=0,\ a_7=-\frac{1}{7}a,\ etc.;$$

hence by substituting these values of a_2, a_3, *etc.*, in equation (g) we obtain

$$\tan^{-1}x=a(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+etc.)\ldots\ldots\ldots\ldots(12).$$

Since $\tan 45° = 1$, when $x = 1$ this equation gives

$$a = \frac{45°}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - etc.} \ldots\ldots\ldots\ldots (13).$$

It is usual to take the angle a as the angular unit, in which case equation (12) becomes

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + etc. \ldots\ldots\ldots\ldots (12');$$

this, therefore, is the simplest form of the required development.

14. To develop the sine and cosine of any angle x in a series of ascending powers of x.

Let
$$\sin x = e_\circ + ex + e_2x^2 + e_3x^3 + etc \ldots\ldots\ldots\ldots (h),$$
$$\cos x = c_\circ + cx + c_2x^2 + c_3x^3 + etc \ldots\ldots\ldots\ldots (i);$$

then by changing x into $-x$, we obtain

$$-\sin x = e_\circ - ex + e_2x^2 - e_3x^3 + etc. \ldots\ldots\ldots\ldots (h'),$$
$$\cos x = c_\circ - cx + c_2x^2 - c_3x^3 + etc. \ldots\ldots\ldots\ldots (i');$$

and by taking half the difference of (h) and (h'), and half the sum of (i) and (i'), we obtain

$$\sin x = ex + e_3x^3 + e_5x^5 + etc. \ldots\ldots\ldots\ldots (k),$$
$$\cos x = c_\circ + c_2x^2 + c_4x^4 + c_6x^6 + etc. \ldots\ldots\ldots\ldots (l).$$

Again, by trigonometry, we have

$$\sin x - \sin z = 2\cos\tfrac{1}{2}(x+z)\sin\tfrac{1}{2}(x-z),$$
$$\cos z - \cos x = 2\sin\tfrac{1}{2}(x+z)\sin\tfrac{1}{2}(x-z),$$

and in virtue of (k) and (l) these become

$$e(x-z) + e_3(x^3 - z^3) + etc. = 2\cos\tfrac{1}{2}(x+z)\left\{e\left(\frac{x-z}{2}\right) + e_3\left(\frac{x-z}{2}\right)^3 + etc.\right\},$$

$$c_2(x^2 - z^2) + c_4(x^4 - z^4) + etc. = -2\sin\tfrac{1}{2}(x+z)\left\{e\left(\frac{x-z}{2}\right) + e_3\left(\frac{x-z}{2}\right)^3 + etc.\right\}.$$

Dividing both sides of each of these by $x-z$, and taking $z=x$ in the results, we get

$$e\cos x = e + 3e_3x^2 + 5e_5x^4 + etc.,$$
$$-e\sin x = 2c_2x + 4c_4x^3 + 6c_6x^5 + etc.$$

and by substituting for $\sin x$ and $\cos x$ their developments (k) and (l), we obtain

$$e(c_\circ + c_2x^2 + c_4x^4 + c_6x^6 + etc.) = e + 3e_3x^2 + 5e_5x^4 + 7e_7x^6 + etc.$$
$$-e(ex + e_3x^3 + e_5x^5 + etc.) = 2c_2x + 4c_4x^3 + 6c_6x^5 + etc.$$

Since these two equations are identical, we have

$$e = ec_\circ,\ 3e_3 = ec_2,\ 5e_5 = ec_4,\ 7e_7 = ec_6,\ etc.,$$
$$2c_2 = -ee,\ 4c_4 = -ee_3,\ 6c_6 = -ee_5,\ etc.;$$

$$\text{hence } c_\circ = 1,\ c_2 = -\frac{e^2}{2},\ e_3 = -\frac{e^3}{1.2.3},\ c_4 = \frac{e^4}{1.2.3.4},$$

$$e_5 = \frac{e^5}{1.2.3.4.5},\ c_6 = \frac{-e^6}{1.2.3.4.5.6},\ etc.;$$

and by substituting these values for the coefficients in the equations (k) and (l), we get the required development

$$\sin x = ex - \frac{e^3x^3}{1.2.3} + \frac{e^5x^5}{1.2.3.4.5} - etc. \ldots\ldots\ldots\ldots (14),$$

$$\cos x = 1 - \frac{e^2x^2}{1.2} + \frac{e^4x^4}{1.2.3.4} - \frac{e^6z^6}{1.2.3.4.5.6} + etc. \ldots (15).$$

15. The value of the coefficient e, in the development of the sine, still remains to be determined. For this purpose let $x = \frac{1}{e}$ in equation (14), then

$$\sin\left(\frac{1}{e}\right) = 1 - \frac{1}{1.2.3} + \frac{1}{1.2.3.4.5} - etc.;$$

hence, if we assume, for the sake of brevity,

$$\lambda = 0{\cdot}84147098, \ etc.,$$

we shall have $\sin\frac{1}{e} = \lambda$, or

$$e = \frac{1}{\sin^{-1}\lambda} \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

In order to simplify the developments of the sine and cosine obtained above, it is usual to take the angle whose sine is λ as the angular unit; in which case we have $e = 1$, and equations (14) and (15) become

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - etc. \ldots\ldots\ldots\ldots (14')$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3\ldots6} + etc. \ldots\ldots (15')$$

By referring to a table of natural sines, we find that the angle here adopted as the angular unit is $57° \ 17' \ 45''$, nearly.

16. The value of the coefficient e may be obtained more directly as follows. By changing x into $ex\sqrt{-1}$ in equation ($3''$) we get

$$\epsilon^{ex\sqrt{-1}} = 1 + ex\sqrt{-1} - \frac{e^2x^2}{1.2} - \frac{e^3x^3\sqrt{-1}}{1.2.3} + \frac{e^4x^4}{1.2.3.4} + \frac{e^5x^5\sqrt{-1}}{1.2.3.4.5}$$
$$- \frac{e^6x^6}{1.2.3..6} - \frac{e^7x^7\sqrt{-1}}{1.2.3..7} + etc.;$$

and by comparing this with equations (14) and (15), we obtain

$$\epsilon^{ex\sqrt{-1}} = \cos x - \sqrt{-1}.\sin x.$$

By taking x negative this becomes

$$e^{-ex\sqrt{-1}} = \cos x - \sqrt{-1}.\sin x,$$

and by dividing the former of these equations by the latter, we have

$$\epsilon^{2ex\sqrt{-1}} = \frac{\cos x + \sqrt{-1}.\sin x}{\cos x - \sqrt{-1}.\sin x}.$$

Taking the Naperian logarithms, we get

$$2ex\sqrt{-1} = l'\left(\frac{\cos x + \sqrt{-1}.\sin x}{\cos x - \sqrt{-1}.\sin x}\right);$$

and developing the second member by means of equation (10), we obtain

$$ex\sqrt{-1} = \sqrt{-1}.\tan x + \tfrac{1}{3}(\sqrt{-1}.\tan x)^3 + \tfrac{1}{5}(\sqrt{-1}\tan x)^5 + etc.$$

or, $$ex = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - etc.$$

The value of e may be found from this equation by taking x equal to any angle whose tangent is known. Thus, if $x = 45°$, we shall have

$$e = \frac{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - etc.}{45°} \quad\ldots\ldots\ldots\ldots(17.)$$

By comparing this with equation (13) we see that the constants e and a are reciprocals of each other; from which it is evident that the angular units adopted in Nos. 13 and 15 are identical.

17. *Scholium.*—It may now be proper to state the reasons which have induced me to make public any further researches on a subject so well known, and one which has occupied the attention of so many eminent writers. Much originality cannot of course be expected in such a case; but all who have been engaged in the business of education are aware, that the slightest improvements in a mathematical process of deduction are often of the utmost importance. From this consideration I have ventured to offer the preceding investigations to the notice of the junior readers of the Mathematician, as establishing the principal elementary developments of algebra by a method more simple and uniform than any that has hitherto appeared.

The determination of the coefficient e in the developments of the sine and cosine, Nos. 15 and 16, is, I think, more direct and satisfactory than any that has yet been given. The ordinary method of determining it from the principle that an arc of a circle, its sine and its tangent, are ultimately equal, when the arc is infinitely small, is extremely indirect, and is always a stumbling-block to beginners. This point, which is perhaps the most original one in the whole paper, acquires considerable importance from its connection with the Differential Calculus, as established by the method of Lagrange. The common method of determining the coefficient e, depends essentially on the theory of limits; and consequently the rule for differentiating the sine* of an angle, as investigated by Lagrange, cannot be considered as independent of that theory. This objection, which I now conceive to be fairly removed, has induced one eminent writer, at least, to abandon, in a great measure, Lagrange's beautiful method of expounding the principles of the differential calculus.

* It is the common practice of all writers on the differential calculus to make the rule for differentiating the sine of an angle, the fundamental one, from which all the rules for the other angular functions are derived. As, however, the investigation of the developments of $\tan^{-1}x$, No. 13, is much simpler than those of the developments of the sine and cosine, it would seem preferable, in Lagrange's method of development, to make the rule for differentiating $\tan^{-1}x$, the fundamental one, from which the rule for $\tan x$ could readily be deduced, and thence those for the other angular functions, by expressing them in terms of the tangent.

Another principal object of this paper is to put the method of conditional coefficients in as clear a point of view as possible. Accordingly I have endeavoured to show that the theorem in No. 2, which constitutes the fundamental principle of that method, is necessarily involved in our conception of a function. The intuitive character of the principle renders its proof very difficult, but by placing it in this general point of view, its truth is perhaps more easily *seen* than in any other in which it has yet been exhibited.

INVESTIGATION OF THE COEFFICIENT OF TERRESTRIAL REFRACTION.

[*Mr. W. Galbraith, M. A., Edinburgh.*]

1. In order to determine the coefficient of terrestrial refraction, by observation of the barometer and thermometer, we have from the known principles adopted by Newton, Laplace, *etc.*, in the theory of the atmospheric refractions,

$$n = \mathrm{RD}\,\frac{r}{4l} \qquad\qquad (1),$$

in which n is the coefficient of refraction, R the refringent power of dry air, D its density, r the mean radius of the earth, and l the height of the homogenous atmosphere.

2. Introducing the effect of aqueous vapour on the density of dry air, and

$$\mathrm{D}' = \mathrm{D}\left(1 - \frac{3f}{8\mathrm{B}}\right) \qquad\qquad (2),$$

where f is the elastic force of aqueous vapour, and B the height of the barometer, as shown in books on physics.

3. In like manner if R′ be the refringent power of the air under given circumstances, while R is that at the pressure B, the standard at which the experiments of Biot were made, or about 0·76 of the metrical barometer, equal to 29·9218 in. of the English barometer,

$$\mathrm{R}' = \mathrm{R}\left(1 + \frac{f}{12\mathrm{B}}\right)\left(\frac{1}{1 - \frac{3f}{8\mathrm{B}}}\right) \qquad\qquad (3).$$

4. Now if Δ be the density of mercury at the freezing point or zero of the centigrade scale, when B is the height of the barometer and l the height of the homogeneous atmosphere,

$$l = \frac{\Delta}{\mathrm{D}}.\mathrm{B} = 10466{\cdot}8\mathrm{B} = 26100 \text{ English feet*} \qquad (4),$$

where $\mathrm{B} = 29{\cdot}9218$ English inches, the pressure at which the value of $\frac{\Delta}{\mathrm{D}} = 10466{\cdot}8$ was determined by Biot and Arago, and reduced to the level of the sea at latitude 45°.

5. The expansion of mercury for one degree of the centigrade thermometer here designated by β, is generally estimated at 0·00018, hence

$$\mathrm{B}' = \mathrm{B}(1 + \beta\mathrm{T}) \qquad\qquad (5),$$

* $l = 27150$ feet at 30 inches of the barometer, and 50° of Fahrenheit's thermometer.

in which T is the temperature by the *attached* thermometer; therefore by substitution for B and D in formula (4),

$$l' = \frac{\Delta}{D} B. \frac{1+\beta T}{1-\frac{3f}{8B}} = l. \frac{1+\beta T}{1-\frac{3f}{8B}} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Recurring to equation (1), and

$$n = R'D'\frac{r}{4l'}, \text{ or by substitution,}$$

$$n = RD\frac{r}{4l}\left(1+\frac{f}{12B}\right)\left(\frac{1-\frac{3f}{8B}}{1+\beta T}\right) \ldots\ldots\ldots\ldots\ldots (7).$$

The value of f may be taken from a table of the elastic force of aqueous vapour, as that of Dalton or Ure.

6. From various experiments on the refringent power of air, and by numerous astronomical observations, the coefficient of the astronomical refractions, a, at 45° of altitude and at the freezing point, with the English barometer at 29·9218 inches,

$$a = 0\cdot000294384 = \tfrac{1}{2}RD, \text{ nearly.}$$

This value of a becomes at 30 inches of the English barometer and 50° of Fahrenheit's thermometer,

$$a = 0.000283 = \tfrac{1}{2}R_1 D_1, \text{ nearly} \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

the standard commonly selected for tables of the astronomical refractions, as that of Ivory.

7. But this coefficient varies directly as the pressure, and inversely as the temperature of the atmosphere, consequently,

$$a' = a\frac{b}{B}.\frac{1}{1+\beta T}.\frac{1}{1+\beta' t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

in which b is the given pressure, B the standard, t the temperature of the atmosphere, and β the expansion of dry air for 1° of the detached centigrade thermometer, commonly assumed at 0·00375.

Combining the results in (8) and (9) in formula (7) and it becomes

$$n = \frac{ar}{2B}.b.\frac{1}{1+\beta T}.\frac{1}{1+\beta' t}\left(1+\frac{f}{12b}\right)\left(\frac{1-\frac{3f}{8b}}{l(1+\beta T)}\right) \ldots\ldots\ldots (10).$$

The value of $l = 26100$ feet was determined on the supposition that the density and temperature were constant throughout the whole atmospheric column, whereas, on the contrary, both density and temperature decrease proportionally to the height. It is therefore evident that l is by far too small, and that the value of the last term of formula (10) should be diminished to correspond with the observed phenomena.

8. By observation combined with the preceding principles, it has been found necessary to modify this expression by considerations depending upon the difference of height, and the expansion of moist air, which last designated by β'' has been found by me to be 0·00230 for 1° of Fahrenheit, or 0.00414 for 1° of the centigrade scale.

From numerous observations on the mountains of Scotland, an elevation of 240 feet depresses Fahrenheit's thermometer 1°, or 430 feet for 1° centigrade. Introducing these into formula (10), and it becomes

$$n = \frac{ar}{2B} \cdot b \frac{1}{1+\beta T} \cdot \frac{1}{1+\beta' t}\left(1 + \frac{f}{12b}\right)\left\{\frac{1 - \frac{3f}{8b}}{l(1+\beta T)} - \frac{\beta'' \delta t}{\delta h(1+\beta T)}\right\} \ldots (11).$$

From the preceding principles the last term of formula (11) becomes $\frac{0{\cdot}000414\delta t}{\delta h(1+\beta T)}$; and since $\delta t = \frac{\delta h}{430}$, the preceding by substitution becomes

$$\frac{0{\cdot}00414\ \frac{\delta h}{430}}{\delta h(1+\beta T)} = \frac{0{\cdot}0000096}{1+\beta T}.$$

Combining the last two terms of (11) by means of the last expression, and they become

$$\frac{1 - \frac{3f}{8b}}{l(1+\beta T)} - \frac{0{\cdot}0000096\, l}{l(1+\beta T)}.$$

Now since $l = 26100$ feet, by (4)

$$\frac{1 - \frac{3f}{8b} - 0{\cdot}0000096\, l}{l(1+\beta T)} = \frac{1 - \frac{3f}{8b} - 0{\cdot}25}{l(1+\beta T)} = \frac{0.75 - \frac{3f}{8b}}{l(1+\beta T)}.$$

Replacing the factor, consisting of the two last terms in formula (11), by this equivalent expression, and it becomes

$$n = \frac{ar}{2B}.b.\frac{1}{1+\beta T} \cdot \frac{1}{1+\beta' t}\left(1 + \frac{f}{12b}\right)\frac{\frac{3}{4} - \frac{3f}{8b}}{l(1+\beta T)}$$

$$= \frac{ar}{2Bl} \cdot b.\left(\frac{1}{1+\beta T}\right)^2 \cdot \frac{1}{1+\beta' t}\left(1 + \frac{f}{12b}\right)\left(\frac{3}{4} - \frac{3f}{8b}\right).$$

If the two last factors be multiplied together the product is $\frac{3}{4} - \frac{5f}{16b} - \frac{f^2}{32b^2}$.

Because $\frac{f^2}{32b^2}$ must always be a very small quantity, it may without sensible error be neglected, and the preceding expression becomes $\frac{3}{4} - \frac{f}{3b}$, very nearly.

By means of these simplifications, equation (11) becomes finally,

$$n = \frac{ar}{2Bl} \cdot b.\left(\frac{1}{1+\beta T}\right)^2 \cdot \frac{1}{1+\beta' t}\left(0{\cdot}75 - \frac{f}{3b}\right) \ldots\ldots\ldots\ldots (12).$$

Adapting this to 30 inches of the English barometer and 50° of Fahrenheit's thermometer, and it becomes

$$n = \frac{ar}{2l} \cdot \frac{b}{B} \cdot \left(\frac{1}{1+\beta(T-50°)}\right)^2 \cdot \left(\frac{1}{1+\beta'(t-50°)}\right)\left(0{\cdot}75 - \frac{f}{3b}\right) \ldots (13).$$

Taking the values of a, β, and l, as formerly stated, and $r = 20922642$ feet $= a$, the radius of the equator equal half the sum of the radius of cur-

vature of the meridian and of the arc perpendicular to the meridian at 45°, which for this and similar purposes may be reckoned the mean radius of curvature of the earth, very nearly.

To simplify the calculation of n, the log of $\frac{ar}{2l}$ may be used as a constant log, when B $= 30$ inches and $t = 50°$ of Fahrenheit's thermometer, the standard pressure and temperature of Ivory's table of refractions.

Log $a =$ log of 0·000283	6·451786
Log $r =$ log of 20922642 feet	7·320616
Ar. Co. log 2	9·698370
$l = 26100$, Ar. Co. log l	5·583360
Log $\frac{ar}{2l} =$ const log	9·054732

where tables to Ivory's refractions are used in the calculation as far as log b extends. It frequently happens, however, that, from the height of the place of the observer, the barometer stands below 27 inches, when log $\frac{b}{\text{B}}$ cannot be taken from the auxiliary refraction tables. In that case, the log of $\frac{ar}{2\text{B}l}$ may be used as a constant, which as formerly becomes......7·57877.

This constant may be combined with the log of the factor $\left(0{\cdot}75 - \frac{f}{3b}\right)$ and inserted in a table to facilitate the calculation of n in a manner similar to that of the astronomical refractions.

It is true, the terrestrial refractions near the horizon are very variable and uncertain, and cannot generally be determined with all desirable accuracy. This fact has induced trigonometrical surveyors to determine the mean effect by reciprocal and simultaneous observations, and to apply this result, under slight modifications, in all ordinary circumstances. Would any practical astronomer employ the mean refractions at low altitudes without corrections from the state of the barometer and thermometer, even though he knows that these corrections are frequently imperfect? Why should the trigonometrical surveyor act in direct opposition to the astronomer? If this short notice should call the attention of mathematicians to the subject, so as to give a more perfect formula, my object will be gained, and I do not think a subject can be beneath their notice which has occupied the attention of such men as M. M. Biot, Puissant, and the very celebrated geometer M. Plana, of Turin.

ON TWO THEOREMS IN THE MODERN GEOMETRY.

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[Soon after the publication of the fourth number of the current volume, in which one of the theorems about to be considered was proposed, a friend informed me, by letter, that the property was due to Brianchon, and a solution had been given by Poncelet, by the method of "Poles and Polars." Of what those eminent mathematicians have done, in respect to that elegant theorem I know nothing,—not having seen the Journals in which their papers were published.]

(I.) If tangents be drawn from each point of a curve of the second order to any other curve of the same order; then will the chords of contact touch another curve of the second order.

Let
$$ay^2+bxy+cx^2+dy+ex+f=0 \ldots\ldots\ldots\ldots(1),$$
$$a_1y^2+b_1xy+c_1x^2+d_1y+e_1x+f_1=0 \ldots\ldots\ldots\ldots(2),$$
be curves of the second order; (1) being the auxiliary curve, or that to which the tangents are drawn, and (2) the curve from which the tangents are drawn.

For brevity of operation, let
$$\left.\begin{array}{ll} A=2cx+by+e, & A_1=2c_1x_1+b_1y_1+e_1 \\ B=2ay+bx+d, & B_1=2a_1y_1+b_1x_1+d_1 \\ C=dy+ex+2f, & C_1=d_1y_1+e_1x_1+2f_1 \end{array}\right\}\ldots\ldots(3).$$
Then if (x_1y_1) be any point in (2), the corresponding chord of contact in reference to (1), is
$$Ax_1+By_1+C=0 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$
Now to find the curve which is constantly touched by (4); differentiate (2, 4), after writing x_1y_1 for xy in (2); then
$$Adx_1+Bdy_1=0\ldots.(5), \qquad A_1dx_1+B_1dy_1=0\ldots.(6).$$
Multiply (5) by the arbitrary λ, add the product to (6), and equate to zero the coefficients of each differential; then
$$\lambda A+A_1=0\ldots\ldots..(7), \qquad \lambda B+B_1=0\ldots\ldots..(8).$$
Multiply (7) by x_1, (8) by y_1, adding and observing (3); therefore
$$\lambda(Ax_1+By_1)+A_1x_1+B_1y_1=0, \text{ or } \lambda C+C_1=0; \therefore \lambda=-\frac{C_1}{C}.$$
By substitution for λ in (7, 8), we get
$$AC_1-A_1C=0, \qquad BC_1-B_1C=0;$$
or restoring for A_1, B_1, C_1 from (3), and arranging with respect to x_1y_1,
$$(Ae_1-2Cc_1)x_1+(Ad_1-Cb_1)y_1+2Af_1-Ce_1=0,$$
$$(Be_1-Cb_1)x_1+(Bd_1-2Ca_1)y_1+2Bf_1-Cd_1=0;$$
which give
$$x_1=\frac{A(4a_1f_1-d^2_1)+B(d_1e_1-2b_1f_1)+C(b_1d_1-2a_1e_1)}{A(b_1d_1-2a_1e_1)+B(b_1e_1-2c_1d_1)+C(4a_1c_1-b^2_1)},$$
$$y_1=\frac{A(d_1e_1-2b_1f_1)+B(4c_1f_1-e^2_1)+C(b_1e_1-2c_1d_1)}{A(b_1d_1-2a_1e_1)+B(b_1e_1-2c_1d_1)+C(4a_1c_1-b^2_1)}.$$
These values of x_1y_1 substituted in (4), give
$$A^2(d^2_1-4a_1f_1)+B^2(e^2_1-4c_1f_1)+C^2(b^2_1-4a_1c_1)+2AB(2b_1f_1-d_1e_1)$$
$$+2BC(2c_1d_1-b_1e_1)+2CA(2a_1e_1-b_1d_1)=0\ldots\ldots(9).$$
Or restoring for A, B, C, we get
$$(d^2_1-4a_1f_1)(2cx+by+e)^2+(e^2_1-4c_1f_1)(2ay+bx+d)^2$$
$$+2(2b_1f_1-d_1e_1)(2cx+by+e)(2ay+bx+d)$$
$$+2(2c_1d_1-b_1e_1)(2ay+bx+d)(dy+ex+2f)$$
$$+2(2a_1e_1-b_1d_1)(2cx+by+e)(dy+ex+2f)$$
$$+(b^2_1-4a_1c_2)(dy+ex+2f)^2=0\ldots\ldots\ldots\ldots(10).$$

This is the equation to the *envelope*, and which is plainly of the second degree.

In *polar* phraseology, the curve (10) is termed the "polar curve" of (2) with regard to (1). Hence, since (x_1y_1) and (4) are conjugate polars, the proposition admits of the following enunciation :—

"If the poles in respect to one conic section move upon another conic section, the corresponding polars will all be tangents to a third conic section."—(*Davies's Hutton*, vol. ii., page 193.)

(2). Before proceeding to investigate the converse of the proposition, I shall give another demonstration, which is founded upon the following general principle :—

"The *line* represented by the equation $\mu^2\text{L} - 2\mu\text{R} + \text{M} = 0$, always touches the curve $\text{LM} = \text{R}^2$."—(*Salmon's Conic Sections*, p. 227.)

Since (x_1y_1) is any point in (2), we have

$$a_1y^2_1+b_1x_1y_1+c_1x^2_1+d_1y_1+e_1x_1+f_1=0 \ldots\ldots\ldots\ldots(a);$$

and from (4) we obtain the relation

$$\text{C}^2 = -\text{C}(\text{A}x_1+\text{B}y_1) = (\text{A}x_1+\text{B}y_1)^2.$$

Multiply the first three terms of (a) by C^2, the fourth and fifth by $-\text{C}(\text{A}x_1+\text{B}y_1)$, and f_1 by $(\text{A}x_1+\text{B}y_1)^2$: then

$$(\text{B}^2f_1-\text{BC}d_1+\text{C}^2a_1)y_1^2+(2\text{AB}f_1-\text{BC}e_1-\text{AC}d_1+\text{C}^2b_1)x_1y_1$$
$$+(\text{A}^2f_1-\text{AC}e_1+\text{C}^2c_1)x^2_1=0.$$

The envelope is therefore

$$(2\text{AB}f_1-\text{BC}e_1-\text{AC}d_1+\text{C}^2b_1)^2=4(\text{A}^2f_1-\text{AC}e_1+\text{C}^2c_1)(\text{B}^2f_1-\text{BC}d_1+\text{C}^2a_1);$$

which being multiplied out and arranged for A, B, C, gives (9) as in the preceding investigation.

(3). The converse of the proposition may be enunciated as follows :—

If a chord be drawn in a curve of the second order, and be a tangent to any other curve of the same order; then the tangents at the extremities of the chord intersect in another curve of the second order.*

Or in polar phraseology :—

"If the polars in respect to one conic section be tangents to a second, the corresponding poles will trace a third conic section."

Let (1) as before be the auxiliary, and

$$a'y^2+b'xy+c'x^2+d'y+e'x+f'=0 \ldots\ldots\ldots\ldots(11)$$

the curve to which the chords are tangents.

Now, if $(x'y')$ be any point from which tangents are drawn to (1), the corresponding chord, or polar of $(x'y')$ in respect to (1) is

$$\text{A}'x+\text{B}'y+\text{C}'=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where

$$\text{A}'=2cx'+by'+e,\quad \text{B}'=2ay'+bx'+d,\quad \text{C}'=dy'+ex'+2f.$$

Eliminating y between (11) and (12); then

$$(\text{A}'^2a'-\text{A}'\text{B}'b'+\text{B}'^2c')x^2+(2\text{A}'\text{C}'a'-\text{B}'\text{C}'b'-\text{A}'\text{B}'d'+\text{B}'^2e')x$$
$$+\text{C}'^2a'-\text{B}'\text{C}'d'+\text{B}'^2f'=0 \ldots\ldots\ldots(13).$$

* This enunciation, for the sake of uniformity, is slightly varied from that given by Brianchon. See Mr. Davies's "Historical Notices," p. 228 of the current volume.

But by the problem, (12) must be tangential to (11); therefore the roots of (13) must be equal: hence the condition

$$(2A'C'a'-B'C'b'-A'B'd'+B'^2e')^2=4(A'^2a'-A'B'b'+B'^2c')$$
$$(B'^2f'-B'C'd'+C'^2a'),$$

which gives, by arranging for A', B', C',

$$A'^2(d'^2-4a'f')+B'^2(e'^2-4c'f')+C'^2(b'^2-4a'c')+2A'B'(2b'f'-d'e')$$
$$+2B'C'(2c'd'-b'e')+2C'A'(2a'e'-b'd')=0.$$

Or, by restoring for A', B', C',

$$(d'^2-4a'f')(2cx'+by'+e)^2+(e'^2-4c'f')(2ay'+bx'+d)^2$$
$$+2(2b'f'-d'e')(2cx'+by'+e)(2ay'+bx'+d)$$
$$+2(2c'd'-b'e')(2ay'+bx'+d)(dy'+ex'+2f)$$
$$+2(2a'e'-b'd')(2cx'+by'+e)(dy'+ex'+2f)$$
$$+(b'^2-4a'c')(dy'+ex'+2f)^2=0 \ldots\ldots\ldots\ldots (14).$$

This condition, which must be satisfied by the point $(x'y')$, is the equation of its *locus*, and is plainly of the second degree; hence the converse of the problem is established.

(4). It may now be shown that the polar curve (14) just determined, is *identical* with the given curve (2). In order to which, let the auxiliary (1), for brevity, be a central curve, and its axes be taken as axes of coordinates; then since $b=0$, $d=0$, $e=0$; (10) reduces to

$$a^2(e^2_1-4c_1f_1)y^2+2ac(2b_1f_1-d_1e_1)xy+c^2(d^2_1-4a_1f_1)x^2$$
$$+2af(2c_1d_1-b_1e_1)y+2cf(2a_1e_1-b_1d_1)x+f^2(b^2_1-4a_1c_1)=0 \ldots (15);$$

and (14) becomes, writing xy for $x'y'$,

$$a^2(e'^2-4c'f')y^2+2ac(2b'f'-d'e')xy+c^2(d'^2-4a'f')x^2$$
$$+2af(2c'd'-b'e')y+2cf(2a'e'-b'd')x+f^2(b'^2-4a'c')=0 \ldots (16).$$

Now since (11) denoted the curve to which the chords were tangents, it also represents (15); hence by comparison of the coefficients, we have

$$a'=a^2(e^2_1-4c_1f_1), \qquad d'=2af(2c_1d_1-b_1e_1),$$
$$b'=2ac(2b_1f_1-d_1e_1), \qquad e'=2cf(2a_1e_1-b_1d_1),$$
$$c'=c^2(d^2_1-4a_1f_1), \qquad f'=f^2(b^2_1-4a_1c_1).$$

Substituting these values of a', b', *etc.* in (16); arranging and dividing each term by

$$16a^2c^2f^2\{b_1(b_1f_1-d_1e_1)+c_1(d^2_1-4a_1f_1)+a_1e^2_1\},$$

we obtain finally the equation

$$a_1y^2+b_1xy+c_1x^2+d_1y+e_1x+f_1=0,$$

which was the object of inquiry.

(5). The curve (14) may be generated as follows:—draw any tangent to (11), that is (10), and take its pole with regard to (1), the *locus* of this pole will be (14), which is (2); and since any tangent to the one curve is the polar of a point on the other, the point of contact of the tangent to the former will be the pole to the tangent through the point on the other. Hence, it follows, that the one curve (10) may be generated from the other (2), in precisely the same manner as the latter would be generated from the

former. The relation between the curves being *reciprocal*; hence the term "reciprocal polars." That is, (2) and (10) are reciprocal polars.

(6). Let the origin be fixed at the centre of the auxiliary (1); then $d = 0$, $e = 0$, and that equation is therefore

$$ay^2 + bxy + cx^2 + f = 0 \dots\dots\dots\dots\dots\dots (17),$$

and (10) reduces to

$$\begin{aligned}&\{b^2(d_1^2-4a_1f_1)+4a^2(e_1^2-4c_1f_1)+4ab(2b_1f_1-d_1e_1)\}y^2\\ &+2\{2bc(d_1^2-4a_1f_1)+2ab(e_1^2-4c_1f_1)+(b^2+4ac)(2b_1f_1-d_1e_1)\}xy\\ &\{4c^2(d_1^2-4a_1f_1)+b^2(e_1^2-4c_1f_1)+4bc(2b_1f_1-d_1e_1)\}x^2\\ &+4f\{2a(2c_1d_1-b_1e_1)+b(2a_1e_1-b_1d_1)\}y\\ &+4f\{b(2c_1d_1-b_1e_1)+2c(2a_1e_1-b_1d_1)\}x\\ &+4f^2(b_1^2-4a_1c_1)=0 \dots (18);\end{aligned}$$

an equation which denotes *an ellipse, hyperbola, or parabola, according as the origin is within, without, or on the reciprocal curve* (2); *and which passes through the origin when* (2) *is a parabola.*

I shall now add a few interesting properties.

I. Let the given curve (2) have a focus at the origin, that is at the centre of the auxiliary (17); then we have the conditions

$$e_1^2 - 4c_1f_1 = d_1^2 - 4a_1f_1, \text{ and } 2b_1f_1 - d_1e_1 = 0;$$

which reduce (18) to

$$(b^2 + 4a^2)y^2 + 4b(a + c)xy + (b^2 + 4c^2)x^2$$
$$+ \frac{4f}{d_1^2 - 4a_1f_1}\left\{(\lambda y + \lambda_1 x) + f(b_1^2 - 4a_1c_1)\right\} = 0,$$

where λ, λ_1 are written for the coefficients of y and x within the braces of (18).

This equation is obviously that of an *ellipse*, which passes through the origin when the condition $b_1^2 - 4a_1c_1 = 0$, is satisfied.

Hence, *when the given curve* (2) *has a focus at the origin, the reciprocal curve is an ellipse, which passes through the origin when* (2) *is a parabola.*

Cor. When the auxiliary is either a circle or an equilateral hyperbola, if its centre coincide with a focus of the given curve, the reciprocal curve is a circle.

This follows by adding the condition $c = \pm a$.

II. Next let the given curve (2) have contact with either of the axes of coordinates; then by taking the condition $e_1^2 - 4c_1f_1 = 0$, or $d_1^2 - 4a_1f_1 = 0$, according as (2) has contact with the axis of x or of y, the equation resulting from (18) readily indicates *an hyperbola* as the reciprocal curve, and passing through the origin when (2) is a parabola; but as neither y^2 nor x^2 is absent, the disposition of its asymptotes is not so obvious. If, however, the auxiliary (17) be referred to a pair of conjugate diameters (which gives $b = 0$); then *the reciprocal will have an asymptote parallel to the diameter, conjugate to that which touches the curve* (2).

Cor. When (2) touches both the axes of coordinates, equation (18) becomes simply

$$(2b_1f_1 - d_1e_1)xy + \frac{f}{c}(2c_1d_1 - b_1e_1)y + \frac{f}{a}(2a_1e_1 - b_1d_1)x$$
$$+ \frac{f^2}{2ac}(b^2_1 - 4a_1c_1) = 0,$$

which represents an hyperbola, *having its asymptotes parallel to the axes of coordinates.*

III. Now if the given curve (2) pass through the origin; then $f_1 = 0$, and the equation to the polar reciprocal (18) may be written

$$\{(bd_1 - 2ae_1)y + (2cd_1 - be_1)x\}^2 + 4f\{(\lambda y + \lambda_1 x) + f(b^2_1 - 4a_1c_1)\} = 0;$$

which indicates a parabola, — the leading terms being a perfect square.

Hence, *when one of the curves is a parabola, the reciprocal curve passes through the origin, and the converse.*

Cor. Let the auxiliary (1) be the circle $x^2+y^2=r^2$, and let the axes of x and y be a tangent and normal, or a normal and tangent to the curve (2); then with $f_1 = 0$, we have the condition $e_1 = 0$, or $d_1 = 0$, and the equation reduces to

$$(d_1x + b_1r^2)^2 = 4(d_1y + a_1r^2)c_1r^2,$$

or

$$(e_1y + b_1r^2)^2 = 4(e_1x + c_1r^2)a_1r^2;$$

each of which denotes a parabola, *having its axis perpendicular to the tangent applied to the reciprocal curve* (2) *at the centre of the auxiliary circle.*

IV. Let (1) and (2) be the concentric curves

$$ay^2 + cx^2 + f = 0, \qquad a_1y^2 + c_1x^2 + f_1 = 0,$$

respectively; then the polar reciprocal is the ellipse

$$\frac{a^2}{a_1}y^2 + \frac{c^2}{c_1}x^2 + \frac{f^2}{f_1} = 0,$$

the disposition and relation of the axes being obvious.

Cor. When we have $x^2 + y^2 = r^2$, as the auxiliary, and $x^2 + y^2 = r_1^2$, as the given curve; then the reciprocal is also a circle, whose equation is

$$x^2 + y^2 = \frac{r^4}{r_1^2};$$

which needs no comment.

There are many other interesting properties, one or two of which I shall enumerate.

(*i.*) *The reciprocal of one circle with regard to another, is a conic section, of which the origin is the focus, and the polar of the centre is the directrix.*

(*ii.*) *If* (2) *be the straight line* $d_1y + e_1x + f = 0$, *and any point* (x_1y_1) *move along this line, its polar always passes through a fixed point, namely, the pole of the line.*

The deduction of these is sufficiently obvious from what has been given above.

HISTORICAL NOTICES RESPECTING AN ANCIENT PROBLEM.

[*Mr. Davies.*]

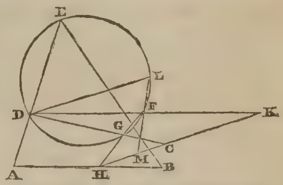
(Concluded from page 233.)

SIMSON, 1731.

SIMSON'S solution has been already spoken of, and it is here given at length, as (at least as far as can be ascertained) it is the earliest successful attempt in modern times to generalize the lemma of Pappus (vii. 117).*

" *Three points* A, B, C, *and a circle* DEF *being given in position; from* A, B, *two of the given points, to draw to the circumference two lines* AE, EB *meeting the circle in* D, G, *so that* D, G, C *shall be in a straight line.*

" ANALYSIS.—Suppose it done. Join AB and draw DF parallel to AB and meeting the circle in F. Join FG and produce it to meet AB in H. Since therefore by the parallels, the angle BHF is equal to the angle DFG, that is, by the circle, to the angle DEG, the triangles AEB, BGH are similar; and therefore the rectangle AB, BH will be equal to the given rectangle EB, BH (Data 92). Also BH is given, and therefore also BH and the point H. Whence, again HC being joined, it is given in position. Let HC,



DF produced meet in K; then the angle DKH will be equal to the angle BHK. Wherefore we have arrived at this condition:—that from the two given points C and H, CG and HG must be drawn to the circumference, so that DF may make a given angle DKH with HC.

" Suppose it done, and let DL be drawn parallel to HC; join LF, and produce it to meet CH in M. Then the point M will be given as before shewn concerning the point H; [namely, since the angle CMF is equal to the angle FLD, that is, to the angle FGC; therefore the angles HMF, HGC are equal, and the triangles HMF, HGC are similar; wherefore also the rectangle CH, HM is equal to the given rectangle FH, HG; and CH being given; therefore also HM and the point M,] and because the angle DKH is given, the angle KDL will be also given; and therefore the straight LF will be given in magnitude (Data, 88.) Since therefore from the given point M to the circle given in position MFL is drawn, making the intercepted part of a given length, ML will be given in position, and therefore also LD, AD and BE.

" COMPOSITION.—Make the rectangle AB, BH equal to the rectangle contained by the segments of any straight line whatever drawn from B to cut the circle, and join HC. Make the rectangle CH, HM equal to the rectangle contained by the segments of any straight line whatever from the point H to the circle. From M draw a line MFL to cut off a segment LEDF containing an angle equal to CHB. Join HF meeting the circumference in G; join BG meeting the circumference in E, and EA meeting it in E; then the points D, G, C will be in a straight line.

" DEMONSTRATION.—Join DF, DL, DG, CG. Then since *(constr.)* the rectangle AB, BH is equal to the rectangle EB, BG, the angle BHG will be equal to the angle E, that is, to the angle DFG; and therefore AB and DF are parallel. Therefore the angle DKH is equal to the angle BHK; that is, by construction, to the angle LDF. Whence DL is

* This solution is taken from Trail's Life of Simson, p. 97;—a work which though not "very scarce," is yet far from being common.

parallel to HK. Also because the rectangle CH, HM is equal to the rectangle FH, HG, the points C, M, G, F are in the circumference of a circle; wherefore the angle CGF is equal to the angle CMF; that is, to the angle DLF; and on account of the circle, the angles DLF and DGF are together equal to two right angles; wherefore the angles CGF, DGF are together equal to two right angles, and therefore the line DGC is a straight line.

"Transcribed Aug. 30th, 1731."

At the close of the solution, Dr. Trail adds the following notice:—"Dr. Simson at the same time gave two variations of the problem; viz., the same things being given, to make the line DG parallel to a given line; and also the same things being supposed, to inflect AE, BE so that DG may make a given angle with the line from D to the third given point, equal to a given angle."

The latter variation so naturally leads to the discovery of the indeterminate case which constitutes props. 55-7 of the posthumous treatise on Porisms, that I felt anxious to consult Simson's solution. I therefore applied to Professor W. Thompson of Glasgow (the Editor of the Cambridge Mathematical Journal) who kindly procured for me a transcript of those and some other of Dr. Simson's solutions. These solutions are interesting, and but for want of room would have been given here; but they do not bear out the anticipations I had formed of some allusion to the porismatic case *necessarily* appearing in them. In fact, there is not the least allusion, either direct or implied, to the indeterminate case. Dr. Simson either had not then seen the porism connected with the problem, or he transferred his remarks and investigations of the entire class of propositions to a separate set of MSS. I am not, however, aware whether there be so much as a single MS. volume, or even a note amongst his papers respecting the great discovery of his life; and as the Treatise was left by him in the state in which he intended to print it, (see the letters quoted in the article "Geometry and Geometers," Phil. Mag., vols. 31, 32) we may infer in the absence of evidence to the contrary, with some probability, that Dr. Simson had himself destroyed all the papers which related to the subject. Whatever may be the fact, I am in no position to do more than offer a conjecture—which however I should be glad to see affirmed or disproved by an appeal to the documents themselves.

Of one thing I am certain:—that any geometer who contemplates an attempt to give a better edition of the seventh book of Pappus will do injustice alike to himself and to his subject, if he neglect to make careful and repeated examinations of that interesting mass of papers. He would also do a gross injustice to the illustrious author if he were to omit a full statement of the suggestions which had been derived from them.

I had thought of giving close translations of Simson's *Porisms* having a more or less direct bearing on the problem, and of those theorems amongst Stewart's *Propositiones Geometricæ* which bore upon the same subject: but as I am anxious to close this series of papers, and as space cannot be allowed me in the present number of the *Mathematician* to accomplish this object, I am compelled to forego that intention. On the whole this may be less regretted, as I am not without the hope that Mr. Potts's translation of the entire series, with valuable explanatory notes and illustrations, will not be long delayed. Simson's Analysis and Composition, too, of these particular porisms are very (and as it turns out from Noble's investigation, unnecessarily) complicated. Simson's investigation is, undoubtedly, "in the

manner of the ancients:" but Noble's is, also, in strict keeping with the *spirit of the ancient geometry*.

Simson's enunciations of the *Porisms* are often couched nearly in the words of Commandine's translation of the original Greek of Pappus into Latin;* and the very ambiguity which rendered the Porisms a mere paradox is still retained by Simson,—though for good and obvious reasons. This difficulty must be removed by fuller expressions in some shape or other in any *useful* translations of the work; and perhaps on this account, it is less to be regretted that Lawson did not complete his translation than it otherwise might have been. For the want of this, we are not unfrequently compelled to read through the entire analysis of one of Simson's Porisms, with hesitation as to its meaning, and uncertainty as to its force of argument, before we can get a clear view of the proposition itself. Nay, not only the analysis, but the composition and demonstration are necessary in certain cases. No wonder, then, that the "Porisms" have been so little studied!

The geometers of the Scottish metropolis, by the attention which they paid to this remarkable work, prevented the discovery of Simson remaining, perhaps to the present hour, what the original porism had so long been—an enigma. Robison, Dugald Stewart, Playfair, Wallace, and Brougham, all entered upon the question with an avidity which apart from the intrinsic merit of the discovery, was quickened by that feeling of nationality which is so peculiarly characteristic of the different clans of the Celtic race—whether insular or continental.

Playfair's explication of the origin and the fundamental principles of the porismatic system was published in the second volume of the Edinburgh Transactions, about 1792. This it was which rendered the system generally intelligible; and it may be fairly questioned, whether before or since the publication of that dissertation, any discussion of the philosophy of a geometrical system, superior in all the essentials to such a discussion, had been laid before the world. This was followed by Wallace's elegant set of *porisms*, together with specimens of their applications to the solution of problems, in the fourth volume of the same Transactions. Lord Brougham, then very young, had a paper on the Porism applied to other loci than the line and circle, in the Philosophical Transactions for 1796: but as its aim was an extension, rather than an explication or illustration of the principle, further reference to it would be out of place here. Robison and Stewart seem to have merged their researches in Playfair's Dissertation.

In an incidental notice like this, it would be impossible to detail, without being irrelevant, an account of the labours of geometers of the present century; and except in the case where that porism has a direct connection with the problem which is the subject of these "Notices," further reference to them will be avoided.

In 1798 or 1799 Mr. Lowry proposed the problem of incribing the tri-

* Simson's enunciation of the principal proposition of the seven which demands our notice are but the subject continued through the two preceding neat propositions under different modifications, is as follows:—

"Quando scilicet recta quæ jungit intersectiones inflexarum et circumferentæ datum continent angulum cum recta quæ ad datum punctum vergit."—Op. Reliq., p. 472.

Wallace's enunciation is, however, compact and very clear:—

"If from two given points A, B, any two straight lines AD, BD be inflected in the circumference of a circle given by position, meeting it again in E and F; the straight line which joins EF shall either contain a given angle with a straight line passing through a given point, or shall be parallel to a line given by position, or shall pass through a given point."—Leyb. Repos., O. S., vol. iii.

angle in the circle, in the old series of *Leybourn's Repository*, vol. ii. This is the earliest printed notice of the problem that I have met with in English books; and I have reason to think from various circumstances that it was *actually* the first. Solutions by himself and Mr. Wallace appeared at pp. 189—96 of the volume just referred to. Mr. Lowry, whose investigations were generally so remarkable for their arrangement, compactness, and completion, scarcely maintains his general character in the details of this solution.* The construction to which his analysis leads him is essentially the same as that of Castillon; whilst his analysis itself, like that of his continental predecessor, is the analysis of it as a determinate problem. He takes, in this solution, no notice of separate cases of the problem; nor is his analysis at all comparable for simplicity with Castillon's. †

Mr. Wallace states that his solution was deduced by considering Simson's corresponding porism. He begins by the solution of three special cases:

1. The points in a straight line.
2. Two of the points in the diameter of the given circle.
3. When two points are in the diameter of the circle and form the points of harmonical division of that diameter.

In the analysis of the problem, Simson's analysis of the porism is as closely followed as the two cases would permit; the same lines are drawn, and the same conclusions, as far as they can be stated in common, are arrived at. The resulting construction is the same as that of Castillon and Lowry.

Mr. Wallace then proceeds to notice one indeterminate case for the triangle, (the earliest notice I have found, and the only one in print); viz. that which is indicated by *prop.* 34, *lib. i.*, *Stewart*, *Propositiones Geometricæ*. This indeterminate case is, however, far more general than for the triangle inscribed in the circle, as will be presently shewn. I will therefore only give here, Mr. Gaskin's view of the triangle inscribed in the conic section, from a MS. with which he has entrusted me.

Let the three points P, Q, R *form a " conjugate triad" (that is such that* P *is the polar of the line* QR, Q *of the line* RP, *and consequently* R *of* PQ): *then if any triangle inscribed in the conic section, two of whose sides pass through* P, Q, *the third side will pass through* R.

The demonstration follows so readily from the familiar properties of reciprocal polars, as to render its detail here altogether unnecessary.

* It is, too, vitiated by several errata, which the reader must correct as he proceeds.

† In an anonymous letter signed from " Clydesdale," this remark is made :—

" Mr. Wallace states distinctly that his [solution] is borrowed from Simson's Porism: and although Mr. Lowry does not even hint at this, it also is *unquestionably adopted from Simson.*"

This remark is both ungenerous and unjust. Had the writer turned to *vol. ii.* p. 57 the matter would have been cleared up to his mind—if indeed, truth rather than partisanship animated his inquiry. Speaking of a " beautiful porism" which he had deduced in the course of his investigation of a collateral problem, he adds—" it was from this consideration that I was first led to the discovery of the beautiful porism exhibited in the above analysis, and at that time I was not aware that it had been thought of by any one. *Since then* [and the question itself which gave rise to this discovery was proposed a year at least subsequent to the solution in which Mr. Lowry is charged with plagiarism] *a friend has favored me with a copy of the Opera Reliqua of Dr. Robert Simson,* where I find a complete investigation is given of this porism." He then gives, after Simson, the porism itself.

To those who are familiar with Mr. Lowry's writings, and still more to those who have lived in habits of friendship with him, no defence against such an unjust imputation is necessary; and it is only for those who infer the entire conclusion from a single fact, the explanation of which they are either too prejudiced or too idle to search, that I have added this note.

Although I have bestown adequate care in the examination of those books to which I could obtain access, that were published about this period, I still feel a suspicion of something having escaped my search. I can, however, only describe such solutions as I have been able to find ; and must leave it to others, or to a future time, to fill up the chasm, if such shall be found to exist.

The next step in this research, then, with which I am acquainted, is the proposition at the end of the volume of the Repository already quoted, given by Mr. Lowry :—

" *In a circle given in position and magnitude, it is required to inscribe a trapezium such that three of its sides shall pass through three given points in the same straight line, and that the fourth shall touch a circle given in magnitude and position.*"—*Rep. ii.*, *p.* 465.

The construction given by Mr. Lowry is remarkably simple and direct ; and the appeal for the proof is to Stewart's *Prop. Geom.* No notice however is taken of special cases : more particularly of that, when the circle is reduced to a point situated in the same line with the other three.

NOBLE, 1806.

Reference has already been made to the paper published by the Rev. Mark Noble, in the first volume of the new series of the Mathematical Repository. The same gentleman also published some researches on porisms in the Mathematical Companion : but as none of them refer to this problem, further notice of them here would be irrelevant. His third porism in the Repository is thus enunciated :—

" *If there be any number of points and a circle given by position ; putting n for the given number, if a polygon of* $n+1$ *sides be inscribed in the circle so that a side may pass through each of the given points, the remaining side shall contain a given angle at the circumference with a straight line tending to a given point.*"—*Math. Repos.*, *N. S.*, *i.* 36.

It will be readily perceived in what the analysis of this porism is so greatly superior to that given by Simson : viz. that a variable chord of the circle through a determinate point (dependent on the variable position of the point in which the lines from the given points meet) is fixed in magnitude but unlimited in position—and therefore that that determinate point is the centre of the circle. Of course I speak here of n being 2 : and this case is in fact the only one of which he has offered an investigation, though he has enunciated the porism generally. There is nothing in the structure of the analysis itself that suggests the investigation to be adapted to the higher values of n. We have no right therefore to infer that the generalised statement was with this geometer so much a matter of demonstration as of probable conjecture. Yet his paper is one of great merit, and worthy of careful study.

LOWRY, 1806.

In the *Repository*, *vol. ii.*, *N. S.*, Mr. Lowry proposed as follows :—

" *In a given ellipse to inscribe a polygon of a given number of sides, so that each side may pass through a given point.*

There cannot be much doubt that the construction given by Mr. Lowry in the following year was suggested from a consideration that the ellipse was

a projection of a circle made by parallel projectors—or conversely that the orthographic projection of the ellipse might be made a circle by the selection of a suitable plane of projection. Nor is he solitary in this mode of viewing the question. This leads him to consider the problem in reference to the circle as the basis of his solution.

A passage, however, in his introduction (p. 189) to the actual solution of the problem has much perplexed me. It is: " To inscribe a polygon in a circle so that its sides shall pass through given points, is *a problem that has often been resolved*; and the solution that I shall give here is not materially different from others that have appeared before, but I shall endeavour to put it in rather a simpler point of view."

It is from the words that I have marked in italics that my difficulty arises: for I have found no solutions of the problem in any English work antecedent to this period. I can only therefore infer that such solutions do exist, or that he refers to those of Ottajano and Malfatti. The method is not so extremely unlike those of the Italian geometers, as to render it impossible to have been founded upon them; whilst if such solutions were to be found in any English work, it is not probable that the question so slightly modified would have been made the " Prize Question" of the year, or that no one else of the able geometers who then corresponded with the Editor of the " Repository" should have left the problem untouched. Yet such was the case; and my view inclines towards the supposition that Mr. Lowry had the Italian solution in his mind.

A few particular cases were discussed, as it were incidentally, in different places during this period: but as they led to no remarkable result, they need not be enumerated. There was indeed *proposed* in the *Mathematical Companion* for 1810, the corresponding *spherical* problem, but with erroneous data. No solution to it, that I am aware of, was offered, even with corrected data. It is the same as that given in the *Mathematician*, (Exercise 151) after Lhuilier.

Two other theorems intimately connected with this problem, also appeared in the " Companion" for the years 1812 and 1813, and were answered each in the following year. They were evidently considered difficult, as they were the "*prize questions*" of the respective next years. Both are signed " Amicus," one of the familiarly-known signatures of Mr. Lowry in that work. They are:—

If from any point in a conic section two straight lines be drawn parallel to two straight lines given by position meeting the curve again in two other points; what is the nature of the curve to which the line joining these points is a tangent?"

" *If a polygon of any number of sides be inscribed in a given circle, so that each of the sides except one may pass through a given point, that side will always touch a given conic section.*"

The former of these has been generalised and demonstrated geometrically in *vol. vi. Math Repos.*, *pp.* 42—7; and I have transferred my own demonstration from that place into the 12th ed. of *Hutton*, *vol. ii*, *p.* 183. The investigations given in the " Companion" are algebraical.

The second is certainly a partial anticipation of the general theorem of Poncelet; and by the adoption of the projective principle, the conic section generally might be substituted for the circle in the enunciation. The demonstrations given by Messrs. Lowry and Gompertz are exceedingly elegant and

strictly geometrical. Neither of these geometers, however, seems to have had the least idea of the truth, that the conic section to which the last side of the polygon was tangential, had itself two points of contact with the original circle; nor, of course, the mode of finding those points of contact, nor of employing them in the construction of the inscription problem, in the way that has since been done by Poncelet and Gaskin.

SWALE, 1824.

The next geometer who discussed this problem was the late Mr. J. H. Swale, of Liverpool, in his "Appollonius:"—a periodical work of which only two numbers were published (1823 and 1824), the latter containing this discussion. A tolerably familiar acquaintance with the works on geometry which have been published in this country during the past and present century, has led to my conviction that Mr. Swale was the most independent and original geometer of his time. Every thing he wrote was more free from the impress of the writings of others (even where the subjects may be the same, and so elementary too that we should hardly suppose it possible for two different methods to be proposed) than I find to be the case with any other geometer, except Dr. Matthew Stewart. Swale's Geometry, in fact, was his own invention; and I have been informed by a mutual friend of his and mine, that he was singularly unacquainted with the writings of other geometers, even of the English school. His "Geometrical Amusements" is, undoubtedly, one of the most original and remarkable books on geometry that has appeared since the time of Stewart and Simson. In this work he reduces by analysis, all the great problems of antiquity to one fundamental problem,—with the construction of which he opens the volume. Unfortunately the work from being of provincial origin, and only a small number having been printed, is become excessively scarce. Its date is 1821.

Mr. Swale gives four propositions as theorems, and adds the composition of the problem for three, four, five, and six given points; adding the special case for the triangle when the given points are in one line, and a special construction for the quadrilateral, of great elegance. The four propositions are:—

1. *If, in a circle, there be drawn* any even number *of chords* AB, BC, CD, *etc., respectively parallel to as many right lines any how given in position: the right line* AG, *drawn through the termination of the* last *chord* FG, *will be a tangent to a circle given in position and magnitude.*

2. *If, in a circle, there be drawn* any odd number *of chords* AB, BC, CD, *etc., respectively parallel to as many right lines any how given in position: the right line* AH, *drawn through the termination of the* last *chord thus drawn, will be parallel to a right line given in position.*

3. *If a polygon of* an odd number *of sides be inscribed in a circle, so that each of its sides shall pass through a given point: a right line may be drawn through* any *angular point of the polygon that shall pass through a* given *point, and touch a circle given in magnitude and position.*

4. *If a polygon of* an even number *of sides be inscribed in a circle so that each of its sides shall pass through a given point: a right line may*

be drawn through any *angular point of the polygon that shall be parallel to a right line given in position.*

The first and second of these are merely subsidiary to the other two, and sufficiently easy of proof by the method of " cases of ease :" but Mr. Swale takes the general case at once, and treats it with his usual adroitness and elegance.

The fourth has some degree of resemblance to the analysis of Ottajano and Malfatti ; but not a close one when details are carefully examined.

The third has scarcely any similarity, in its essential character, to any thing which preceded it.

It would be incompatible with the plan which I have been compelled to adopt in this *final* list of " notices," to give further account of Mr. Swale's labours ; and I can only refer my readers to the " Apollonius" itself for additional information.

As we have arrived at the conclusion of this class of investigations for the circle, it may not be improper to express the surprise I have felt that the very obvious mode of constructing the points F, G, *etc.*, by means of circles through ABT, BCQ, *etc.*, should never appear to have been thought of by any writer ; although some very interesting properties, besides those which were the immediate object of research, would have resulted. Want of room, and the complexity of the resulting figure prevent my giving a few of those which I have observed, in this place. They are not difficult—and may form a good exercise for geometrical students ; and some slight indications of the method to be pursued may be seen at p. 145 of the present volume of this work.

HEARN, 1846.

I have often wondered, since looking into this history, that though the problem attracted so much of the attention of the continental geometers, many of whom were eminently skilled in the use of the Cartesian system, no one seems to have made any attempt (at least so far successful as to induce publication) to apply the coordinate method to either one or other of these dual problems. The earliest attempt of this kind that I have been able to discover, was made by Mr. Hearn, of the Royal Military College, in his valuable little work entitled " *Researches on Curves of the Second Order*," about the middle of 1846. He was led to consider the subject in consequence of the slight notice of Castillon's and Lagrange's solutions given by Montucla in his " *Histoire des Mathématiques* ;" and as he had not the opportunity of consulting either of these solutions, " nor indeed any other," he was induced (from Montucla's description of the great difficulty of the problem) to undertake it in his own way. In this he has been eminently successful, as far as Castillon's form of the problem is concerned ; and Mr. Hearn does not seem to have contemplated its extension to polygons in general inscribed in conic sections or circumscribed about them.

Mr. Hearn's solution may be considered as purely coordinate. He commences by quoting a property of the related pole and polar, which he does not prove—deeming it, of course, unnecessary ; since that, and kindred properties, had been discussed by the coordinate method by Mr. Fenwick so copiously in the " *Mathematician*." He thereby reduces the problem of the *inscription* to its dual, of *circumscription* ; thus taking the same view of it that Gergonne has been described to have done before.

He then obtains equations for the circumscription of a triangle about the given circle whose angular points shall lie upon the polars of the three given points, and thus solves the problem.

As I presume Mr. Hearn's work is in the hands of most of my readers (at least it ought to be in the hands of all) it will not be necessary for me to give further details, but simply to refer to pp. 15—21 of the "*Researches.*"

GASKIN, 1847-8.

It would appear that Mr. Gaskin was led to turn his attention to this problem, partly from what Mr. Hearn had done, and partly because it formed one of the "Problems" in the paper of 1833, given in St. John's College, solutions to which Mr. Gaskin contemplated printing as models for the undergraduates of Cambridge University, in the formation of such solutions.

Mr. Gaskin's first solution ("*Solutions of Geometrical Problems,*" *App.* 1, *p.* 166) is wholly by the coordinate method, and manifests the most perfect mastery over his instrument of research. It is altogether different in principle from Mr. Hearn's, and instead of the circle, he takes the ellipse, hyperbola, and parabola successively as the curve in which the triangle is to be inscribed. If a conjecture may be indulged as to Mr. Gaskin's career in this inquiry, I should be led to think that *projective considerations* first led him to the solution for the ellipse from equations previously obtained (but not given) for the circle; and that *the forms of the equations* in this case suggested a trial of its extension to the hyperbola and parabola. I have, however, none but the evidence afforded by the solutions themselves for this inference :—nor is it very material.

Mr. Gaskin seeing that in general the problem admits of two solutions, was led to the happy thought of finding the equation of the line in which the corresponding vertices of the two solutional triangles were situated; and hence by constructing this line, its intersections with the curve give the two positions of the vertex, and hence the two triangles themselves. The same thing, however, was virtually done in Poncelet's construction, derived from considerations altogether different,—as has been already described.

The reproductive character of the equations which express the conditions, upon successively eliminating the unknown (which had been shewn also by Lhuilier and Carnot to take place in their solutions) naturally led to a similar inference in Mr. Gaskin's method; and shewed that *any polygon* whatever may be inscribed in *any conic section.* The calculations required are extremely laborious, but always of the most simple kind.

The solutions of Mr. Hearn for the triangle and circle, and Mr. Gaskin for the polygon and conic section, are, as far as I can discover, the only ones that have been given to the public, which are effected by the Cartesian method. By the use of the polar equation of the straight line and of the conic sections, I have myself succeeded in obtaining a general solution, which I may hereafter publish, but for which room cannot be spared here.

Mr. Gaskin has also given in the same appendix some other properties of the system of lines involved in this problem, which will be spoken of hereafter. Amongst them is the general theorem which arises out of Exercise 154, a *geometrical* proof of which he some time ago communicated to me privately, and which I very much regret my inability to lay before my readers in this place.

I have reason to believe that Mr. Gaskin was unacquainted with all that had been done towards the solution of the problem, except what is contained

in the "Researches" of Mr. Hearn; and hence what he published in his first appendix has all the merit of complete originality, notwithstanding that he had been anticipated as regards the idea of the extensions of the problem. His method of investigation is at least not only original to himself, but unanticipated by any geometer. The entire appendix (as well as the following one) will be an improving study for the young "analytical geometer."

Just upon finishing the printing of the work, Mr. Gaskin met with a stray number of *Swale's Apollonius*, containing the general solution already described. This suggested a new train of thought; and some (though not very important) additions were made in *Appendix iii.* to what was done in that already spoken of.

Mr. Gaskin's work was reviewed in the *Mechanics' Magazine*, in a very decided but still not unfriendly spirit. This led Mr. Gaskin to send a strictly geometrical investigation of the general problem to the Editor of that work. Circumstances accidentally caused me to become acquainted with that solution, and as I had been engaged in a similar attempt, I felt it to be due to Mr. Gaskin and to my own character, to cease from further prosecution of the research—especially as I found our views ran almost entirely in the same direction. A modification, therefore, of Mr. Gaskin's original solution (with one or two short notes of my own) was published in *vol.* 48, *p.* 396.

More than a year ago, Mr. Gaskin put into my hands a solution (founded on the same principles) of the general problem, and appended to this some investigations respecting the indeterminate cases—the case of Brianchon (MATH. p. 226) amongst them. Besides this he furnished me with a coordinate solution, and an investigation of the still more general problem, proposed by Ottajano. As, however, the original solution in the *Mechanics' Magazine* is accessible to all my readers, and I have no material (and only a few formal) alterations to suggest, I have not deemed it necessary to repeat that solution here; and hence content myself with a few remarks upon it.

I shall asume that the reader has before him the *Mechanics' Magazine*, No. 1289, for April 22, 1847.

It will be remarked then that Swale first *finds by construction*, F, G, *etc.*, and then *proves* that EK is parallel to PT, *etc.* Now the particular relation which gives F, G, *etc.*, in the circle does not hold good in the other conic sections; and it is by no means probable *à priori* that any relation whatever can subsist, the same through all the successive steps, which shall produce this parallelism in the conic sections. Nearly all the projective properties that we know are confined to contacts and intersections; at least when our data are very few, as in this case. As a general rule we have no reason to expect that the projections of lines shall have the same ratio as the segments of the lines themselves, with fewer divisions than are requisite for the harmonic or anharmonic ratio; nor for angles, except under the same conditions. Mr. Gaskin, therefore, to ascertain this, reversed the process, drawing his lines A_1B_1, B_1B_2, *etc.*, parallel to P_1P_2, Q_1P_3, *etc.*: and by means of the proposition which he had investigated by coordinates found that Q_1, Q_2, *etc.*, were *given points*. This changed the entire current of the subsequent investigation: and rendered his solution more correspondent to the Italian ones than to Swale's.

I had, however, objected to his use of the proposition quoted by him, except he gave a geometrical demonstration of it; and he furnished me with one rigidly so. It recurred to my memory, however, afterwards that Dr. Matthew Stewart had given the same property, amongst those which

were printed in the *second* edition of Simson's conic sections (1750, p. 164,) and that it had been noticed by M. Chasles as a very important one in connection with the doctrine of the involution of Desargues.*

The next difference is much more remarkable. Swale's is strictly an *analysis of the problem*, the required polygon being as usual supposed to exist, or to have been constructed. Gaskin investigates an inscribed property of n chords drawn through n given points, which is true whether the polygon "closes" or not. It is in fact a porism in everything but form; and an important one amongst the very few we possess respecting the conic sections. He then examines what takes place when the polygon is a closed one; or, in other words, in the figure which resolves the problem; and thence deduces his construction. It is certainly true that the analysis which he has applied generally might have been specially and solely applied to the hypothetically existing polygon which the problem requires: but it does not appear to me that the method of actual construction which he has given, could have been deduced, in so simple and direct a manner as that which forms his *prop. vi.*

Again, the doctrine of "reciprocal polars" has here been *directly* brought into play, for the first time, for the *inscription* of any figure beyond the triangle.

The remarkable construction of Poncelet (enunciated at p. 232) is also deduced with great simplicity from the analysis before us.

The last of his propositions is perhaps the most remarkable of all, inasmuch as it furnishes the means of finding the conditions amongst the points in order that the polygon may become indeterminate.

He has given me privately a neater version of this investigation,—which however must be reserved for another place.

The "Exercise 154" was *intentionally* proposed in the form of a *problem* for the purpose of leading the contributors to that department of the "*Mathematician,*" to a consideration of the peculiar circumstances which rendered an even-sided polygon indeterminate, and that when it was not indeterminate, it was altogether impossible. Several answers were sent by different correspondents; only a few of which indicated a full conception of the fact as regarded the quadrilateral, and no one that took the property in all its generality. This would, hence, be an appropriate place to give Mr. Gaskin's general investigation, did circumstances permit; especially as he adds some remarks of great interest upon certain ambiguous cases that arise, and which require great delicacy and discrimination in their discussion.

Were I not confident of Mr. Gaskin's general analytical solution of this problem, and of the general problem of Ottajano, being published at no distant date, I should deeply regret having been the unintentional cause of their being so long delayed.

* In fact this theorem is identical with the theorem of Desargues when applied to a quadrilateral inscribed in a conic section; and since the principle is general for all inscribed polygons of an even number of sides, we are at once led by it to the general conclusion respecting the indeterminateness or impossibility of the problem in Brianchon's case. Great caution is, however, necessary in the application of a general principle to special cases where particular relations subsist amongst the data. The expression may become illusory; the construction porismatic; the theorem indeterminate, doubly or triply; or the process itself be excluded, from the reasoning being inapplicable to the case under consideration. Such cases do exist in reference to this problem.

TOWNSEND, 1848.

Mr. Townsend (Trinity College, Dublin) has given a beautiful application of the principle of *anharmonic ratio*, by its application to investigate the envelope of the free side, and to establish the truth of Poncelet's construction. (See *the Rev. George Salmon's excellent work on the Conic Sections*, pp. 266-7). This is one proof more of the fecundity and power of the principle itself; and I gather from a letter from M. Chasles, that he has been enabled to so shape the idea, as to render it available for even *elementary geometry*. At any rate, the application of the principle to the demonstration of Poncelet's construction deserves our especial notice, from its being totally different from all others, and likewise as (in conjunction with Mr. Salmon's other applications of it) shewing how simply some of the most recondite properties of the conic sections may be established. Mr. Salmon's proof of Pascal's theorem at p. 268 is also interesting, not only for its brevity, but because it exhibits that connection between the anharmonic ratio and the hexagram, in a direct and immediate form, which Sir William Hamilton imagined (*Phil. Mag.*, vol. 29, p. 327) to be capable of exhibition only through the medium of his system of "Quaternions."* What, however, is more singular in connection with Sir William Hamilton's remark, is, that M. Chasles himself deduces by means of anharmonic ratios, the very proposition in question. He gives it in note 15, under the aspect that Maclaurin viewed it—which is one of the most elegant of the many forms under which Pascal's hexagram has been exhibited. Mr. Townsend, however, investigates it under the usual form. The claim advanced in favour of the peculiar virtues of the Quaternion system, in respect to this case at least, must have been the result of oversight, — and to such oversights every one is liable under the same circumstances.

My task is finished:—imperfectly enough, I am well aware. It has proved so much more difficult to execute than I had foreseen, that had I anticipated a tenth part of the labour it has cost me, I should have been deterred from undertaking it. Any want of method, or oversights of any kind, will be more readily excused by those who have had experience in such attempts, than perhaps by those who have never engaged in such undertakings. It has been throughout my most especial desire to appreciate and to represent the labours of the different geometers who have discussed this problem with strict impartiality. If I shall be found to have done justice to every one whose labours have been passed under review, my great object will have been attained; and all minor considerations will cease to be sources of the least anxiety to me.

* Of the courteous manner (strongly contrasted with the manner in which some other Mathematicians of the Dublin school who have referred to my writings, have been pleased to speak of them) in which Mr. Salmon has referred to a paper of mine in the "Mathematician," I feel duly sensible: but it is not on that account that I so strongly recommend his work to my younger geometrical readers:—it is because they will find treated very luminously and with adequate fulness for the purpose, most of the more recent theories and methods of continental geometers.

Of the question respecting the legitimacy of M. Chasles' demonstration, to which I had called attention, and on which Mr. Salmon disagrees with me, I cannot (for want of space) speak here: but I may hereafter give a note in the "Mathematician" on the subject.

Littleheath, Charlton, March 16*th*, 1849.

SOLUTIONS OF MATHEMATICAL EXERCISES.

CLXVIII. *Mr. John Walker, Trin. Coll., Dublin.*

If the line joining the centre of the inscribed circle with the vertex of any angle of a triangle be a mean proportional between the diameter of that circle and the radius of the circumscribed circle, prove that the three sides of the triangle are in arithmetical progression.

[First Solution. *By Function; and similarly by Mr. William Mortimer, and Mr. Thomas Wilkinson, Burnley.*]

The line AO joining the centre of the inscribed circle and the vertex of the triangle ABC is

$$\mathrm{AO} = \sqrt{\left\{\frac{bc(s-a)}{s}\right\}} = \sqrt{\left\{\frac{abc(s-a)}{as}\right\}} = \sqrt{\left\{\frac{4\mathrm{R}r(s-a)}{a}\right\}},$$

(since $4\mathrm{R}r = \dfrac{abc}{s}$, *Hind's Trig.* art. 80.) Then by the condition

$$2\mathrm{R}r = \mathrm{AO}^2 = \frac{4\mathrm{R}r(s-a)}{a}: \text{ therefore we have}$$

$$a = 2(s-a) = b + c - a, \text{ or } 2a = b + c;$$

therefore the sides b, a, c are in arithmetical progression.

Mr. Wilkinson deduced from his solution some neat properties.

[Second Solution. *Mr. Andrew Roy, Dundee Academy; and similarly by Mr. Stephen Watson, Castleside.*]

Let a, b, c be the sides, and A, B, C the opposite angular points of the given triangle, and G the centre of the inscribed circle; then if $a+b+c=2s$, we have

$$\text{the diameter of inscribed circle} = 2\left(\frac{(s-a)(s-b)(s-c)}{s}\right)^{\frac{1}{2}} \ldots\ldots(1),$$

$$\text{and radius of circumscribed circle} = \frac{abc}{4\{(s-a)(s-b)(s-c)\}^{\frac{1}{2}}} \ldots\ldots(2),$$

$$\mathrm{AG} = \sqrt{\frac{bc(s-a)}{s}},\ \ \mathrm{BG} = \sqrt{\frac{ac(s-b)}{s}},\ \text{ and } \mathrm{CG} = \sqrt{\frac{ab(s-c)}{s}}.$$

Now if the product of (1) and (2) $= \mathrm{AG}^2$, then will $2a = b + c$,

if .... $= \mathrm{BG}^2$, ... $2b = a + c$,

if .... $= \mathrm{CG}^2$, ... $2c = a + b$.

[Third Solution. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*]

Let a, b, c be the sides of the triangle ABC; O the centre of the inscribed circle; r its radius, and R that of the circumscribed circle: then by known properties of the "*Horæ Geometricæ*,"

$$\frac{\mathrm{AO}^2}{bc} = \frac{-a+b+c}{a+b+c},\quad \frac{2\mathrm{R}r}{bc} = \frac{a}{a+b+c}.$$

Now, by the exercise, $\mathrm{AO}^2 = 2\mathrm{R}r$; therefore $a = -a + b + c$, or $2a = b + c$.

Similarly, when $BO^2 = 2\,Rr$, $2b = a + c$; and when $CO^2 = 2\,Rr$, $2c = a + b$.

I shall add one or two propositions, which may perhaps possess interest; and for brevity, shall assume the sides of the triangle to be in arithmetical progression.

(*A*). Three times the radius of the inscribed circle is equal to the radius of the escribed circle, or that which touches externally the *side* which is a mean between the other two.

For generally,

$$a(r_1 + r) = (b + c)(r_1 - r),$$
$$b(r_2 + r) = (a + c)(r_2 - r),$$
$$c(r_3 + r) = (a + b)(r_3 - r);$$

and therefore, when $2a = b + c$, $r_1 = 3r$; and so for others.

(*B*). The perpendicular drawn from the angle to the opposite side is equal to the radius of the (opposite) escribed circle.

For let Δ denote the area of the triangle: then

$2\Delta = ap_1 = r_1(b+c-a)$; therefore when $2a = b+c$, $p_1 = r_1$, *etc.*

(*C*). The radii of the escribed circles are in harmonic progression.

For generally,

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}: \text{ hence } (A), \text{ when } 2a = b + c,$$

$$\frac{3}{r_1} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \text{ or } \frac{2}{r_1} = \frac{1}{r_2} + \frac{1}{r_3};$$

and similarly for others.

Note.—This proposition might have been established with the aid of (*B*) and the known property, $\dfrac{2}{p_1} = \dfrac{1}{r_2} + \dfrac{1}{r_3}$.

Messrs. Ashton, A. D. B., Bolton, Burns, P. Elliott, Gibb, Greenup, Johnson, Kincella, W. H. Levy, Sherret, Smyth, Walker, the proposer, and C. A. Weddle, sent good solutions.

CLXIX. *Mr. Robert Harley, Mathematical Master at Mr. Hoole's Academy, Blackburn.*

Find the centre of a given circle, by means of straight lines only.

[SOLUTION. *Mr. Patrick Kincella, Leighlin Bridge; and similarly by Dr. Burns, Rochester.*]

Let ABCD be the given circle, in which draw any chord AC; bisect it in P (Leslie's Geometry, Appendix, p. 201, prop. 2). Draw DBP at right angles to AC (prop. 4, ibid), meeting the circumference in D and B; bisect DB in O (prop. 2), and O will be the centre of the circle, and it has been determined by straight lines only. The demonstration is too obvious to require insertion.

Ingenious constructions were received from Messrs. W. S. Ashton, Darwen; T. Greenup, Newcastle; W. H. Gibb, Manchester; Joseph Harrap, Ashton-under-Lyne; Thomas Johnson, St. Helens; W. H. Levy, Shalbourne; A. Roy, Dundee; N. Smyth, Galway; C. A. Weddle, Training School, Durham; and Thomas Wilkinson, Burnley.

CLXX. *Mr. W. H. Levy, Shalbourne.*

Let ρ_1, ρ_2, ρ_3 be the radii of the three circles described in a plane triangle to touch the inscribed circle whose radius is r and two of the sides, and r_1, r_2, r_3 the radii of the escribed circles: then will

$$\frac{2\sqrt{r\rho_1}+2\sqrt{r\rho_2}+2\sqrt{r\rho_3}}{\sqrt{(r_1r_2+r_2r_3+r_1r_3)}}+\frac{\rho_1}{r_1}+\frac{\rho_2}{r_2}+\frac{\rho_3}{r_3}=1.$$

[First Solution. *Mr. Peter Elliott, Lamesley*].

Let s_1, s_2, s_3 be, as usual, the segments of the sides made by the points of contact of the inscribed circle, then, by similar triangles, we have

$$r : \sqrt{s_1^2+r^2} : : \rho_1 : \sqrt{s_1^2+r^2}-(r+\rho_1);$$

hence $(r-\rho_1)\sqrt{s_1^2+r^2}=r(r+\rho_1)$, and therefore

$$s_1^2=\frac{r^2(r+\rho_1)^2}{(r-\rho_1)^2}-r^2=\frac{4r^3\rho_1}{(r-\rho_1)^2},$$

which gives $$2\sqrt{r\rho_1}+\frac{\rho_1 s_1}{r}=s_1.$$

Similarly, $$2\sqrt{r\rho_2}+\frac{\rho_2 s_2}{r}=s_2, \text{ and } 2\sqrt{r\rho_3}+\frac{\rho_3 s_3}{r}=s_3.$$

Summing these three equations, and reducing by the properties $s_1+s_2+s_3=s$, and $\dfrac{s_1}{r}=\dfrac{s}{r_1}$, *etc.*, we get

$$2\sqrt{r\rho_1}+2\sqrt{r\rho_2}+2\sqrt{r\rho_3}+s\left(\frac{\rho_1}{r_1}+\frac{\rho_2}{r_2}+\frac{\rho_3}{r_3}\right)=s;$$

dividing by s, or its equivalent $\sqrt{r_1r_2+r_1r_3+r_2r_3}$, we get finally,

$$\frac{2\sqrt{r\rho_1}+2\sqrt{r\rho_2}+2\sqrt{r\rho_3}}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}+\frac{\rho_1}{r_1}+\frac{\rho_2}{r_2}+\frac{\rho_3}{r_3}=1.$$

[Second Solution. *Mr. J. W. Elliott, Stockton-on-Tees.*]

Let ABC be any plane triangle, O the centre of the inscribed circle, and P, R, Q the centres of the circles ρ_1, ρ_2, ρ_3 respectively: then

$$\text{AO}\sin\tfrac{1}{2}\text{A}=r,\quad \text{AP}\sin\tfrac{1}{2}\text{A}=\rho_1,\quad \text{AO}-\text{AP}=\text{OP}=r+\rho_1;$$

hence $\sin\tfrac{1}{2}\text{A}=\dfrac{r-\rho_1}{r+\rho_1}$; and therefore $\cos\tfrac{1}{2}\text{A}=\dfrac{2\sqrt{r\rho_1}}{r+\rho_1}$.

Whence, by division,

$$2\sqrt{r\rho_1}\tan\tfrac{1}{2}\text{A}+\rho_1=r \ldots\ldots\ldots\ldots (1).$$

Moreover, by the "*Horæ Geometricæ,*" $\tan\tfrac{1}{2}\text{A}=\dfrac{r_1}{s}$; therefore (1) by a trifling reduction, becomes

$$\frac{2\sqrt{r\rho_1}}{s}+\frac{\rho_1}{r_1}=\frac{r}{r_1} \ldots\ldots\ldots\ldots (2).$$

By a similar operation,

$$\frac{2\sqrt{r\rho_2}}{s}+\frac{\rho_2}{r_2}=\frac{r}{r_2},\quad \frac{2\sqrt{r\rho_3}}{s}+\frac{\rho_3}{r_3}=\frac{r}{r_3} \ldots\ldots (3, 4),$$

and by the addition of (2, 3, 4), observing that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}, \text{ and } \sqrt{r_1r_2+r_2r_3+r_3r_1}=s,$$

the proposition is established.

Cor. From (1) we have the relation

$$r=2\sqrt{r\rho_1}\tan\tfrac{1}{2}A+\rho_1=2\sqrt{r\rho_2}\tan\tfrac{1}{2}B+\rho_2=2\sqrt{r\rho_3}\tan\tfrac{1}{2}C+\rho_3.$$

Or (2, 3, 4),

$$\Delta=2r_1\sqrt{r\rho_1}+s\rho_1=2r_2\sqrt{r\rho_2}+s\rho_2=2r_3\sqrt{r\rho_3}+s\rho_3;$$

where Δ denotes the area of the triangle ABC.

Note.—There is an intimate relation between this elegant theorem and that proposed by Mr. Woolhouse as the "Prize Question," in the "*Ladies' Diary*," for 1839.

[THIRD SOLUTION. *Mr. W. Mortimer, Seedley, Manchester.*]

Since (Mathematician, p. 216) $\sin\frac{A}{2}=\frac{r-\rho_1}{r+\rho_1}$, we have $\cot\frac{A}{2}=\frac{2\sqrt{r\rho_1}}{r-\rho_1}$;

and in a similar manner $\cot\frac{B}{2}=\frac{2\sqrt{r\rho_2}}{r-\rho_2}$ and $\cot\frac{C}{2}=\frac{2\sqrt{r\rho_3}}{r-\rho_3}$;

$$\therefore\quad \frac{2\sqrt{r\rho_1}+2\sqrt{r\rho_2}+2\sqrt{r\rho_3}}{s}=\frac{r-\rho_1}{s\tan\frac{1}{2}A}+\frac{r-\rho_2}{s\tan\frac{1}{2}B}+\frac{r-\rho_3}{s\tan\frac{1}{2}C}\ldots\ldots(1).$$

But since $r_1=s\tan\frac{1}{2}A$, $r_2=s\tan\frac{1}{2}B$, $r_3=s\tan\frac{1}{2}C$; and $s=\sqrt{(r_1r_2+r_1r_3+r_2r_3)}$; therefore by transposing (1),

$$\frac{2\sqrt{r\rho_1}+2\sqrt{r\rho_2}+2\sqrt{r\rho_3}}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}+\frac{\rho_1}{r_1}+\frac{\rho_2}{r_2}+\frac{\rho_3}{r_3}=\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3}=1.$$

Excellent Solutions were received from Messrs. Ashton, Harrap, Johnson, Kincella, Levy, (the proposer), Smyth, and Roy. Dr. Burns, of Rochester, and A. D. B., of Hammersmith, sent good solutions.

CLXXI. *Lieut. A. R. Clarke, R. E.*

A semicircle is divided into two parts by an ordinate at right angles to the diameter. In these two spaces circles are described; required the locus of the middle point of the line joining their centres.

[SOLUTION. *Messrs. Peter Elliott, Lamesley; J. W. Elliott, Greatham; C. A. Weddle, Training School, Durham, and Function.*]

Take the diameter of the semicircle for the axis of x, and one of its extremities for the origin. Let $(m\,n)$ and (m_1n_1) be the coordinates of the centres of a pair of inscribed circles drawn as directed, and (xy) the corresponding point in the required locus; then we have the following conditions.

$$m_1-m=n+n_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$x=\tfrac{1}{2}(m_1+m)\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$y\left(=\frac{n+n_1}{2}\right)=\tfrac{1}{2}(m_1-m)\ldots\ldots\ldots\ldots(3),$$

$$(r-m)^2+n^2=(r-n)^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$(r-m_1)^2+n_1^2=(r-n_1)^2\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Eliminating n, n_1 by means of (1), (4), (5) we obtain

$$m_1 - m = \frac{2r(m_1 + m) - (m_1^2 + m^2)}{2r};$$

from which $$m_1 = \pm \sqrt{(4rm - m^2)} \ldots\ldots (6).$$

The half sum and half difference of (2) and (3) give respectively,

$$x + y = m_1, \text{ and } x - y = m.$$

Substituting these values of m_1 and m in (5), we get

$$x + y = \pm \sqrt{4r(x-y) - (x-y)^2}.$$

Squaring and reducing

$$x^2 + y^2 - 2r(x - y) = 0,$$

$$\text{or} \quad (x - r)^2 + (y + r)^2 = 2r^2,$$

the equation to a circle passing through the origin, whose radius $= r\sqrt{2}$, and the coordinates of its centre $(r, -r)$.

The solution of the proposer, Lieut. A. R. Clarke, R. E., is very elegant, and is omitted with regret.

Solutions were also received from Messrs. W. S. Ashton, Darwen; Thomas Greenup, Newcastle; W. Mortimer, Manchester; N. Smyth, Galway; W. Sherret, Cuminestone, and A. Roy, Dundee; Dr. Burns, Rochester, also sent a good solution.

CLXXII. *By M. Poncelet.*

If a hexagon have its non-contiguous sides meeting in one point, and the three others meeting in another point; then the three lines which join the pairs of opposite summits will also meet in one point.

[FIRST SOLUTION. *Mr. Peter Elliott, Lamesley.*]

Let L, M, N, O, P, Q be the six sides of the hexagon taken consecutively; then since L, N, P are convergent, their equations may be represented by

$$A = 0, \quad B = 0, \quad A - kB = 0.$$

And because M, O, Q are convergent, we may denote their equations by

$$\alpha = 0, \quad \beta = 0, \quad \alpha - l\beta = 0.$$

Also the equation of a line passing through the points ($\alpha = 0$, $B = 0$), and ($A - kB = 0$, $\beta = 0$), which are opposite summits, is

$$\alpha(A - kB) - B\beta = 0 \ldots\ldots (1);$$

for it is satisfied, either by putting $\alpha = 0$, $B = 0$, or $A - kB = 0$, $\beta = 0$. Again, the equation of the line passing through the points ($\alpha = 0$, $A = 0$) and ($A - kB = 0$, $\alpha - l\beta = 0$) is

$$A(\alpha - l\beta) - \alpha(A - kB) = 0 \ldots\ldots (2).$$

And of the line through ($A=0$, $\beta=0$), ($B=0$, $\alpha - l\beta=0$) the equation is

$$B\beta - A(\alpha - l\beta) = 0 \ldots\ldots (3).$$

But the sum of the left hand members of equations (1), (2), (3) is zero, therefore the lines represented by them are convergent; that is, the lines drawn through the opposite summits of a hexagon, under the given limitations, meet in a point.

[SECOND SOLUTION. *Mr. Nicholas Smyth, Galway; and similarly by Mr. W. S. Ashton, Over Darwen.*]

Let A, B, C, D, E, F be the angular points of the hexagon. Let the non-contiguous sides BC, DE, FA meet in P, and the other sides in Q. Let PA and PQ be the axes of x and y, and the distance PQ $= d$; then if the equations of PED and PBC be

$$y = ax \ldots\ldots\ldots\ldots(1), \qquad y = bx \ldots\ldots\ldots(2);$$

and the equations of QBA, QEF, QCD be respectively

$$y = mx + d \ldots(3), \qquad y = nx + d \ldots(4), \qquad y = px + d \ldots(5),$$

we shall have the coordinates of the six angular points A, B, C, D, E, F as follows :—

$$\text{Points}\begin{cases} \text{A} \ldots x = -\dfrac{d}{m}, y = 0, \\ \text{B} \ldots x = \dfrac{d}{b-m}, y = \dfrac{bd}{b-m}, \\ \text{C} \ldots x = \dfrac{d}{b-p}, y = \dfrac{bd}{b-p}, \end{cases} \qquad \text{Points}\begin{cases} \text{D} \ldots x = \dfrac{d}{a-p}, y = \dfrac{ad}{ap}, \\ \text{E} \ldots x = \dfrac{d}{a-n}, y = \dfrac{ad}{a-n}, \\ \text{F} \ldots x = -\dfrac{d}{n}, y = 0. \end{cases}$$

Whence we derive, as usual, the following equations of the diagonals,

$$(\text{AD}) \ldots\ldots\ldots\ldots (a + m - p)y = a(mx + d) \ldots\ldots\ldots\ldots\ldots (6),$$

$$(\text{BE}) \ldots\ldots (a - b + m - n)y = (am - bn)x + (a-b)d \ldots. (7),$$

$$(\text{CF}) \ldots\ldots\ldots (b + n - p)y = b(nx + d) \ldots\ldots\ldots\ldots\ldots (8).$$

Finally, since any one of the equations (6), (7), (8) is derivable from the other two, by subtraction, it follows that these diagonals have a common origin or intersection.

CLXXVI. *Mr. R. H. Wright, M. A., London.*

A fluid mass revolves round an axis with a given angular velocity, and is attracted to a centre of force in that axis varying as the distance : find the form the fluid will assume, and the equation which represents it.

[SOLUTION *Mr. Nicholas Smyth, Galway; and similarly by the proposer.*]

Let the attractive force at an unit's distance $= m$, and the given angular velocity $= \omega$. Let the centre of force be the origin of coordinates, the given axis or fixed line, being the axis of z; then if xyz be the rectangular coordinates of a molecule whose distance from the origin is r, the components of the attractive force will evidently be

$$-mx, \qquad -my, \qquad -mz.$$

Again the centrifugal force in the direction of the axis of z is $= 0$, and in the plane of xy, it will be $= \dfrac{(vel)^2}{r} = \dfrac{(\omega r)^2}{r} = \omega r^2$; which resolved in the directions of x and y gives $\omega^2 x$ and $\omega^2 y$, and as these resolved forces tend to increase the coordinates of x and y, they will be both positive ; hence the components of the centrifugal force are

$$\omega^2 x, \qquad \omega^2 y, \qquad 0.$$

Now if X, Y, Z be the total forces in the directions of the axes, we have

$$\text{X} = \omega^2 x - mx, \qquad \text{Y} = \omega^2 y - my, \qquad \text{Z} = -mz.$$

But the equation of a free surface is

$$\text{X}dx + \text{Y}dy + \text{Z}dz = 0,$$

and if the values of X, Y, Z, obtained above, be substituted in the equation to the surface, we shall have

$$(\omega^2-m)xdx+(\omega^2-m)ydy-mzdz=0;$$

or
$$(m-\omega^2)xdx+(m-\omega^2)ydy+mzdz=0.$$

Integrating, we get

$$(m-\omega^2)x^2+(m-\omega^2)y^2+mz^2=c,$$

the equation to a prolate spheroid, whose centre is the centre of force, and its principal axis is the axis of rotation. The major and minor axes of the generating ellipse are $2\left(\frac{c}{m-\omega^2}\right)^{\frac{1}{2}}$ and $2\left(\frac{c}{m}\right)^{\frac{1}{2}}$.

Mr. A. Roy, of Dundee, gave two very good solutions of this exercise, which are omitted with much reluctance.

CLXXVII. *Mr. Thomas Weddle.*

From the centre O of a surface of the second degree, and any point P, draw perpendiculars to the polar plane of P, and let the former perpendicular meet the said plane in G, and the latter intersect the diametral plane containing any two principal diameters in H. The rectangle of the perpendiculars OG, and PH, is equal to the square of half the third principal diameter.

[First Solution. *Mr. Thomas Weddle, the Proposer.*]

Let the equation to the surface be

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1\text{.....................(1).}$$

Also let $(x'y'z')$ denote the point P. The equation of the polar plane is

$$\frac{x'x}{a^2}\pm\frac{y'y}{b^2}\pm\frac{z'z}{c^2}=1\text{...................(2),}$$

and OG being the perpendicular on this plane from the origin, we have

$$\frac{1}{\text{OG}^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}\text{.................. (3).}$$

The equations to the line PH which passes through the point $(x'y'z')$ and is perpendicular to the plane (2), are

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\pm\frac{y'}{b^2}}=\frac{z-z'}{\pm\frac{z'}{c^2}}\text{................(4).}$$

The point H is situated in one of the coordinate planes—in that of yz, suppose; hence if it be denoted by $(oy''z'')$ we shall (4) have

$$y'-y''=\pm\frac{a^2}{b^2}y',\text{ and }z'-z''=\pm\frac{a^2}{c^2}z'\text{............(5);}$$

$$\begin{aligned}\therefore\ \text{PH}^2&=x'^2+(y'-y'')^2+(z'-z'')^2\\&=(5),\ x'^2+\frac{a^4}{b^4}y'^2+\frac{a^4}{c^4}z'^2\\&=a^4\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}\right)\\&=(3),\ \frac{a^4}{\text{OG}^2}.\end{aligned}$$

$$\therefore\ \text{OG.PH} = a^2.$$

Cor. If the perpendicular from P on the polar plane meet the other principal diametral planes in H_1 and H_2; we shall in like manner have $\text{OG.PH}_1 = b^2$, and $\text{OG.PH}_2 = c^2$, and hence

$$\text{PH} : \text{PH}_1 : \text{PH}_2 :: a^2 : b^2 : c^2.$$

Note. This exercise may be enunciated in a somewhat different manner when P is a point in a given surface, for then the polar plane becomes the tangent plane at P, and the perpendicular drawn from P coincides with the normal to the surface at that point.

[SECOND SOLUTION. *Mr Peter Elliott, Lamesley, Durham.*]

Let

$$Px^2 + P'y^2 + P''z^2 = H \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

be the equation to a central surface of the second degree, then the polar to the point P, whose coordinates are $(x'y'z')$, with respect to (1) is

$$Px'x + P'y'y + P''z'z = H \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and the equation to the line PH perpendicular to (2) is

$$\frac{x - x'}{Px'} = \frac{y - y'}{P'y'} = \frac{z - z'}{P''z'} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The coordinates of H, the point of intersection of (3) and the plane in which the principal diameters $\frac{2H^{\frac{1}{2}}}{P^{\frac{1}{2}}}$, $\frac{2H^{\frac{1}{2}}}{P'^{\frac{1}{2}}}$ are contained, that is, the coordinate plane of (xy), are found by making $z = 0$ in (3), to be

$$x = x' - \frac{P}{P''}x', \quad y = y' - \frac{P'}{P''}y.$$

Hence,

$$\text{PH} = \frac{1}{P''}(P^2x'^2 + P'^2y'^2 + P''^2z'^2)^{\frac{1}{2}}$$

Also OG, the length of the perpendicular from the origin upon (2),

$$= \frac{H}{(P^2x'^2 + P'^2y'^2 + P''^2z'^2)^{\frac{1}{2}}};$$

$\therefore\ \text{OG.PH} = \frac{H}{P''}$, the square of half the principal diameter perpendicular to the coordinate plane of (xy).

[THIRD SOLUTION. *Mr. William Mortimer, Manchester.*]

Let the equation to the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Then the equation to the polar plane of $P(x_1y_1z_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1,$$

whence we obtain $\text{OG} = \dfrac{1}{\left(\dfrac{x^2_1}{a^4} + \dfrac{y^2_1}{b^4} + \dfrac{z^2_2}{c^4}\right)^{\frac{1}{2}}}.$

Also the equation to PH is

$$\frac{a^2(x-x_1)}{x_1}=\frac{b^2(y-y_1)}{y_1}=\frac{c^2(z-z_1)}{z_1}=r;$$

therefore for the point H (xyo) we have

$$r=\left\{(x-x_1)^2+(y-y_1)^2+z^2_1\right\}^{\frac{1}{2}}=c^2\left\{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right\}^{\frac{1}{2}}=\text{PH};$$

$$\therefore\quad \text{OG.PH}=c^2.$$

A good solution was also received from Mr. Nicholas Smyth, of Galway.

[A solution of Exercise 173 is given at page 305 of the present number. Other solutions were received from Messrs. P. Elliott, T. Greenup, and A. Roy.]

CLXXVIII. *Mr. A. Roy, Dundee Academy.*

A tetrahedron is divided by a plane making a section which is a rhombus; to find in how many different ways this may be done, and the relations between the segments into which it may be divided.

[SOLUTION. *Dr. Burns, Rochester.*]

Since the section must be parallel to two of the opposite edges, and only one section parallel to each pair will give the relations that subsist when the section is a rhombus, there are only three ways in which the section can be made.

In the figure to the first solution of Exercise 159, let a plane pass through EF and the vertex D, and let the whole tetrahedron $=a$, the pyramid of which BCFE is the base and the vertex D $=b$, and the part cut off by the rhombus EFGH towards BC $=c$; then

$$a:b::\Delta\text{ABC}:\text{BCFE, and } b:c::\Delta\text{BCD}:\text{BCGH};$$

$$\therefore a:c::\text{ABC}\times\text{BCD}:\text{BCFE}\times\text{BCGH, and}$$

$$a-c:c::\text{ABC}\times\text{BCD}-\text{BCFE}\times\text{BCGH}:\text{BCFE}-\text{BCGH}.$$

Solutions were also received from the proposer and Messrs. A. Roy, Nicholas Smyth, and T. Greenup.

CLXXIX. *Mr. W. H. Levy, Shalbourne.*

If the radii of the escribed circles of any plane triangle be in harmonical progression, then will the sides of the triangle be in arithmetical progression.

[FIRST SOLUTION. *Mr. Thomas Wilkinson, Burnley, Lancashire.*]

By harmonical progression we have $r_1:r_3::r_1-r_2:r_2-r_3$, whence

$$r_2(r_1+r_3)=2r_1r_3\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again (*Horæ Geom.* Prop. xx) we have

$$sa=r_1(r_2+r_3),\quad sb=r_2(r_1+r_3),\quad sc=r_3(r_1+r_2):$$

$$\therefore\quad sa+sc=r_1(r_2+r_3)+r_3(r_1+r_2)$$

$$=r_2(r_1+r_3)+2r_1r_3=4r_1r_3,\text{ by (1).}$$

But $2sb=4r_1r_3$ by the same equation (1); therefore

$$sa+sc=2sb,\text{ or } a+c=2b.$$

[SECOND SOLUTION. *Mr. John Osborne, Londonderry; Mr. Nicholas Smyth, Galway; and Lieut. A. R. Clarke, R. E.*]

Let a, b, c be the sides of the triangle, r_1, r_2, r_3 the radii of the escribed circles, Δ the area of the triangle, and s half the sum of the sides; then (*Thomson's Euclid*, IV. 4, cor. 7)

$$r_1 = \frac{\Delta}{s-a},\quad r_2 = \frac{\Delta}{s-b},\quad r_3 = \frac{\Delta}{s-c};$$

but since r_1, r_2, r_3 are in harmonical proportion, their reciprocals are in arithmetical proportion; hence

$$\frac{s-a}{\Delta} + \frac{s-b}{\Delta} = \frac{2(s-c)}{\Delta},\quad \therefore a+b=2c.$$

Very good solutions were sent by Messrs. W. S. Ashton, Darwen; W. S. Berry, and Mortimer Dover Tait, at Mr. Hoole's Academy, Blackburn; Thomas Johnson, St. Helens; J. W. Elliott,; John Fraser and T. Greenup, Percy Street Academy, Newcastle-on-Tyne; W. H. Levy, the proposer; A. Roy, Dundee; Stephen Watson, Castle Side, Durham; and also by Dr. Burns, and A. D. B., of Hammersmith.

CLXXX. *Mr. Septimus Tebay, Gas Works, Preston.*

Three circles, radii r_1, r_2, r_3 touch each other on a plane. Let three other circles, radii ρ_1, ρ_2, ρ_3 be described each touching two of the former circles and a common tangent; then will

$$\frac{1}{\sqrt{\rho_1}} + \frac{1}{\sqrt{\rho_2}} + \frac{1}{\sqrt{\rho}} = \frac{2}{\sqrt{r_1}} + \frac{2}{\sqrt{r_2}} + \frac{2}{\sqrt{r_3}}.$$

[SOLUTION. *Mr. Septimus Tebay, the Proposer; and Mr. Stephen Watson, Castle Side.*]

Let α and β denote the segments of the common tangent to the circles r_1, r_2 made by the point of contact of the circle ρ_3; then we shall have

$$2\sqrt{(r_1r_2)} = \alpha+\beta,\quad 2\sqrt{(r_1\rho_3)} = \alpha,\quad 2\sqrt{(r_1\rho_3)} = \beta;$$

$$\therefore\ 2\sqrt{(r_1r_2)} = 2\sqrt{(r_1\rho_3)} + 2\sqrt{(r_2\rho_3)},\ \text{or}\ \frac{1}{\sqrt{\rho_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}.$$

Similarly, $\dfrac{1}{\sqrt{\rho_1}} = \dfrac{1}{\sqrt{r_1}} + \dfrac{1}{\sqrt{r_3}}$, and $\dfrac{1}{\sqrt{\rho_1}} = \dfrac{1}{\sqrt{r_2}} + \dfrac{1}{\sqrt{r_3}}$;

therefore $$\frac{1}{\sqrt{\rho_1}} + \frac{1}{\sqrt{\rho_2}} + \frac{1}{\sqrt{\rho_3}} = \frac{2}{\sqrt{r_1}} + \frac{2}{\sqrt{r_2}} + \frac{2}{\sqrt{r_3}}.$$

The solutions sent by Messrs. W. S. Ashton; W. S. Berry; J. W. Elliott; John Fraser; Thomas Greenup; J. Harrop, Ashton-under-Lyne; W. H. Levy; M. D. Tait; A. Roy; N. Smyth; and Lieut. A. R. Clarke, were exactly similar to the above.

CLXXXI. *Lieut. A. R. Clarke, R. E.*

If the angular points of a triangle be joined with the opposite points of contact of the escribed parabolas, the three lines will meet in a point, and the sum of its distances from the sides of the triangle is

$$\frac{1}{D}\cdot\frac{a^2b^2+a^2c^2+b^2c^2}{ab+ac+bc},$$

where D is the diameter of the circumscribed circle.

[SOLUTION. *Lieut. A. R. Clarke, the proposer, and Mr. Nicholas Smyth.*]

Let ABC be the triangle, F the focus of the escribed parabola opposite A, and AEF the line bisecting the angle A, cutting BC in E. Let M be the middle point of BC, and D the point of contact of the parabola with BC.

It will readily be seen that F is the middle point of the arc of the circumscribed circle cut off by BC; therefore FM is perpendicular to BC. Consequently by the property of the parabola, MD=ME, and therefore CE=BD. Consequently,

$$BD = \frac{ab}{b+c}.$$

Similarly, $$AK = \frac{ac}{a+b}, \text{ and } AL = \frac{ab}{a+c};$$

where K, L are the middle points of AB, AC.

Taking AB, AC as axes of x and y, the equations of AD, BL, CK are

$$y = x\frac{b^2}{c^2} \quad \text{.........................}(1),$$

$$\frac{x}{c} + y\frac{a+c}{ab} = 1 \quad \text{.........................}(2),$$

$$x\frac{a+b}{ac} + \frac{y}{b} = 1 \quad \text{.........................}(3).$$

The difference of (2), (3) gives (1); consequently the three lines meet in a point.

From (1), (2) we have $y = \frac{ab^2}{ab+ac+bc}$; whence ($\delta_1$, δ_2, δ_3 denoting the distances specified in the enunciation)

$$\delta_3 = y\sin A = 2y\frac{\Delta}{bc} = \frac{ab}{c}.\frac{2\Delta}{ab+ac+bc} = \frac{2\Delta}{abc}.\frac{a^2b^2}{ac+bc+ab}.$$

Similarly, the expressions for δ_1 and δ_2 are found, and hence

$$\delta_1+\delta_2+\delta_3 = \frac{2\Delta}{abc}\,\frac{a^2b^2+b^2c^2+a^2c^2}{ab+ac+bc} = \frac{1}{D}\,\frac{a^2b^2+a^2c^2+b^2c^2}{ab+ac+bc}.$$

Cor. 1. Since $ab+ac+bc = D(p_1+p_2+p_3)$, where p_1, p_2, p_3 are the perpendiculars from the angles A, B, C upon the opposite sides; therefore

$$\delta_1+\delta_2+\delta_3 = \frac{a^2b^2+a^2c^2+b^2c^2}{D^2(p_1+p_2+p_3)}.$$

Cor. 2. Hence also, $\delta_1+\delta_2+\delta_3 = \frac{bcp_1+acp_2+abp_3}{ab+ac+bc} = \frac{bcp_1+acp_2+abp_3}{D(p_1+p_2+p_3)}$,

$$\text{or, } \delta_1+\delta_3+\delta_2 = \frac{p_1^2+p_2^2+p_3^3}{p_1+p_2+p_3}, \text{ (since } bc = Dp_1, \textit{etc.})$$

a remarkable relation between these two systems of perpendiculars.

The solutions of Messrs. Roy and Fraser were very elegant; they are omitted with much regret.

CLXXXIV. *Dr. Rutherford.*

If the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ has two roots of the form $+\sqrt{-\beta}$, and $-\sqrt{-\beta}$, there exists among the coefficients a, b, c, d, e, the relation

$$\frac{ab - c}{ad - e} = \frac{ad - e}{cd - be}, \quad \text{or} \quad \frac{abc - a^2d - c^2}{ab^2 - 2ad - bc - e} = \frac{e}{d}.$$

[First Solution. *Mr. Thomas Wilkinson, Burnley; and similarly by W.H. Levy.*]

By art. 7, *Dr. Rutherford's Numerical Equations*, the value of a, the rational part of the roots, will be zero, and the values of A, B, C, D, E, given in equation (3), will reduce to

$$A = a, \quad B = b, \quad C = c, \quad D = d, \quad E = e.$$

Hence if $m = 5$ in equations (6), art. 7, *ibid.* we shall have

$$\beta^2 - b\beta + d = 0,$$

$$a\beta^2 - c\beta + e = 0.$$

Eliminating β from these equations, we obtain

$$(ab - c)(cd - be) = (ad - e)^2,$$

$$\text{or} \quad \frac{ab - c}{ad - e} = \frac{ad - e}{cd - be}; \text{ the relation required.}$$

[Second Solution. *Mr. Nicholas Smyth, Galway: and similarly by Mr. Andrew Roy, Dundee Academy.*]

Assume

$$(x - \sqrt{-\beta})(x + \sqrt{-\beta})(x^3 + ax^2 + Ax + B) = \text{the given equation.}$$

Then multiplying out and comparing coefficients, we have

$$A + \beta = b, \quad B + a\beta = c, \quad A\beta = d, \text{ and } B\beta = e \ldots\ldots\ldots\ldots(1).$$

This system of simple equations (consisting of four in number) enables us to eliminate A, B and β, and thence the enunciated relation is obtained. Thus from the first and third of (1),

$$A = b - \beta, \text{ and } A = \frac{d}{\beta}.$$

Also from the second and fourth of the relations (1),

$$B = c - a\beta, \text{ and } B = \frac{e}{\beta}.$$

Hence $\quad b - \beta : c - a\beta :: d : e$, or $\beta = \dfrac{cd - be}{ad - e}$. Whence

$$A = \frac{d}{\beta} = \frac{ad - e}{cd - be}.d, \text{ and } B = \frac{e}{\beta} = \frac{ad - e}{cd - be}.e.$$

Substituting these values in the equation

$$A + \beta = b,$$

we get by obvious reductions,

$$(ad-e)^2=(ab-c)(cd-be), \text{ or } \frac{ab-c}{ad-e}=\frac{ad-e}{cd-be},$$

which is the relation required.

Good solutions were also received from Messrs. W. S. Ashton, Darwen, S. Berry, Mr. Hoole's Academy, Blackburn, and John Fraser.

CLXXXVI. *Dr. Burns.*

Find a multiplier which will make $\sqrt{11}-\sqrt{7}+\sqrt{5}-\sqrt{3}$ a rational quantity, and show how a multiplier may be found which will make $\sqrt{a}\pm\sqrt{a_1}\pm\sqrt{a_2}\pm\sqrt{a_3}\pm\ldots\pm\sqrt{a_n}$, a rational quantity.

[First Solution. *Messrs. A. Roy, Dundee, and N. Smyth, Galway.*]

First multiply by $\sqrt{11}+\sqrt{5}+\sqrt{7}+\sqrt{3}$, and the product is

$$6+2\sqrt{55}-2\sqrt{21}.$$

Next multiply by $6+2\sqrt{55}+2\sqrt{21}$, and the result is $172+24\sqrt{55}$.

Lastly, multiply by $172-24\sqrt{55}$, and we have $172^2-55.24^2$, a rational quantity. Hence the multipliers required to make the expression rational, are $\sqrt{11}+\sqrt{5}+\sqrt{7}+\sqrt{3}$, $6+2\sqrt{55}-2\sqrt{21}$, and $172-24\sqrt{55}$.

In a similar way the expression

$$\sqrt{a}\pm\sqrt{a_1}\pm\sqrt{a_2}+\ldots\ldots$$

is made rational. (See *Hirsch's Algebra*, page 186.)

[Second Solution. *Mr. W. S. Ashton, Darwen.*]

First, multiplying

$$(\sqrt{11}-\sqrt{7})+(\sqrt{5}-\sqrt{3}) \text{ by } (\sqrt{11}-\sqrt{7})-(\sqrt{5}-\sqrt{3}),$$

we have

$$(\sqrt{11}-\sqrt{7})^2-(\sqrt{5}-\sqrt{3})^2, \text{ or } 10-2(\sqrt{77}-\sqrt{15}).$$

Again, multiplying $10-2(\sqrt{77}-\sqrt{15})$ by $10+2(\sqrt{77}+\sqrt{15})$, we have $100-4(\sqrt{77}-\sqrt{15})^2$, or $8\sqrt{1155}-268$.

Lastly, multiplying

$8\sqrt{1155}-268$ by $8\sqrt{1155}+268$, we have $(64\times1155)^2-268^2$, or 2096.

If then we multiply $\sqrt{11}-\sqrt{7}+\sqrt{5}-\sqrt{3}$ by

$$\{\sqrt{11}-\sqrt{7}-\sqrt{5}+\sqrt{3}\}\{10+2(\sqrt{77}-\sqrt{15})\}\{8\sqrt{1155}+268\},$$

the result will be rational.

The reason of the above selection of factors is obvious, and a similar method will plainly apply to the rationalizing of every expression of the form

$$\sqrt{a}\pm\sqrt{a_1}\pm\sqrt{a_2}\pm\sqrt{a_3}\pm\ldots\ldots\pm\sqrt{a_n}.$$

A good solution was sent by Mr. John Fraser.

CLXXXIX. *Mr. Septimus Tebay, Preston.*

Three planes xy, xz, yz intersect at right angles. Let six straight lines be drawn through their common point of intersection; two at right angles in the plane xy; two at right angles in the plane xz; and two at right angles in the plane yz; prove that the sum of the moments of inertia of any body about these six lines as axes, is

$$4\Sigma(x^2m) + 4\Sigma(y^2m) + 4\Sigma(z^2m);$$

x, y, z being the coordinates of any particle m of the solid.

[Solution. *By the proposer, Mr. Septimus Tebay.*]

Let op be any axis passing through the origin o, in the plane xy; mp perpendicular to op, and the angle $pox = a$. We have

$$mp^2 = x^2 + y^2 + z^2 - (x \cos a + y \sin a)^2$$
$$= x^2 \sin^2 a + y^2 \cos^2 a + z^2 - xy \sin 2a;$$

$$\therefore \qquad K^2M = \Sigma(mp^2.m)$$
$$= \sin^2 a \Sigma(x^2m) + \cos^2 a\Sigma(y^2m) + \Sigma(z^2m) - \sin 2a\Sigma(xym) \ldots\ldots(1).$$

Similarly about the axes in the planes xz, yz, we shall find (continuing the notation)

$$K^2M = \sin^2\beta\Sigma\,(x^2m) + \cos^2\beta\Sigma(z^2m) + \Sigma\,(y^2m) - \sin 2\beta\Sigma(xzm)\ldots.(2),$$
$$K^2M = \sin^2\gamma\Sigma\,(y^2m) + \cos^2\gamma\Sigma(z^2m) + \Sigma\,(y^2m) - \sin 2\gamma\Sigma(yzm)\ldots.(3).$$

Hence for axes at right angles to those in (1), (2), (3), write $a + 90°$, $\beta + 90°$, $\gamma + 90°$ for a, β, γ; then

$$K^2M = \cos^2 a\Sigma(x^2m) + \sin^2 a\Sigma(y^2m) + \Sigma(z^2m) + \sin 2a\Sigma(yzm)\ldots\ldots(4),$$
$$K^2M = \cos^2\beta\Sigma(x^2m) + \sin^2\beta\Sigma(z^2m) + \Sigma(y^2m) + \sin 2\beta\Sigma(xzm)\ldots\ldots(5),$$
$$K^2M = \cos^2\gamma\Sigma(y^2m) + \sin^2\gamma\Sigma(z^2m) + \Sigma(x^2m) + \sin 2\gamma\Sigma(xym)\ldots\ldots(6).$$

Assume

$$M_1 = (1) + (4) = \Sigma(x^2m) + \Sigma(y^2m) + 2\Sigma(z^2m),$$
$$M_2 = (2) + (5) = \Sigma(x^2m) + 2\Sigma(y^2m) + \Sigma\,(z^2m),$$
$$M_3 = (3) + (6) = 2\Sigma(x^2m) + \Sigma\,(y^2m) + \Sigma\,(z^2m).$$
$$\therefore \qquad M_1 + M_2 + M_3 = 4\Sigma(x^2m) + 4\Sigma(y^2m) + 4\Sigma(z^2m),$$

which establishes the property proposed.

In addition to the above we also have

$$-M_1 + M_2 + M_3 = 2\Sigma(x^2m) + 2\Sigma(y^2m),$$
$$M_1 - M_2 + M_3 = 2\Sigma(x^2m) + 2\Sigma(z^2m),$$
$$M_1 + M_2 - M_3 = 2\Sigma(y^2m) + 2\Sigma(z^2m).$$

Correct and elegant solutions were also received from Lieutenant A. R. Clarke, R. E.; Mr. Roy, Dundee; and Mr. N. Smyth, Galway.

SUPPLEMENTARY NUMBER

TO VOL. III.

MATHEMATICIAN.

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NEW GEOMETRICAL DEMONSTRATION OF PASCAL'S THEOREM.

[*James Dalmahoy, Esq., F.R.S.E.*]

All the *geometrical* demonstrations of Pascal's Theorem which have come to my knowledge are cited below:* and it is after having examined these that I venture to offer the following one as new.

It may be remarked that the truth directly established by this demonstration is literally the one enunciated by Pascal, in the third Lemma of his "Essai pour les Coniques."† Pascal's enunciation and the modern one are merely two modes of expressing the same truth; but the difference becomes interesting if we conjecture that the form which he adopted might possibly have had some relation to his method of demonstration.

LEMMA.

If a polygon, having an even number of sides, be inscribed in a conic section, the relation between the segments resulting from the mutual intersection of the opposite sides, will be the same as would subsist if the sides of the polygon were cut by a straight line passing through the points in which the opposite sides intersect each other.

* Geometrical demonstrations limited to the case of the circle:—Maclaurin, Fluxions, vol. ii. art. 623. Carnot, Théorie des Transversales, p. 92. Earl Stanhope, Sir James Ivory, Mr. Lowry, Mr. Nicholson, in Leybourn's Mathematical Repository, new series, vol. ii., p. 42—44.

General Geometrical demonstrations:—Carnot, Géométrie de Position, art. 397. Sir John Leslie, Geometry of Curve Lines, book ii., prop. xv. Mr. Davies, in Hutton's Mathematics, 12th edition, vol. ii., pp. 190, 233. Walker's Conic Sections, prop. xxv. An author whose signature is J. B. B. C., (Dr. James Booth,) in the Lady's and Gentleman's Diary, for 1843, p. 64. The second of Mr. Davies' demonstrations above quoted, originally appeared in the same Diary, p. 66.

For a list of the Analytical demonstrations, see Note to Mr. Weddle's demonstration, in Cambridge Mathematical Journal, new series, vol. v., p. 285; to which add two demonstrations in Philosophical Magazine for 1829, p. 249, and for 1842, p. 37, the former by Sir J. Lubbock, and the latter by Mr. Davies.

† Œuvres de Blaise Pascal, t. iv., p. 2.

As the case of the quadrilateral figure requires no proof, let ABCDEF* be a hexagon inscribed in a conic section, and let the opposite sides, produced if necessary, intersect in the points G,H,K, it is to be proved that GF.GC.HA.HD.KB.KE = GA.GD.HB.HE.KC.KF, which, by the theory of transversals, is the relation which would subsist if the points G,H,K, or rather, since these points are double, if the points G,G,H,H,K,K, lay in one straight line.

Join the opposite angles of the figure by drawing the diagonals AD, BE, CF, intersecting each other, two and two, in the points L, M, N; and parallel to these diagonals respectively draw the semi-diameters OP, OR, OQ.†

Since the triangles ALF and CLD are cut respectively by the transversals GCD and GAF, the former triangle furnishes the relation GF.AD.CL = GA.CF.LD; the latter, the relation GC.AD.FL=GD.CF.AL; and the equality compounded of these two is $GF.GC.AD^2.CL.FL = GD.GA.CF^2.AL.LD$. But AD being parallel to OP and CF to OQ, it follows that $AL.LD : CL.FL :: OP^2 : OQ^2$; hence the above expression, after having substituted OP^2 for AL.LD, and OQ^2 for CL.FL, will furnish this analogy $GF.GC : GD.GA :: CF^2.OP^2 : AD^2.OQ^2$.

By considering, in a similar manner, the pairs of triangles AMB, DME, and BNC, ENF, there are obtained these two other analogies

$$HA.HD : HE.HB :: AD^2.OR^2 : BE^2.OP^2,$$

and $$KB.KE : KF.KC :: BE^2.OQ^2 : CF^2.OR^2;$$

and if the corresponding terms of the three analogies be compounded, there results the equality GF.GC.HA.HD.KB.KE=GD.GA.HE.HB.KF.KC, or the analogy GF.GC.HA.HD : GD.GA.HE.HB :: KF.KC : KB.KE.

The demonstration with respect to the octagon, and other polygons with an even number of sides, is precisely similar.

Cor. Let the quadrilateral BCEF be completed by drawing BF and CE, and let them be produced to meet in V. Since a transversal through the points K,V cuts the four sides of the quadrilateral BECF, we have the relation KF.KC : KB.KE :: VF.VC : VE.VB; hence, by the lemma and by equality of ratios

$$GF.GC.HA.HD : GD.GA.HE.HB :: VF.VC : VE.VB :: \frac{VF}{VB} : \frac{VE}{VC}.$$

PASCAL'S THEOREM.

Let ABCDEF ‡ be a hexagon inscribed in a conic section; let the opposite sides, produced if necessary, intersect in G,H,K, these three points lie in a straight line; or, according to Pascal's own enunciation, the lines GH, BC, FE, meet in one point K.

The theorem may be conveniently divided into two cases,

1. When the lines AD, BE, CF, joining the opposite angles, intersect in one point.
2. When they intersect, two and two, in the points L, M, N.

* See Diagrams. The point W is the intersection of FV with GC instead of FV with GH as in the diagram.

† In the parabola draw tangents parallel to the diagonals, also, circumscribing these, draw other two tangents, and let the portions of the three tangents intercepted by the circumscribing tangents be denoted by OP, OR, OQ.

‡ See Diagrams.

Case 1. Let lines be drawn through A,C and D,F meeting in S; through A,E and B,D meeting in T; and through B,F and C,E meeting in V. The points L, M, N are respectively the poles of the polars drawn through G,S, through H,T, and through K,V;* but when the poles L,M,N coincide, so must the polars GS, HT and KV; for, from the definition of their relation, one pole can have only one polar; hence G,H, K are in a straight line.

Case 2. Join G,H, any two of the points in which the opposite sides intersect, and produce the line GH indefinitely towards the third point K.

As, by hypothesis, the lines AD, BE, CF joining the corresponding angles of the triangles ABF and DEC do not intersect in one point, these triangles are neither co-polar† nor co-axial, and therefore the line GH does not pass through V, the intersection of BF and CE, but cuts the former in W and the latter in X. Further, as the sides of the triangles ABF and DEC are cut by the transversal GWXH, the former furnishes the relation GF.WB.HA = GA.WF.HB; the latter the relation GC.XE.HD = GD.XC.HE; and the equality compounded of these two is

$$GF.GC.HA.HD.XE.WB = GD.GA.HE.HB.WF.XC;$$

from which is derived the analogy

$$GF.GC.HA.HD : GD.GA.HE.HB :: WF.XC : XE.WB :: \frac{WF}{WB} : \frac{XE}{XC}.$$

But by the *Cor.* to the lemma

$$GF.GC.HA.HD : GD.GA.HE.HB :: \frac{VF}{VB} : \frac{VE}{VC},$$

therefore by equality of ratios $\frac{VF}{VB} : \frac{WF}{WB} :: \frac{VE}{VC} : \frac{XE}{XC}.$ The lines BF and CE, intersecting in V, are therefore divided in the same anharmonic ratio‡, F,E; B,C; W,X; and V being the corresponding points: consequently the lines FE, BC, WX intersect in one point;|| but FE and BC intersect in K, hence WX or GH passes through K, and the points G,H,K are in a straight line.

The demonstrations of the lemma and of the theorem are applicable, without altering a word, to all the simple hexagons of which the complete§ hexagon is composed, provided the consecutive angles are always denoted by the consecutive letters. They also apply to all the conic sections.

With respect to the total number of simple hexagons, it is evident that if there be six points on a curve, and if any given consecutive order of these points be denoted by six consecutive letters, the positions of the letters admit of twelve permutations, in each of which the consecutive order of the letters still indicates the same consecutive order of the points, for each letter may be attached to each point in the direct and also in the

* Hutton, 12th edition, vol. ii., p. 178. † Ibid, p. 222.

‡ Hutton, 12th edition, vol. ii., p. 239.— Mathematician, vol. i., p. 171.—Aperçu Historique, sur l'Origine et le Developpement des Méthodes en Géométrie, par M. Chasles, note ix., p. 302. || Ibid, note ix., p. 303, l. 20—28.

§ The term complete hexagon is here applied to the figure composed of all the hexagons having the same six angular points. It is however more correctly applicable to the figure composed of all the hexagons having the same six lines for their sides. But in both senses and in all polygons there evidently subsists the distinction between the complete and simple figure.

inverse order: but as only those hexagons differ from each other in which the consecutive orders of the points differ, it follows that the total number* of simple hexagons is equal to one-twelfth of the permutations of which six things are capable; that is, it is equal to $\frac{1.2.3.4.5.6}{12} = \frac{720}{12} = 60$.

Hence the points in which the opposite sides of all the hexagons intersect, lie, three and three, on sixty different straight lines.

But in this arrangement it is obvious that one point belongs to more than one line; for the combinations of six things, two and two, amount to fifteen; consequently this is the whole number of lines of which the sixty hexagons are composed; but since five of these fifteen lines pass through each angular point of the hexagon, the lines meeting in any of these points can never be combined, two and two, as opposite sides of a hexagon, while there is nothing to forbid all the other lines being so combined; hence it follows that the number of points in which the opposite sides of the hexagons mutually intersect amounts to

$$\frac{15.14}{2} - 6.\frac{5.4}{2} = 105 - 60 = 45,$$

and therefore one point is common to four different lines.

In this estimate and in that respecting the number of directions in which three points lie in a straight line, it is supposed that in none of the hexagons do the diagonals intersect in one point.

In conclusion, as there are forty-five points in which two lines intersect mutually, there must be ninety points of intersection on the lines taken singly; but as the whole lines amount only to fifteen, it is evident that on each of them there must be six of the forty-five points. Hence Pascal's theorem seems to furnish a solution to the following problem:—It is required to arrange forty-five points in such a manner that in sixty different directions three of them shall lie in a straight line; and in fifteen different directions six of them shall lie in a straight line.

* If n be the number of sides or angles of a polygon, and N the whole number of simple polygons of either kind, it follows from the reasoning with respect to the hexagon, that

$$\mathrm{N} = \frac{1.2.3\ldots.(n-1)n}{2n} = \frac{1.2.3\ldots\ldots(n-1)}{2}.$$

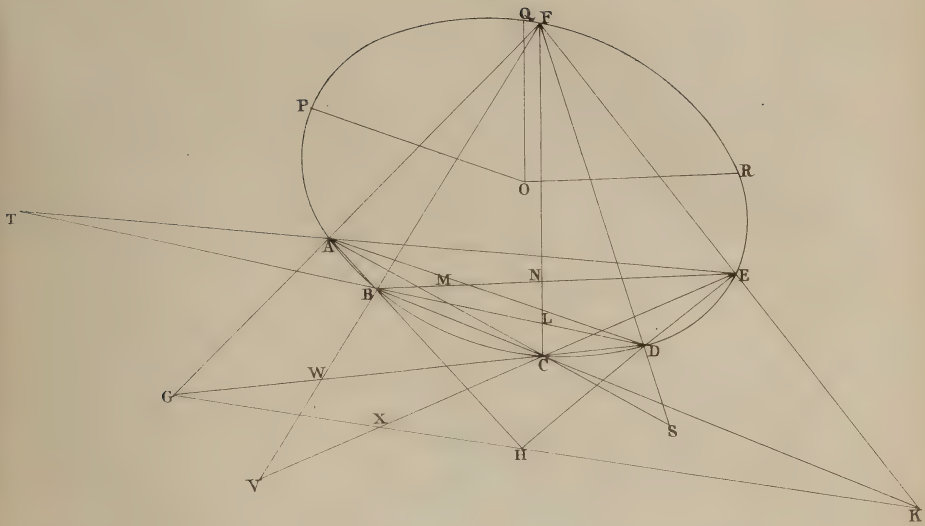
SUPPLEMENT TO THE REMARK ON THE GENERAL EQUATION OF THE SECOND DEGREE.

(See Vol. III. p. 154.)

[*Professor De Morgan.*]

A notation may be given, by which the general equation both of the curve and surface of the second order, may be written down in a manner easy to remember.

Let P_Q signify, *in a diagram*, the point at which the central conjugate of a line P meets the line Q; *in an equation*, the distance of that point from the origin. Then, D and E being the distances at which the polar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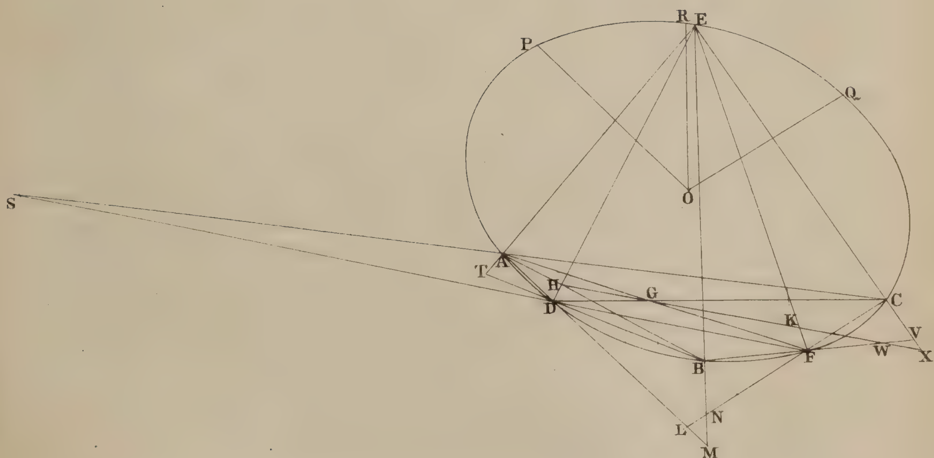
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line of the origin meets the axes of y and x, we have $DY_X = EX_Y$, and for the equation of the curve

$$\frac{y^2}{DY_Y} + \frac{2xy}{DY_X \text{ or } EX_Y} + \frac{x^2}{EX_X} - \frac{2y}{D} - \frac{2x}{E} + 1 = 0.$$

In the surface of the second order, let O be the origin, L, M, N the points at which the polar plane of the origin meets the axes (OL being L, *etc.*, in the equation) and X_Y the point at which the central conjugate plane of the axis of x cuts the axis of y, *etc.* (OX_Y being denoted by X_Y in the equation, *etc.*) Then the theorem answering to that in my former communication is as follows. First, LM and $X_Y\,Y_X$ are parallel, and also MN and $Y_Z\,Z_Y$, and also NL and $Z_X\,X_Z$. Or,

$$LX_Y = MY_X, \quad MY_Z = NZ_Y, \quad NZ_X = LX_Z,$$

and the equation of the surface is

$$\frac{x^2}{LX_X} + \frac{y^2}{MY_Y} + \frac{z^2}{NZ_Z} + \frac{2yz}{MY_Z \text{ or } NZ_Y} + \frac{2zx}{NZ_X \text{ or } LX_Z} + \frac{2xy}{LX_Y \text{ or } MY_X}$$
$$- \frac{2x}{L} - \frac{2y}{M} - \frac{2z}{N} + 1 = 0.$$

ON ANALYTICAL GEOMETRY OF THREE DIMENSIONS RELATIVE TO OBLIQUE AXES.

[*Mr. Thomas Weddle, F.R.A.S., Training Institution, Battersea*]

(Continued from Vol. III., page 242.)

CHAPTER III.

On the Transformation of Coordinates.

In this chapter I shall, except in prop. iii., treat only of the change of *direction* in the coordinate axes, the change of *origin* being too simple to require consideration here.

PROP. I. *To pass from one system of oblique axes to another system also oblique.*

Let $(l_1m_1n_1)$, $(l_2m_2n_2)$, $(l_3m_3n_3)$ be the projecting coefficients of the new axes relative to the old, and $(l'l''l''')$, $(m'm''m''')$, $(n'n''n''')$ those of the old axes relative to the new. Also let Γ and Γ' be the coefficients of cubature relative to the old and new axes respectively. Now, (xyz) and $(x'y'z')$ denoting the same point with respect to the old and new axes, it is clear that the coordinate projections (relative to the old axes) of the broken lines $x+y+z$ and $x'+y'+z'$ must be equal; moreover the coordinate projection of the former on the axis of x is evidently x_1, and that of the latter $l_1x'+l_2y'+l_3z'$,

$$\left.\begin{array}{rl} \because & x = l_1x' + l_2y' + l_3z' \\ \text{Similarly,} & y = m_1x' + m_2y' + m_3z' \\ \text{and,} & z = n_1x' + n_2y' + n_3z' \end{array}\right\} \dots\dots\dots\dots\dots (93),$$

and these equations, as soon as l_1, m_1, n_1.... shall have been determined, will enable us to effect the transformation required.

It may sometimes be convenient to have the values of the new coordinates in terms of the old. By taking successively the coordinate projections of the broken lines $x'+y'+z'$ and $x+y+z$ on the axes of x', y', and z', we shall get

$$\left.\begin{array}{l} x' = l'x + m'y + n'z \\ y' = l''x + m''y + n''z \\ z' = l'''x + m'''y + n'''z \end{array}\right\} \dots\dots\dots\dots\dots\dots (94).$$

Eliminate y and z from (94), by cross multiplication,

$$\therefore t'.x = (m''n''' - m'''n'')x' + (m'''n' - m'n''')y' + (m'n'' - m''n')z' \ldots (a),$$

where $t' = l'm''n''' - l'm'''n'' + l''m'''n' - l''m'n''' + l'''m'n'' - l'''m''n'$.

From the origin O measure on the axes of x, y and z the distances OL, OM and ON each equal to unity. It is evident that the points L, M, N will, to the new axes, be denoted by $(l'l''l''')$, $(m'm''m''')$ and $(n'n''n''')$ respectively; and hence the volume V of the tetrahedron of which OL, OM and ON are three contiguous edges will, (91), be

$$V = \pm \frac{1}{6} \Gamma' (l'm''n''' \dots\dots\dots - l'''m''n') = \pm \frac{1}{6} \Gamma' t'.$$

But, (92), the volume of this solid is also

$$V = \frac{1}{6} \Gamma.\mathrm{OL.OM.ON} = \frac{1}{6} \Gamma.$$

Equating these two volumes of V we have t' or

$$l'm''n''' - l'm'''n'' + l''m'''n' - l''m'n''' + l'''m'n'' - l'''m''n' = \pm \frac{\Gamma}{\Gamma'} \ldots (95).$$

Now it is evident that (a) after division by t' must be identical with the first equation of (93); hence equating the coefficients of $x'y'z'$, and performing similar operations by eliminating first x and z, and then x and y from (94), we shall have

$$\left.\begin{array}{l} l_1 = \pm \frac{\Gamma'}{\Gamma} (m''n''' - m'''n''),\ l_2 = \pm \frac{\Gamma'}{\Gamma} (m'''n' - m'n'''),\ l_3 = \pm \frac{\Gamma'}{\Gamma} (m'n'' - m''n') \\ m_1 = \pm \frac{\Gamma'}{\Gamma} (l'''n'' - l''n'''),\ m_2 = \pm \frac{\Gamma'}{\Gamma} (l'n''' - l'''n'),\ m_3 = \pm \frac{\Gamma'}{\Gamma} (l''n' - l'n'') \\ n_1 = \pm \frac{\Gamma'}{\Gamma} (l''m''' - l'''m''),\ n_2 = \pm \frac{\Gamma'}{\Gamma} (l'''m' - l'm'''),\ n_3 = \pm \frac{\Gamma'}{\Gamma} (l'm'' - l''m') \end{array}\right\} \ldots (96).$$

By operating in a similar manner on (93) instead of (94), we shall get

$$l_1m_2n_3 - l_1m_3n_2 + l_2m_3n_1 - l_2m_1n_3 + l_3m_1n_2 - l_3m_2n_1 = \pm \frac{\Gamma'}{\Gamma} \ldots\ldots (97),$$

and,

$$\left.\begin{array}{l}
l' = \pm \frac{\Gamma}{\Gamma'}(m_2n_3 - m_3n_2),\ m' = \pm \frac{\Gamma}{\Gamma'}(l_3n_2 - l_2n_3),\ n' = \pm \frac{\Gamma}{\Gamma'}(l_2m_3 - l_3m_2) \\
l'' = \pm \frac{\Gamma}{\Gamma'}(m_3n_1 - m_1n_3),\ m'' = \pm \frac{\Gamma}{\Gamma'}(l_1n_3 - l_3n_1),\ n'' = \pm \frac{\Gamma}{\Gamma'}(l_3m_1 - l_1m_3) \\
l''' = \pm \frac{\Gamma}{\Gamma'}(m_1n_2 - m_2n_1),\ m''' = \pm \frac{\Gamma}{\Gamma'}(l_2n_1 - l_1n_2),\ n''' = \pm \frac{\Gamma}{\Gamma'}(l_1m_2 - l_2m_1)
\end{array}\right\} \ldots (98).$$

These formulas are analogous to, and inclusive of Lagrange's relations among the directing cosines of three straight lines which are perpendicular to each other; and by their means, if l_1m_1.. be known, $l'l''$... may be found, and conversely. Moreover if the equations of the new axes to the old ones be given l_1m_1.. may be calculated, (14, 15); or if the equations of the old axes to the new be known, we can determine $l'l''$...; so that in either case all the $l\ m\ n$ may be computed.

Besides the preceding relations among $l_1m_1n_1..l'l''l'''$.., and those that may be derived from (12), there are several others which it will be expedient to put down.

Take the values of $l'x + m'y + n'z$, $l''x + m''y + n''z$, and of $l'''x + m'''y + n'''z$ from (93), and compare the results with (94),

$$\left.\begin{array}{r}
\therefore\ l_1l' + m_1m' + n_1n' = 1 \\
l_2l'' + m_2m'' + n_2n' = 1 \\
l_3l''' + m_3m''' + n_3n''' = 1
\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (99).$$

and

$$\left.\begin{array}{r}
l'\,l_2 + m'\,m_2 + n'\,n_2 = 0 \\
l'\,l_3 + m'\,m_3 + n'\,n_3 = 0 \\
l''\,l_1 + m''\,m_1 + n''n_1 = 0 \\
l''\,l_3 + m''\,m_3 + n''n_3 = 0 \\
l'''l_1 + m'''m_1 + n'''n_1 = 0 \\
l'''l_2 + m'''m_2 + n'''n_2 = 0
\end{array}\right\} \ldots\ldots\ldots\ \ldots\ldots (100).$$

Also taking the values of $l_1x' + l_2y' + l_3z'$, $m_1x' + m_2y' + m_3z'$, and $n_1x' + n_2y' + n_3z'$ from (94), and comparing the results with (93), we have,

$$\left.\begin{array}{r}
l_1l' + l_2l'' + l_3l''' = 1 \\
m_1m' + m_2m'' + m_3m''' = 1 \\
n_1n' + n_2n'' + n_3n''' = 1
\end{array}\right\} \ldots\ldots\ldots\ldots\ldots (101).$$

and,

$$\left.\begin{array}{r}
l_1m' + l_2m'' + l_3m''' = 0 \\
l_1n' + l_2n'' + l_3n''' = 0 \\
m_1l' + m_2l'' + m_3l''' = 0 \\
m_1n' + m_2n'' + m_3n''' = 0 \\
n_1l' + n_2l'' + n_3l''' = 0 \\
n_1m' + n_2m'' + n_3m'' = 0
\end{array}\right\} \ldots\ldots\ \ldots\ldots (102).$$

Also f, g, h, and f', g', h' being the cosines of ordination relative to the old and new axes respectively, we shall, (by considering the directing cosines of the two systems) have, (16), the following formulas

$$\left.\begin{array}{l} l_1 + hm_1 + gn_1 = l' + h'l'' + g'l''' \\ hl_1 + m_1 + fn_1 = m' + h'm'' + g'm''' \\ gl_1 + fm_1 + n_1 = n' + h'n'' + g'n''' \\ l_2 + hm_2 + gn_2 = h'l' + l'' + f'l''' \\ \qquad etc. \qquad\qquad etc. \end{array}\right\} \ldots\ldots \ldots (103).$$

Of course the preceding formulas apply when one or both of the systems of axes are rectangular, and various simplifications may then be made. When the old axes are rectangular $l_1m_1n_1l_2$...become the directing cosines of the new axes and $\Gamma = 1$. If the new axes be rectangular $l'm'n'l''$...become the directing cosines of the axes of x', y' and z', and $\Gamma' = 1$. Also when both systems are rectangular, $l_1m_1n_1l_2$..coincide with $l'm'n'l''$..and $\Gamma = \Gamma' = 1$.

The last of these cases does not come within my plan, and it does not seem necessary to say any thing further upon the first, especially since it is sufficiently considered (or nearly so) in *Gregory's Solid Geometry*. I feel disposed however to put down the preceding formulas modified so as to be adapted to the second case. I shall have to use them in this form hereafter, and (the old axes being oblique) this case is that which will, I think, be most wanted. It is to be observed however, that this case includes the first by merely considering the axes of $x'y'z'$ to be the old ones, and those of xyz to be the new.

PROP. II. *To pass from oblique axes to rectangular.*

Let $l_1m_1n_1$; $l_2m_2n_2$; $l_3m_3n_3$ be the projecting coefficients of the new axes of x', y', z', and $\lambda_1\mu_1\nu_1$; $\lambda_2\mu_2\nu_2$; $\lambda_3\mu_3\nu_3$ the directing cosines of the same.

Then,
$$\left.\begin{array}{l} x = l_1x' + l_2y' + l_3z' \\ y = m_1x' + m_2y' + m_3z' \\ z = n_1x' + n_2y' + n_3z' \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (104).$$

$$\left.\begin{array}{l} x' = \lambda_1x + \mu_1y + \nu z \\ y' = \lambda_2x + \mu_2y + \nu_2z \\ z' = \lambda_2x + \mu_3y + \nu_3z \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (105).$$

$$\lambda_1\mu_2\nu_3 - \lambda_1\mu_3\nu_2 + \lambda_2\mu_3\nu_1 - \lambda_2\mu_1\nu_3 + \lambda_3\mu_1\nu_2 - \lambda_3\mu_2\nu_1 = \pm\,\Gamma \ldots\ldots\ldots (106).$$

$$\left.\begin{array}{l} l_1 = \pm\dfrac{1}{\Gamma}(\mu_2\nu_3-\mu_3\nu_2),\ l_2 = \pm\dfrac{1}{\Gamma}(\mu_3\nu_1-\mu_1\nu_3),\ l_3 = \pm\dfrac{1}{\Gamma}(\mu_1\nu_2-\mu_2\nu_1) \\ m_1 = \pm\dfrac{1}{\Gamma}(\lambda_3\nu_2-\lambda_2\nu_3),\ m_2 = \pm\dfrac{1}{\Gamma}(\lambda_1\nu_3-\lambda_3\nu_1),\ m_3 = \pm\dfrac{1}{\Gamma}(\lambda_2\nu_1-\lambda_1\nu_2) \\ n_1 = \pm\dfrac{1}{\Gamma}(\lambda_2\mu_3-\lambda_3\mu_2),\ n_2 = \pm\dfrac{1}{\Gamma}(\lambda_3\mu_1-\lambda_1\mu_3),\ n_3 = \pm\dfrac{1}{\Gamma}(\lambda_1\mu_2-\lambda_2\mu_1) \end{array}\right\} \ldots (107).$$

$$l_1m_2n_3 - l_1m_3n_2 + l_2m_3n_1 - l_2m_1n_3 + l_3m_1n_2 - l_3m_2n_1 = \pm\frac{1}{\Gamma} \ldots\ldots (108).$$

$$\left.\begin{array}{lll}\lambda_1 = \pm\Gamma(m_2n_3 - m_3n_2), & \mu_1 = \pm\Gamma(l_3n_2 - l_2n_3), & \nu_1 = \pm\Gamma(l_2m_3 - l_3m_2)\\ \lambda_2 = \pm\Gamma(m_3n_1 - m_1n_3), & \mu_2 = \pm\Gamma(l_1n_3 - l_3n_1), & \nu_2 = \pm\Gamma(l_3m_1 - l_1m_3)\\ \lambda_3 = \pm\Gamma(m_1n_2 - m_2n_1), & \mu_3 = \pm\Gamma(l_2n_1 - l_1n_2), & \nu_3 = \pm\Gamma(l_1m_2 - l_2m_1)\end{array}\right\}\dots(109).$$

$$\left.\begin{array}{l} l_1\lambda_1 + m_1\mu_1 + n_1\nu_1 = 1\\ l_2\lambda_2 + m_2\mu_2 + n_2\nu_2 = 1\\ l_3\lambda_3 + m_3\mu_3 + n_3\nu_3 = 1\end{array}\right\}\dots\dots(110).$$

$$\left.\begin{array}{l} \lambda_1 l_2 + \mu_1 m_2 + \nu_1 n_2 = 0\\ \lambda_1 l_3 + \mu_1 m_3 + \nu_1 n_3 = 0\\ \lambda_2 l_1 + \mu_2 m_1 + \nu_2 n_1 = 0\\ \lambda_2 l_3 + \mu_2 m_3 + \nu_2 n_3 = 0\\ \lambda_3 l_1 + \mu_3 m_1 + \nu_3 n_1 = 0\\ \lambda_3 l_2 + \mu_3 m_2 + \nu_3 n_2 = 0\end{array}\right\}\dots\dots(111).$$

$$\left.\begin{array}{l} l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3 = 1\\ m_1\mu_1 + m_2\mu_2 + m_3\mu_3 = 1\\ n_1\nu_1 + n_2\nu_2 + n_3\nu_3 = 1\end{array}\right\}\dots\dots(112).$$

$$\left.\begin{array}{l} l_1\mu_1 + l_2\mu_2 + l_3\mu_3 = 0\\ l_1\nu_1 + l_2\nu_2 + l_3\nu_3 = 0\\ m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = 0\\ m_1\nu_1 + m_2\nu_2 + m_3\nu_3 = 0\\ n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 = 0\\ n_1\mu_1 + n_2\mu_2 + n_3\mu_3 = 0\end{array}\right\}\dots\dots(113).$$

$$\left.\begin{array}{l} \lambda_1 = l_1 + hm_1 + gn_1\\ \mu_1 = hl_1 + m_1 + fn_1\\ \nu_1 = gl_1 + fm_1 + n_1\\ \lambda_2 = l_2 + hm_2 + gn_2\\ \quad etc.\end{array}\right\}\dots\dots(114).$$

Other relations may also be got from (12, 18, 19) and (22...26).

PROP. III. *To pass from any system of axes to another, when the equations to the new coordinate planes are given.*

Let the equations to the (new) coordinate planes of $y'z'$, $x'z'$, and $x'y'$ be

$$A_1x + B_1y + C_1z = D_1 \dots\dots (115),$$

$$A_2x + B_2y + C_2z = D_2 \dots\dots (116),$$

and, $$A_3x + B_3y + C_3z = D_3 \dots\dots (117).$$

Let $K_1^2 = (1-f^2)A_1^2 + (1-g^2)B_1^2 + (1-h^2)C_1^2$
$+ 2(gh-f)B_1C_1 + 2(fh-g)A_1C_1 + 2(fg-h)A_1B_1,$
$P = A_1B_2C_3 - A_1B_3C_2 + A_2B_3C_1 - A_2B_1C_3 + A_3B_1C_2 - A_3B_2C_1,$
$L_1^2 = (B_2C_3 - B_3C_2)^2 + (A_3C_2 - A_2C_3)^2 + (A_2B_3 - A_3B_2)^2$
$+2f(A_3C_2-A_2C_3)(A_2B_3-A_3B_2)+2g(B_2C_3-B_3C_2)(A_2B_3-A_3B_2)$
$+2h(B_2C_3-B_3C_2)(A_3C_2-A_2C_3)$, $L_2^2 =$ *etc.*, $L_3^2 =$ *etc.*;

also let $\alpha\beta\gamma$ be the values of xyz that satisfy (115, 116, 117) simultaneously, so that $\alpha\beta\gamma$ are coordinates of the new origin.

The equations (to the old axes) of the axis of x' are (116, 117),

$$\frac{x-\alpha}{B_2C_3-B_3C_2} = \frac{y-\beta}{A_3C_2-A_2C_3} = \frac{z-\gamma}{A_2B_3-A_3B_2},$$

hence, (14, 15), the projecting coefficients of the axis of x' are

$$\frac{B_2C_3-B_3C_2}{L_1},\quad \frac{A_3C_2-A_2C_3}{L_1} \text{ and } \frac{A_2B_3-A_3B_2}{L_1}.$$

Moreover, (52, 53, 115), the directing cosines of the normal to the plane of $y'z'$, are $\frac{\Gamma A_1}{K_1}$, $\frac{\Gamma B_1}{K_1}$ and $\frac{\Gamma C_1}{K_1}$; hence the cosine of the angle which this normal makes with the axis of x', is, (22),

$$\frac{\Gamma A_1(B_2C_3-B_3C_2)+\Gamma B_1(A_3C_2-A_2C_3)+\Gamma C_1(A_2B_3-A_3B_2)}{K_1L_1} = \frac{\Gamma P}{K_1L_1}.$$

Hence, taking the orthogonal projections of the broken lines $x'+y'+z'$ and $(x-\alpha)+(y-\beta)+(z-\gamma)$, (which terminate in the same points), on the normal to the plane of $y'z'$, we shall have

$$\frac{\Gamma P}{K_1L_1}x' = \frac{\Gamma A_1}{K_1}(x-\alpha) + \frac{\Gamma B_1}{K_1}(y-\beta) + \frac{\Gamma C_1}{K_1}(z-\gamma)$$
$$= (115),\ \frac{\Gamma(A_1x+B_1y+C_1z-D_1)}{K_1};$$

$$\left.\begin{aligned} \therefore\quad & \frac{P}{L_1}x' = A_1x + B_1y + C_1z - D_1 \\ \text{Similarly,}\quad & \frac{P}{L_2}y' = A_2x + B_2y + C_2z - D_2 \\ \text{and}\quad & \frac{P}{L_3}z' = A_3x + B_3y + C_3z - D_3 \end{aligned}\right\}\ \ldots\ldots (118).^*$$

These equations determine the new coordinates in terms of the old; and thence, by their solution, the old coordinates may be found in terms of the new.

July 3rd, 1849.

* If my memory does not deceive me, the formulas (118) are given, (the primitive axes being *rectangular*) in Monge's *Analyse*, but I have not at present the means of consulting that work.

ON CERTAIN PROPERTIES OF MERCATOR'S CHART,

MORE ESPECIALLY WITH REFERENCE TO THE CURVE FORMED BY THE PROJECTION OF A GREAT CIRCLE OF THE SPHERE UPON IT.

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The following paper, as its title imports, relates principally to the peculiar curve formed by the projection of a great circle of a sphere upon a Mercator's chart. It has been thought expedient, however, to commence with a short exposition of the principles of the projection of the sphere, with which the name of GERARD MERCATOR is associated, both because an acquaintance with the nature of the chart is necessary to the understanding of what follows, and because being placed here the reader is saved the inconvenience of reference to other works.

This is followed by a short comparison between this projection and the projection of the sphere upon an enveloping cylinder.

As to the notation employed, x is used to represent portions both of the equator on the globe, and on the corresponding projection of it; and to express that the true latitude of any point on the sphere, and the representation of this latitude on the meridian of the chart, are functions of each other, they are represented by Y and y respectively.

The increased attention paid by nautical men to the problem of great circle sailing, and the invariable use of Mercator's charts for laying down the track which a ship describes, are sufficient to invest the inquiry which forms the leading feature of this paper with some interest, and it is hoped also that some of the results will show that this curve has certain properties which render it worthy of attention as a pure mathematical figure; its simple arcs being always represented by an elliptic function of the first order.

In conclusion, will be found a few remarks on another curve usually given in illustration of these expressions.

On Mercator's projection of the sphere, the equator is represented by a right line, and the meridians also by right lines perpendicular to the equator, the equator being divided into equal parts as on the globe for the purpose of marking the longitudes of the places depicted on the chart. The condition requisite for the proper division of the meridians, for the determination of the latitudes of the various points on the chart, is, that the indefinitely small portions of the sphere, and the corresponding portions of the projections shall be every where similar, so that the angle formed by any two lines on one shall be equal to the angle formed by the corresponding lines on the other.*

If dp and dx respectively represent indefinitely small portions of a parallel and of the equator, between the same two meridians, and Y the latitude of the parallel; then

$$\frac{dp}{dx} = \cos \mathrm{Y}.$$

* The general problem of the projection of any one surface on another surface, so that these conditions may be fulfilled, was discussed by the celebrated Professor Gauss. A translation of this memoir was given in the Philosophical Magazine for August and September 1828. In it the general formulæ are applied to the particular cases of the projection of a plane upon another plane, the surface of a cone on a plane, of a sphere and spheroid on a plane, and of an ellipsoid of revolution on the surface of a sphere.

Now on Mercator's chart, dp and dx are represented by lines both equal to dx; if we call them dp' and dx, then

$$\frac{dp'}{dx} = 1, \text{ and therefore also } \frac{dp'}{dp} = \frac{1}{\cos Y}.$$

Consequently if dY and dy represent corresponding infinitesimal portions of the meridians adjacent to dp and dp' on the sphere and on the projection, it follows from the principle enunciated above, that

$$\frac{dy}{dY} = \frac{1}{\cos Y} = \sec Y \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From this the early table of "meridional parts," as they are called, was formed by the continual addition of the natural secants of the successive latitudes from 0 to 90°, assuming dy or the difference of the successive latitudes to be 1′ of a great circle of the sphere. But the formula usually employed is obtained as follows, let $z = \sin Y$, then

$$\frac{dz}{dY} = \cos Y, \text{ or } \frac{dY}{dz} = \frac{1}{\cos Y}.$$

And this being multiplied into equation (1) gives

$$\frac{dy}{dz} = \frac{1}{\cos^2 Y} = \frac{1}{1 - \sin^2 Y}, \text{ or } \frac{dy}{dz} = \frac{1}{1 - z^2}.$$

Integrating this expression, substituting sin Y for z, and making a few easy trigonometrical substitutions, we obtain

$$y = \log_e \cot \left(45° - \frac{Y}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Or if common logarithms are employed and multiplied by 7915·705, the product of the modulus of Briggs' logarithms, and the nautical miles in the radius of the earth, then

$$y \text{ in nautical miles} = 7915{\cdot}705 \log \cot (\tfrac{1}{2} \text{comp}^{t} \text{ of lat.}),$$

or $$\log y = 3{\cdot}898489 + \log (\log \cot \tfrac{1}{2} \text{colat.} - 10.)$$

As a practical illustration of the use of this formula, let it be required to find the projected latitude in miles corresponding to the true latitude 37°48′ on the sphere; the work stands thus,

Lat. = 37° 48′, colat. = 52° 12′, ½ colat. = 26° 6′,

Log cot 26° 6′ — 10 = 0·309897 log..........$\bar{1}$·491216

Constant log......................................3·898489

Mer. parts 2453 ....3·389705

So that the projected latitude is equal to 2453′, or 40° 88′, and this being taken from the divided right line which represents the equator, and measured upwards and downwards from the equator on any meridian; then lines drawn through the points thus ascertained, parallel to the equator, will represent on the projection the parallels of 37° 48′, N. and S. respectively.

The appearance of the chart thus constructed, closely resembles one which would be obtained by the projection of the points of the sphere upon

a cylinder equal to it in diameter, and having their axes coincident; the particulars of the spherical surface being transferred to the cylinder, where it is met by the radii produced, the cylinder being afterwards unrolled. In this projection the true latitude (Y) is represented by its tangent, or $\tan Y = Y'$, if Y' be taken for the projected latitude. Let us enquire how this agrees with Mercator's chart. Resuming equation (2) we evidently have

$$e^y = \cot\left(45^\circ - \frac{Y}{2}\right) = \frac{1+\tan\frac{1}{2}Y}{1-\tan\frac{1}{2}Y}; \quad \therefore \frac{e^y-1}{e^y+1} = \tan\tfrac{1}{2}Y.$$

And this value of $\tan\frac{1}{2}Y$ being substituted in the elementary formula $\tan Y = \dfrac{2\tan\frac{1}{2}Y}{1-\tan^2\frac{1}{2}Y}$, and reduced to its simplest form, gives

$$\tan Y = \tfrac{1}{2}\left(e^y - e^{-y}\right);$$

or the relations between $\tan Y$ and y may be represented thus:

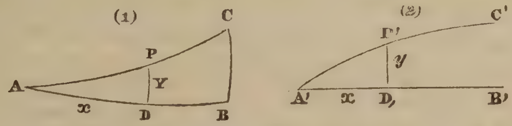
$$\tan Y = -\sqrt{-1}\sin\left(y\sqrt{-1}\right).$$

Or again, expanding $\sin(y\sqrt{-1})$, and multiplying by $-\sqrt{-1}$, the imaginary factors disappear, and

$$\tan Y = y + \frac{1}{2.3}y^3 + \frac{1}{2.3.4.5}y^5 + etc.$$

And this establishes the relation between the meridional divisions in the two projections contrasted. If $y = 1$, that is 57·29578 degrees of a great circle of the sphere, Y is easily found to be 49° 36′ 18″; but if $Y' = 1$, $Y = 45°$, so that these true latitudes are respectively represented, on the two projections, by lines equal to the radius of the sphere.

If AB represent a quadrant of the equator, and AC a quadrant of any other great circle of the sphere, the position of any point P in AC may be determined by the coordinates AD, DP formed by drawing a perpen-



dicular from P on AB; these, in the remarks which follow will be represented by x and Y, x is the longitude of P estimated from the intersection of the circles, and Y is the latitude of the point; and angle A is measured by the greatest latitude through which the circle AC passes.

If in the right angled triangle ADP, A be known, then with successive values of x from 0 to 90°, corresponding values of Y can be computed from the equation obtained from the right angled triangle, viz.

$$\tan Y = \sin x \tan A \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now on Mercator's projection, x is represented by a right line equal in length to the arc x, and the projected value of Y corresponding to the successive values of x, may be determined by equation (2), or may be taken in minutes of arc from a table of meridional parts, such as is found in every collection of tables for nautical purposes.

The projected value of Y being represented by y; then
$\tan Y = \sin x \tan A$......(3), and $y = M \log \cot (45° - \frac{1}{2}Y)$......(2)
furnish the means of computing x and y, the rectilinear coordinates of the curve on Mercator's chart, which corresponds to the great circle AC. Equation (3) is the equation to the great circle itself on the sphere, and M may serve both to reduce the common to Naperian logs, and to express y in units corresponding with those in which x is estimated.

If y be then measured from the divided line which represents the equator, and applied to its proper meridian which is determined by the value of x, a series of points on the required projected curve are most easily obtained.

It is next proposed to find the expression for the radius of curvature. The similarity of the indefinitely small parts of the sphere, and their representation on the chart, gives

$$\frac{dy}{dx} = \frac{dY}{dp},$$

where dp is an elementary part of a parallel of latitude on the sphere.

Now $\frac{dY}{dp} = \cot P$; $\therefore \frac{dy}{dx} = \cot P$, and differentiating with respect to x, gives

$$\frac{d^2y}{dx^2} = -\operatorname{cosec} P . \frac{dP}{dx}.$$

And by the spherical triangle APD, $\cos P = \cos x \sin A$; therefore $\frac{dP}{dx} \sin P = \sin x \sin A$; but $\sin x = \sin AP \sin P$; therefore $\frac{dP}{dx} = \sin AP \sin A = \sin Y$; hence $\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 P \sin Y$; and also $1 + \frac{dy^2}{dx^2} = 1 + \cot^2 P = \operatorname{cosec}^2 P$. Substituting these values in the expression for the radius of curvature, we have, since $\sin P = \frac{\cos A}{\cos Y}$,

$$r = \frac{1}{\sin P \sin Y} = \cot Y \sec A \text{ ....} (6).$$

Consequently at the vertex, or point of greatest latitude, where $A = Y$, $r = \operatorname{cosec} A$; and as Y diminishes from A to 0, the radius increases from cosec A to ∞. When $A = 90°$, $r = \infty$, and when $A = 0$, $Y = 0$, and $r = \infty$. In the first of these two cases, the curve is the meridian, and in the second it coincides with the equator, which are both represented by right lines.

To determine the coordinates of the evolute, the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ must be substituted in the equations

$$\left.\begin{aligned} (y-\beta)\frac{dy}{dx} + (x-\alpha) &= 0, \\ \text{and } (y-\beta)\frac{d^2y}{dx^2} + \left(\frac{dy^2}{dx^2}+1\right) &= 0, \end{aligned}\right\}$$

which thus become

$$\left.\begin{array}{r}(y-\beta)\cot P+(x-a)=0\\ \text{and }-(y-\beta)\sin Y\,\text{cosec}^2 P+\text{cosec}^2 P=0\end{array}\right\}.$$

From the second of these we get

$$y-\beta=\frac{1}{\sin Y}=\text{cosec } Y,\ \text{ or } \beta=y-\text{cosec } Y \ldots\ldots\ldots\ldots(7),$$

and from the first

$\frac{\cot P}{\sin Y}+x=a$, but $\sin Y=\tan x\cot P$, by the spherical triangle,

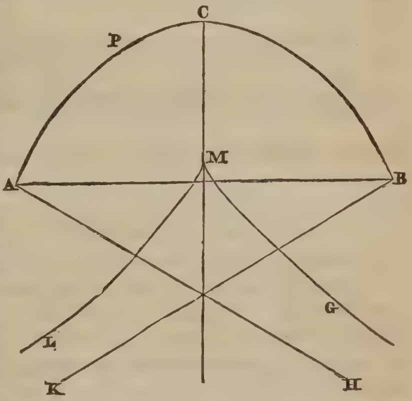
$$\therefore a=x+\cot x\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Thus the coordinates of the evolute are determined. It is to be observed that a is reckoned on the axis of x from the origin, and β on the axis of y, that is on the meridian; and if from any point x_1y_1 in the curve there be taken in the direction of the parallel a line $=\cot x$, and from its extremity in the direction of the meridian, another line be measured $=-$ cosec Y, the point thus ascertained is the centre of curvature of the point x_1y_1. Hence also it follows that

$$r^2=\text{cosec}^2 Y+\cot^2 x=\cot^2 Y+\text{cosec}^2 x.$$

The accuracy of these results is easily verified by comparison with the value of r in (6).

By (8) as x varies from 0 to $\frac{\pi}{2}$, a varies from ∞ to $\frac{\pi}{2}$; and by (7) as Y varies from 0 to A (these correspond to the limits assigned to x) β changes from $-\infty$ to y_1- cosec A, y_1 representing the value of y when Y becomes A, and as the quadrants of the curve only differ in position, it is easily seen that the form of the evolute is such as represented in the annexed diagram, where AC is a quadrant of the projected curve, AB the equator and axis of x; MG and ML branches of the evolute corresponding to AC and CB; and the straight lines AH and BK which make with AB angles equal to the complement of the greatest latitude are asymptotes to MG and ML.



With respect to the length of the projected curve,—let ds represent an element of it, and dz an element of the great circle itself; then it follows that $\frac{dz}{dp}=\frac{ds}{dx}$, dp and dx, as before, being elementary portions of a parallel of latitude on the sphere, and on the projection respectively; but $\frac{dz}{dp}=\text{cosec } P$, therefore $\frac{ds}{dx}=\text{cosec } P=\frac{1}{(1-\cos^2 P)}$, and since $\cos P=\sin A\cos x$, by the spherical triangle APD, we have

$$\frac{ds}{dx}=\frac{1}{(1-\sin^2 A\cos^2 x)^{\frac{1}{2}}}\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Now $\cos x = \sin(90° - x) = \sin\phi$, say; then $-\dfrac{dx}{d\phi}\sin x = \cos\phi$, therefore $-\dfrac{dx}{d\phi} = 1$. Multiplying this and equation (8), and making $\sin^2 A = k^2$,

$$-\frac{ds}{d\phi} = \frac{1}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}}.$$

Here ϕ is the longitude reckoned from the meridian of the vertex, and the sign we attach to it is quite arbitrary; observing then that $\{\sin(-\phi)\}^2 = \sin^2\phi$, we have

$$\frac{ds}{d\phi} = \frac{1}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}}, \text{ and } s = \mathrm{F}k\phi;$$

and thus the arcs of this curve are constantly expressed by an elliptic function of the first order, no constant being required.

It has been shewn that $s = \int_0^\phi \frac{d\phi}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}}$; hence when $k = 1$, or the maximum latitude $= 90°$,

$$s = \int_0^\phi \frac{d\phi}{\cos\phi} = \log_e \cot\left(45 - \frac{\phi}{2}\right);$$

a result which is certainly remarkable when viewed in connection with the fundamental equation to Mercator's projection.

From this equation too, we can readily obtain that of the curve to which it belongs, for assuming the vertex for the origin (the pole in this case), ϕ is one of its coordinates in common with all the curves of which it is the limiting form.

Let y_1, then, represent the other coordinate, taken on the meridian of the vertex; then

$$\frac{ds}{d\phi} = \left\{\frac{dy_1^2}{d\phi^2} + 1\right\}^{\frac{1}{2}} = \frac{1}{\cos\phi};$$

$$\therefore \quad \frac{dy_1}{d\phi} = \tan\phi, \quad \text{and} \quad y_1 = \log_e \sec\phi.$$

which is the equation to the curve required. It is evident when $\phi = 90°$, y_1 is infinite.

The radius of curvature and coordinates of the evolute determined as before are as follows.

The radius of curvature $= \sec\phi$.

Coordinates $\begin{cases} \text{From the pole on the meridian of the vertex } \beta = 1 + y_1 \\ \text{Parallel to the equator} \ldots\ldots\ldots\ldots\ldots\ldots \alpha = \phi - \tan\phi. \end{cases}$

The last being independent of the meridional divisions, is the same for all the curves, and coincides with the value of a deduced before.

* And since when $\cos\phi = \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2(1 - k^2\sin^2\phi)^{\frac{1}{2}}$; $\mathrm{F}\phi_1 + \mathrm{F}\phi_2 = \mathrm{F}\phi$; then if $\phi_1 = \phi_2$, and $\phi = 90$, we have

$$0 = \cos^2\phi_1 - \sin^2\phi_1(1 - k^2)^{\frac{1}{2}};$$

* See article, Definite Integrals, in the Encyclopedia Metropolitana, by the Rev. H. Moseley, F. R. S.

and substituting for k^2 its value $\sin^2 A$, we obtain from this

$$\cos A = \cot^2\phi_1 \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

The value of ϕ deduced from this corresponds to the middle point of the projected quadrant, for

$$2F\phi_1 = F_1 k = \text{curve from } \phi = 0 \text{ to } \phi = 90^\circ.$$

Also on the sphere, if the quadrant AC is bisected in P, then $\cos A = \tan x \cot AP$, or $\cos A = \cot\phi_3$, if we represent this particular value of $90^\circ - x$, or ϕ by ϕ_3.

There is a remarkable relation of the maximum latitude with the true latitude and longitude of the middle point of the projected quadrant.

It has been shewn that if P be the middle point of the projected quadrant, then $\cos A = \tan^2 x$, or *the cosine of maximum latitude is equal to the square of the tangent of the longitude from the point of intersection in the equator.* Now for the corresponding point on the sphere, if Y is the true latitude of the point, we have

$$\tan Y = \sin x \tan A, \text{ or } \tan^2 Y = \sin^2 x \tan^2 A.$$

But from the equation $\cos A = \tan^2 x$, we get $\frac{1}{\cos A} = \cot^2 x$, and adding 1 to both sides,

$$\frac{1 + \cos A}{\cos A} = \operatorname{cosec}^2 x = \frac{1}{\sin^2 x}; \quad \therefore \sin^2 x = \frac{\cos A}{1 + \cos A};$$

hence $$\tan^2 Y = \sin^2 x \tan^2 A = \frac{\cos A}{1 + \cos A} \cdot \frac{\sin^2 A}{\cos^2 A} = \frac{\sin A}{1 + \cos A} \cdot \frac{\sin A}{\cos A};$$

$$\text{or } \tan^2 Y = \tan A \tan \frac{A}{2}; \quad \therefore 1 + \tan^2 Y = 1 + \tan A \tan \frac{A}{2},$$

or $\sec^2 Y = \frac{1}{\cos A}$; therefore $\cos A = \cos^2 Y = \tan^2 x$, from above; hence the cosine of the true latitude of the middle point is equal to the tangent of the longitude from the point where the curve meets the equator, and the square of each is equal to the cosine of the maximum latitude. It can be easily shewn also, that *if the apex of the curve is placed in any parallel, with the branches of the curve upwards, it is the projection also of a less circle which passes through the pole.*

It may be interesting to give the computed elements of one of these curves, the following belong to the projection of a great circle of the sphere, the maximum latitude (A) of which is 60°.

They are given in degrees for the sake of the facility this affords in measuring them from the divided equatorial line.

The 2nd column contains the true latitudes of those points of the circle which correspond to the successive values of x, recorded in the 1st.

The 3rd gives the corresponding values of the projected latitudes, or meridional parts.

The 4th, 5th, and 6th are the coordinates of the evolute and radius of curvature.

x	True Latitudes.	Projected Latitudes.	a	β	γ
5°	8° 35′	8°·616	659°·9	—375·284	759·09
10	16 44	16 ·976	334 ·94	—182·484	381·00
15	24 9	24 ·898	228 ·83	—115·142	255·62
20	30 39	32 ·226	177 ·42	—80·164	193·44
25	36 12	38 ·881	147 ·87	—58·010	156·54
30	40 54	44 ·894	129 ·239	—42·616	132·32
35	44 48	50 ·216	116 ·827	—31·097	115·35
40	48 4	54 ·958	108 ·283	—22·060	102·93
45	50 46	59 ·110	102 ·296	—14·860	93·564
50	53 0	62 ·729	98 ·077	—9·013	86·365
55	54 49	65 ·814	95 ·119	—4·290	80·766
60	56 18	48 ·437	93 ·080	—0·432	76·394
65	57 30	70 ·634	91 ·719	+2·699	72·999
70	58 26	72 ·394	90 ·889	+5·148	70·405
75	59 8	73 ·745	90 ·352	+6·995	68·493
80	59 37	74 ·694	90 ·103	+8·285	67·180
85	59 54	75 ·256	90 ·013	+9·029	66·412
90	60 0	75 ·456	90 ·000	+9·296	66·160

Also by formula (9) the value of x for the middle point = 35° 16′.

Below also is the whole computation required for the coordinates of the curve which represents the great circle when the greatest latitude is 40° 40′; y being taken from a table of meridional parts.

$$A = 40° 40'; \quad \log \tan A = 9{\cdot}934056.$$

x	log sin x	log tan Y	Y	y
5°	8·940296	8·874352	4° 17′	257′ = 4°·31
10	9·239670	9·173726	8 29	511 = 8·51
15	9·412996	9·347052	12 33	759 = 12·65
20	9·534052	9·468118	16 22	996 = 16·60
25	9·625948	9 560004	19 57	1222 = 20·37
30	9·698970	9·633026	23 15	1435 = 23·91
35	9·758591	9·692647	26 14	1632 = 27·2
40	9·808067	9·742123	28 54	1813 = 30·21
45	9·849485	9·783541	31 17	1976 = 32·91
50	9·884254	9·818310	33 21	2125 = 35·41
55	9·913365	9·847421	35 8	2254 = 37·5
60	9·937531	9·871587	36 39	2366 = 39·43
65	9 957276	9·891332	37 54	2461 = 41·01
70	9·972986	9·907042	38 55	2538 = 42·3
75	9·984944	9·919000	39 42	2599 = 43·31
80	9·993351	9·927407	40 14	2641 = 44·0
85	9·998344	9·932400	40 34	2667 = 44·45
90			40 40	2675 = 44·58

On this it is only necessary to remark that the logs in the column marked log tan x are added severally to log tan A, and the sums form the column marked log tan Y.

The curve which is touched by all the perpendiculars to the extremities of the successive radii vectores of an ellipse, is known to possess this property, that the *difference between any arc and that part of the tangent intercepted by the curve and the ellipse*, can be expressed by an elliptic function of the first order; the coordinates of this curve are

$$\left.\begin{aligned} x &= \frac{\cos\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}\{1-2k^2\sin^2\phi\} \\ y &= \frac{\sin\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}\{1+k^2-2k^2\sin^2\phi\} \end{aligned}\right\}.$$

The axis of x coinciding with the semi-minor axis of the ellipse, and that of y with the major-axis, the centre being taken as the origin; k also is the ratio of the eccentricity to the semi-axis major, and ϕ the angle between the radius vector and the semi-axis minor. The semi-axis minor is also considered as unity, and k of course less.

This curve in ellipses of small eccentricity is of a simple form, touching the ellipse internally at the extremities of the principal axes; and when $k=0$ they coincide, both then become a circle.

But its form is materially changed when the eccentricity is greater than the semi-axis minor; it then cuts the major-axis and forms a cusp whose second branch touches the ellipse externally at the extremity of the major-axis. It is evident that x vanishes when $\cos\phi=0$, or when $1-2k^2\sin^2\phi=0$, that is, when $\phi=90°$, or when $\sin\phi=\dfrac{1}{k\sqrt{2}}=\dfrac{\frac{1}{2}\sqrt{2}}{k}$; now this value of $\sin\phi$ is possible whenever k is not less than $\frac{1}{2}\sqrt{2}$, or sin 45°; and when $k=\frac{1}{2}\sqrt{2}$, the minor semi-axis equals the eccentricity.

Whenever k is greater than $\frac{1}{2}\sqrt{2}$, x vanishes before ϕ becomes 90°, and again on ϕ attaining 90°, when y always equals the semi-axis major, as the equations plainly shew.

By differentiating the equations, and reducing, we find the following expressions:—

$$\frac{dx}{d\phi}=\frac{-\sin\phi}{(1-k^2\sin^2\phi)^{\frac{5}{2}}}(1-3k^2+2k^4+4k^2\cos^2\phi-2k^4\cos^2\phi)=-\text{Q}\sin\phi \;..(m),$$

$$\frac{dy}{d\phi}=\frac{\cos\phi}{(1-k^2\sin^2\phi)^{\frac{5}{2}}}(1-3k^2+2k^4+4k^2\cos^2\phi-2k^4\cos^2\phi)=\text{Q}\cos\phi \ldots(n),$$

$$\therefore\ \frac{dx}{dy}=-\tan\phi, \text{ and } \frac{d^2x}{dy^2}=-\frac{1}{\cos^2\phi}\cdot\frac{d\phi}{dy}=-\frac{1}{\text{Q}\cos^3\phi}\;\ldots\ldots\ldots\ldots(o).$$

$$\text{And also}\;\ldots\ldots\;\frac{d^2y}{dx^2}=\ldots\;\ldots\ldots\ldots-\frac{1}{\text{Q}\sin^3\phi}\ldots\ldots\ldots\ldots(p).$$

If $\text{Q}=0$, $\cos^2\phi=\dfrac{3k^2-2k^4-1}{4k^2-2k^4}$, this gives the value of ϕ at the cusp,

where also

$$x = \frac{2}{3\sqrt{3}.k} \cdot \frac{(2k^2 - 1)^{\frac{3}{2}}}{1 - k^2}, \text{ and } y = \frac{2}{3\sqrt{3}.k} \cdot \frac{(1 + k^2)^{\frac{3}{2}}}{(1 - k^2)^{\frac{1}{2}}}.$$

When $k = \sin 60° = \frac{1}{2}\sqrt{3}$, the ellipse has then its eccentricity $= \sqrt{3}$, and its semi-axis major $= 2$, x vanishes when $\sin\phi = \frac{\frac{1}{2}\sqrt{2}}{k} = \sqrt{\frac{2}{3}}$, and this makes $y = \sqrt{3}$ = the eccentricity. Hence in this particular case, the curve passes through the focus, and at the cusp $\phi = 75° \; 2' \; 10''$.

All circles of the sphere are included in one or other of the three following classes: viz.

1. Those whose circumferences do not include one of the poles (fig. 1.)
2. Those whose circumferences pass through one of the poles.
3. Those whose circumferences include one of the poles (fig. 2.) To this class great circles belong.

Fig.1

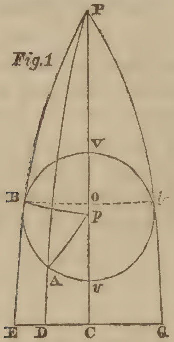
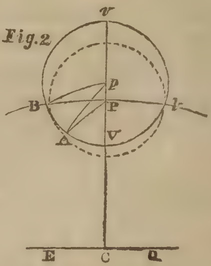


Fig.2



In each of the figures P represents one of the poles of the earth, and p the pole of the circles BVbv, the projections of which are required; V the point nearest to the pole, is called the vertex, and PVC the meridian of the vertex. The longitude of any point A from the meridian PVC and its latitude, are considered as the spherical coordinates of the point.

1. To begin with fig. 1. It is evident there is a maximum longitude east and west of the meridian of the vertex where the meridians PBE and PbQ touch the circle; and its value is readily found when we know pB and pP, by the right-angled triangle BpP, thus,

$$\sin \text{EC or } \sin \text{CPE} = \frac{\sin p\text{B}}{\sin p\text{P}}.$$

And the latitude also of the points B, b, to which this longitude belongs, is determined by this equation,

$$\sin \text{BE} = \frac{\cos p\text{P}}{\cos p\text{B}}.$$

Let EC be represented by X, and any less longitude, as CD, by x, and $\angle$ BAP between the less circle, and the meridian by θ, then

$$\frac{\sin pA}{\sin pP} = \frac{\sin pB}{\sin pP} \text{ or } \frac{\sin pPA}{\sin pAP} = \sin CPE,$$

$$\text{or } \frac{\sin x}{\cos\theta} = \sin X, \text{ and } \frac{1}{\cos\theta} = \frac{\sin X}{\sin x}.$$

Now in Mercator's projection the angle BAP or θ is preserved of the same magnitude on the plane as it is on the sphere; and therefore representing by dy an element of the rectilinear meridians on the chart, and by ds an element of the projected curve, we have

$$\frac{ds}{dy} = \frac{1}{\cos\theta} = \frac{\sin X}{\sin x} \text{ or } \left\{\frac{dx^2}{dy^2}+1\right\}^{\frac{1}{2}} = \frac{\sin X}{\sin x};$$

$$\text{whence} \qquad -\frac{dy}{dx} = \frac{\sin x}{(\sin^2 X - \sin^2 x)^{\frac{1}{2}}}.$$

The negative sign is taken, because as the longitude from the vertex increases, the latitude diminishes.

And integrating $y = \log_e C.\{\cos x + (\cos^2 x - \cos^2 X)^{\frac{1}{2}}\}$.

x has of course the same signification on the chart as on the sphere, with however this addition, that any of the parallels may be taken as the axis of x, since they are all made equal to the equator. Taking the parallel which corresponds to BOb, when x becomes X, $y = 0$, for OB is represented by X on the chart;

$$\text{whence} \qquad C = \frac{1}{\cos X}, \text{ and } y = \log_e \left\{\frac{\cos x}{\cos X} + \left(\frac{\cos^2 x}{\cos^2 X} - 1\right)^{\frac{1}{2}}\right\}.$$

Or putting $\dfrac{\cos x}{\cos X} = \sec x_1$; $y = \log_e\{\sec x_1 \pm \tan x_1\}$ according as x_1 is taken $+$ or $-$; or $y = \pm \log_e \cot(45° - \frac{1}{2}x)$;

Also if x be taken $\pm$ the formulæ remain unchanged. Hence in projecting this circle on Mercator's chart, we obtain a curve symmetrically placed about the point which corresponds to O, and of an elliptical form.

When $x = 0$, $x_1 = X$, and $Y = \pm \log_e(\cot 45° - \frac{1}{2}X)$; hence the least and greatest semi-axes of the projected figure are the maximum longitude X and the *meridional parts* corresponding to the same.

It is observable also, that X, x and x_1 are related to each other in the same manner as the hypothenuse, base and perpendicular of a right-angled spherical triangle.

2. When the circle to be projected includes one of the poles. In fig. 2, vPV being the meridian which passes through the points of greatest and least latitudes of the circle vBVb, let PB be the meridian which cuts PC at right-angles. Join pB, and take any point A in the circle bBV, and join pA, PA. Call $<p$BP, a; pAP, θ; and CPA, the longitude of A from CP, x as before, then $\dfrac{\sin pP}{\sin pB} = \sin a$; and sin lat. of B $= \dfrac{\cos Bp}{\cos Pp}$; also

$$\frac{\sin pP}{\sin pB} = \frac{\sin pP}{\sin pA}, \text{ or } \sin a = \frac{\sin\theta}{\sin x} \quad \therefore \sin\theta = \sin a . \sin x.$$

Now in the projection of this circle, $\frac{ds}{dy} = \frac{1}{\cos \text{PAV}} = \frac{1}{\sin\theta} = \frac{1}{\sin a .\ \sin x}$.

Whence, as before, we determine $-\frac{dy}{dx} = \frac{1}{(\operatorname{cosec}^2 a - \sin^2 x)^{\frac{1}{2}}}$;

an expression of the same form as before, $\operatorname{cosec}^2 a$ standing in place of $\sin^2 X$; and integrating this, making the parallel which corresponds to that of B the axis of x, which consists in introducing the constant so that y may vanish when $x = 90°$, we have

$$y = + \log_e \{ (1 + \tan^2 a \cos^2 x)^{\frac{1}{2}} + \tan a \cos x \};$$

or putting $\tan a . \cos x = \tan x_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$

$$y = + \log_e \cot \{45° - \tfrac{1}{2} x_1\},$$

x_1 is $+$ or $-$ according as $\cos x$ is $+$ or $-$, that is as x is less or greater than 180°; and according as x_1 is $\pm$

$$y = \pm \log_e \cos \{45° - \tfrac{1}{2} x_1\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and for values of x differing by 180° we have equal and opposite values of y.

In fact, the projected form of this circle is the same as that which results from the projection of a great circle, as may be immediately seen by comparing the equations (1) and (2) with those before given for the great circle.

Thus, too, the curves will have the same form at the limit, when they pass through the pole; therefore all circles which pass through the pole are represented by the equations before given of the curve to which the projected great circles tend as their highest latitude increases.

Thus there are three forms assumed by the projected circles of the sphere according as they include, do not include, or pass through one of the poles.

1. The first similar to the projection of a great circle, a waving form, symmetrically placed above and below a certain parallel of latitude, the latitude of which can be easily determined.
2. The second an elliptical form, the arcs BVv and Bvb (fig. 1,) being represented as equal in every respect above and below the right line which represents the parallel BOb.
3. The third a curve of two infinite branches, the vertex being placed on the parallel of the point of least latitude. The meridians 90° distant from the meridian of the vertex being asymptotes of the curve.

It is readily seen from the value of $\frac{dy}{dx}$, deduced before, for curves of the second form, namely,

$$\frac{dy}{dx} = \frac{-\sin x}{(\sin^2 X - \sin^2 x)^{\frac{1}{2}}} \text{ that, } \left\{ \frac{dy^2}{dx^2} + 1 \right\}^{\frac{1}{2}} = \frac{ds}{dx} = \frac{1}{\left(1 - \frac{\sin^2 x}{\sin^2 X}\right)^{\frac{1}{2}}}.$$

And because $\frac{\sin x}{\sin X}$ is necessarily less than 1, let $\frac{\sin x}{\sin X} = \sin\phi$, or $\sin x = \sin X \sin \phi$, or if the constant $\sin X$ is represented by k, $\sin x = k \sin\phi$; then making the necessary substitutions,

$$\frac{ds}{d\phi} = \frac{k}{(1 - k^2 \sin^2\phi)^{\frac{1}{2}}} \qquad \text{or,} \quad s = k . \mathrm{F} k \phi.$$

And thus the length of these curves is also expressed by an elliptic function of the first order.

Lastly, the radius of curvature and coordinates of the evolute are easily found, and are represented by the following remarkably simple expressions; viz.

Radius of curvature $$r = \frac{\sin X}{\cos x} \dots\dots (1)$$

Abscissa of evolute estimated on the parallel of the points where the longitude is greatest, X,

$$\alpha = x - \tan x \dots\dots (2),$$

Ordinate on the meridians,

$$\beta = y - \sin x \dots\dots (3).$$

From the expression for the radius of curvature, it is evident that it varies from sin X to tan X, or from the sine to the tangent of the maximum longitude.

It is also remarkable that the expression for α is common to all three forms of these curves.

ON EQUAL TANGENTS TO CURVES OF THE SECOND ORDER.

[*Mr. Fenwick.*]

To find the condition that the two tangents drawn from the point (α, β) *to the parabola* $y^2 = 4mx$, *may be equal.*

The equation of the chord of contact corresponding to the point (α, β) in reference to the parabola $y^2=4mx$, is (see any treatise on the conic sections),

$$\beta y = 2m(x+\alpha) \dots\dots (1).$$

Hence the equation of the perpendicular to this line from the point (α, β) is

$$y - \beta = -\frac{\beta}{2m}(x - \alpha) \dots\dots (2).$$

Eliminating y between (1) and (2), there results for the abscissa of the point of intersection,

$$x = \frac{2m\beta^2 + \alpha\beta^2 - 4\alpha m^2}{\beta^2 + 4m^2} \dots\dots (3).$$

In order therefore that the two tangents drawn from the point (α, β) to the parabola $y^2 = 4mx$, may be equal, the point of which x in (3) is the abscissa, *must be the middle of the chord of contact;* that is, if x_1, x_2, be the abscissæ of the points of contact,

$$x_1 + x_2 = 2x = \frac{2(2m\beta^2 + \alpha\beta^2 - 4\alpha m^2)}{\beta^2 + 4m^2} \dots\dots (4).$$

Again, eliminating y between (1) and the equation of the curve $y^2=4mx$, we get for the abscissæ of the points of contact,

$$mx^2 + (2\alpha m - \beta^2)x + m\alpha^2 = 0.$$

Whence by the theory of roots,

$$x_1 + x_2 = \frac{\beta^2 - 2\alpha m}{m} \dots\dots (5).$$

Comparing (4) and (5) we get finally,

$$(\beta^2-2am)(\beta^2+4m^2)=2m(2m\beta^2+\alpha\beta^2-4am^2),$$

or $$\beta^2(\beta^2-4am)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

This equation is satisfied either by $\beta=0$, or $\beta^2=4am$; the latter is the locus of the point (α,β) when the two tangents coalesce in the point of contact, that is, when (α,β) is in the curve, and consequently the part of the tangent between (α,β) and the point of contact vanishes; the former or $\beta=0$ (which evidently denotes the axis of x) is the required condition. Hence *two equal tangents can only be drawn to a parabola from a given point without it when that point is in the principal axis produced.*

Cor. The condition corresponding to (6) in reference to the ellipse $a^2y^2+b^2x^2=a^2b^2$, is

$$\alpha\beta^2(a^2-b^2)(a^2\beta^2+b^2\alpha^2-a^2b^2)=0,$$

which is satisfied by

$$a^2=b^2,\quad a^2\beta^2+b^2\alpha^2=a^2b^2,\quad \alpha=0,\ \text{or}\ \beta=0.$$

The first shews that in the particular case of the ellipse when $a=b$, that is, when the ellipse becomes a *circle*, two equal tangents can be drawn to the circle from any point without it, a well known property; the second is the locus of (α,β) when it is in the curve: the third or $\alpha=0$, which denotes the axis of y, shews that a pair of equal tangents may be drawn to the ellipse from any point in the axis of y without the curve; and similarly for the fourth which denotes the axis of x. Hence *two equal tangents can only be drawn to the ellipse from a given point without it when that point is in the major or minor axis produced.* A similar property is also easily shewn to belong to the hyperbola.

Scholium.—The preceding investigation whilst it exhibits the very comprehensive character of analytical or coordinate geometry as an instrument of research, points out at the same time the necessity of retaining and of subsequently examining, every factor that may enter into an investigation. If in the preliminary reductions we had suppressed the factor α or β^2, we should not have possessed the means at last of determining the most important properties of the locus. On the other hand, every case that can possibly arise is indicated, both in reference to the parameters of the curves and the position of the point from which the equal tangents are to be drawn.

June 15*th*, 1850.

INVESTIGATION OF EXERCISE 150.

[*The Rev. Thomas Gaskin, M. A., Jesus College, Cambridge.*]

Prop. I. *Let* AB, BC *be any two chords in a conic section, which pass through the fixed points* P *and* Q *respectively: draw* AD *parallel to* PQ, *and join* DC *cutting* PQ *in* R: *then* R *will be a fixed point.*

From Q draw any arbitrary but fixed line QE, cutting the curve in E, F; and, in the ellipse and hyperbola, let OG, OH, OK be semidiameters drawn parallel to AD, BC, EQ respectively; and in the parabola let OG, OH, OK be chords drawn through the focus parallel respectively to AD, BC, EQ.

Then in the ellipse and hyperbola,

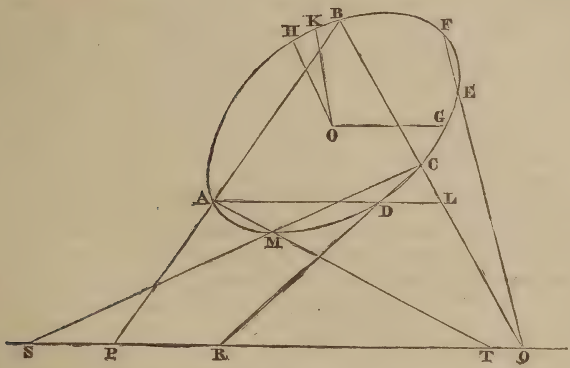
$$QR : QC :: DL : LC,$$

$$QP : QB :: AL : LB; \text{ whence}$$

$$QR.QP : QB.QC :: DL.AL : BL.CL :: OG^2 : OH^2.$$

But $$QB.QC : QE.QF :: OH^2 : OK^2,$$

and hence $$QR.QP : QE.QF :: OG^2 : OK^2.$$



From this it follows that QR.QP is a given rectangle, and QP is a given line; and, therefore, that R is a given point.

In like manner, in the parabola, QR.QP : QE.QF :: OH : OK; and hence in this case also R is a given point.

PROP. II. *If three sides of a quadrilateral figure inscribed in a conic section always pass through three points situated in a straight line, the fourth side will also pass through a fixed point situated in the same straight line.*

Let AB, BC, CM pass through the three points P, Q, S in the same straight line. Draw AD parallel to PQ, and join DC meeting PQ in R.

Then *(prop. i.)* R is a fixed point.

Again, MC, CD are drawn through two fixed points S, R, and AD is parallel to RS; hence, again *(prop. i.)* AM passes through a fixed point T in the line RS, or PA.

PROP. III. *If a hexagon be inscribed in a circle, and five of its sides pass through five fixed points in the same straight line, then the sixth side will always pass through a fixed point in the same straight line.*

Draw any one of the diagonals of the hexagon, then this diagonal being taken as a side of each quadrilateral, the hexagon itself is divided into two quadrilaterals.

Since the three sides of the first quadrilateral pass through given points in the straight line, the fourth side (diagonal of the hexagon) will also pass through a fixed point in that line. This diagonal and two of the sides of the quadrilateral passing through fixed points in that line, the remaining side (or sixth side of the hexagon) will also pass through a fixed point in that line. In the same way as in the preceding case, the corresponding property is readily proved for the hexagon, the decagon, *etc.*, till we arrive at a polygon of any given *even* number of sides.

It will follow from what has been proved that the proposed problem is always either indeterminate or impossible. That is :—if the $2n^{\text{th}}$ point have a specific position, depending on the positions of the other and upon the magnitude and position of the conic section, innumerable quadrilaterals (or $2n$-gons) may be drawn to fulfil the conditions. If, on the other hand, the fourth (or $2n^{\text{th}}$) point have a position in the straight line different from that specific one above mentioned, no quadrilateral (or $2n$-gon) can exist capable of fulfilling the conditions of the problem.

These, however, are not the only cases of indetermination in the general problem. Conditions have been assigned by Mr. Gaskin amongst the points which will convert the problem generally into a porism : but they are omitted here owing to their forming part of a general investigation which ought not to be broken up into fragments. They will, however, be published hereafter : but it is due to that gentleman here to state—that in the algebraic investigation he has succeeded in shewing how the indeterminate and imaginary cases are connected together, in a very remarkable manner. Any very clear account of it would take more room than can be here spared.*

I may moreover remark here, as an interesting fact, that this theorem in all its generality may be at once deduced from Desargues' general theorem on *Involution*—provided the lines in which the points are situated either cut or touch the conic section. For the present hypothesis can only vary one single point in Desargues' line, and as that point *must fulfil* the conditions of involution, it can have only one single position for all deformations of the polygon, only keeping it subject to the general hypothesis.

When the line neither cuts nor touches the conic section, the theorem of Desargues requires modification to adapt it to this demonstration. Two modes of doing this present themselves.

(α). To establish Desargues' theorem as applicable to the intersections of the line with the sides of the inscribed polygon, independently of its intersections with the conic section. This is not only true but easily proved.

(β). To find two points in the line itself which will represent the intersections with a subsidiary conic when the line neither cuts nor touches the given one ; and which, when it does touch, merge in the point of contact, and

* It is due to Mr. Gaskin to state that what I have here called "an investigation" of a special theorem, is only a very limited exposition of his general method – that being all for which space could be allowed me here. For the very *form* in which it is now given I am responsible; and I feel it to be due to my esteemed friend to say, that it presents but a very imperfect idea of his careful and masterly discussion of the general principle—and none whatever of the more delicate and critical of the imaginary and porismatic cases of the proposition which are contained in his paper. These, however, will not be *ultimately* lost.—
T. S. D., June 20th, 1850.

which become the points of actual intersection with the given conic when the line and curve intersect at all. The only question turns upon the possibility of finding such points: but that will be at once removed by consulting the demonstration of the beautiful theorem of Dr. Matthew Stewart contributed to the second edition of Simson's Conic Sections, 1750, p. 152, (*case* 3.) Stewart was there, indeed, upon the very point of opening out the whole system of Desargues: but he evidently knew nothing of what the illustrious Lyonnese geometer had done. In fact it was equally unknown in France at that time.

The solution by the proposer of the problem, which is marked by great purity of geometrical taste, was intended for insertion here. It is extended as above to polygons of $2n$ sides, but is investigated only for the circle. As, however, the steps arrived at by *Disïota* in his process are so exactly correspondent with those of Mr. Gaskin, the publication of it would appear to be superfluous.

Mr. Weddle's discussion of the general case of $2n$-sided polygons was also confined to the case of the circle. We omit it with regret.

The case of the quadrilateral in any conic section was discussed by the coordinate method in this work (vol. i. p. 289); and it has been discussed by several continental geometers. Whether it has been extended to the cases of $2n$-sided polygons by any of the foreign writers, we do not know; but we have no recollection of such an attempt having come before us.

Some very neat geometrical investigations respecting the particular case proposed, were received from Mr. Nicholas Smyth of Galway. Mr. Elliott of Lamesley refers to our first volume as above quoted; and Messrs. Mortimer of Seedley, and Roy of Dundee, gave algebraical solutions of the proposed case. The latter gentleman has also communicated some further discussions of the even-sided polygon in the circle which have considerable merit.

ON THE THEORY OF SYMMETRIC FUNCTIONS.

[*James Cockle, Esq., M.A., Barrister-at-Law.*]

(Concluded from Vol. III., p. 249.)

1. Let us recapitulate some of our results, first observing that, since $f_n\, o = n$, there is *no* critical function of the first degree.

2. We have seen, (Vol. III. p. 177,) that

$$2np_2-(n-1)p_1^2 \text{ (or } a)$$

is the critical function of the second degree.

3. We have also seen (Ib. p. 247,) that

$$3n^2p_3-3n(n-2)p_2p_1+(n-1)(n-2)p_1^3 \text{ (or } \beta')$$

is the critical function of the third degree.

4. And, on making the corrections given in the note below*, it will be

* The reader will be pleased to make the following corrections in my last paper on Symmetric Functions:—

Vol. III., p. 248, lines 14, 15, and 16 from the top; and also at the bottom line, and the 8th line from the bottom of that page; and also at line 4 of p. 249;—*for* "10" *read* 6.

perceived that the function of the fourth degree given at page 249 of the third volume of this Journal takes the simpler form

$$24n^3p_4-12n^2(n-3)p_3p_2+(n-1)(n-2)(n-3)p_1^4,$$

in consequence of $-2\ (n-3)$ being the corrected value of λ. Call this four dimensioned function δ, then (see Vol. III., pp. 248-9) δ contains no power of P higher than the first.

In order to ascertain the existence of a critical function of the fourth degree, I shall form the function

$$\delta+4\lambda'\beta'(n-3)p_1,$$

λ' being a disposable multiplier. This last function only contains one dimension of P, and may be written as follows:—

$$24n^3p_4+12n^2(n-3)(\lambda'-1)\,p_3p_1+\mathrm{L}p_2p_1^2+\mathrm{L}'p_1^4,$$

where

$$\mathrm{L}=-12n(n-2)(n-3)\lambda',$$

and

$$\mathrm{L}'=(n-1)(n-2)(n-3)(4\lambda'+1).$$

Now, in $24n^3p_4$, the most general term in the development of the coefficient of P is of the form

$$24n^3 \int_n oqrs,$$

where, as we know (Vol. III., p. 249),

$$\int_n oqrs=(n-3)\int_n qrs\,;$$

so, the corresponding term in p_3p_1 is

$$n\int_n qrs+\{\int_n q\int_n ors+\int_n r\int_n oqs+\int_n s\int_n oqr\},\ \text{or}$$

$$\{n\int_n qrs+(n-2)(\int_n q\int_n rs+\int_n r\int_n qs+\int_n s\int_n qr)\}\,;$$

the term in $p_2p_1^2$ is

$$2\{\int_n qr\int_n o\int_n s+\int_n qs\int_n o\int_n r+\int_n rs\int_n o\int_n q+\int_n os\int_n q\int_n r$$
$$+\int_n or\int_n q\int_n s+\int_n oq\int_n r\int_n s\},$$

or

$$2n\Sigma\{\int_n q\int_n rs\}+6(n-1)\int_n q\int_n r\int_n s\,;$$

and that in p_1^4 is

$$24\int_n o\int_n q\int_n r\int_n s=24n\int_n q\int_n r\int_n s.$$

Hence $\delta+4\lambda'(n-3)\beta'p_1$ becomes, on reduction and substitution, equal to

$$\mathrm{EP}+\text{terms free from P},$$

where

$$\mathrm{E}=a\int_n qrs+b\Sigma\{\int_n q\int_n rs\}+c\int_n q\int_n r\int_n s,$$

and

$$\begin{aligned}
a&=n^3(n-3)\{24+12(\lambda'-1)\},\\
b&=12n^2(n-2)(n-3)(\lambda'-1)+2n\mathrm{L}\\
&=12n^2(n-2)(n-3)\{\lambda'-1-2\lambda'\},\\
c&=6(n-1)\mathrm{L}+24n\,\mathrm{L}'=6\{(n-1)\,\mathrm{L}+4n\mathrm{L}'\}\\
&=6n(n-1)(n-2)(n-3)\{-12\lambda'+4(4\lambda'+1)\}\\
&=24n(n-1)(n-2)(n-3)\{-3\lambda'+4\lambda'+1\},
\end{aligned}$$

and the conditions requisite in order that E may vanish independently of any relations among the quantities whose symmetrical functions are considered, are that a, b, and c should so vanish. This will be the case provided that

$$24+12(\lambda'-1)=0, \quad \lambda'-1-2\lambda'=0, \quad -3\lambda'+4\lambda'+1=0;$$

but, these conditions are all satisfied by the value $\lambda'=-1$. With this value of λ', then, $\delta+4\lambda'(n-3)\beta' p_1$ (or, as we may now write it, $\delta-4(n-3)\beta' p_1$) is a *critical* function, which if we call δ' we have, on substitution

$$\delta'=\{24n^3p_4-24n^2(n-3)p_3p_1+12n(n-2)(n-3)p_2p_1^2$$
$$-3(n-1)(n-2)(n-3)p_1^4\},$$

which is the critical function of the fourth degree.

5. If we divide α by $2n$, β' by $3n^2$, and δ by $24n^3$, the respective results will be

$$p_2-\frac{n-1}{2n}p_1^2$$

$$p_3-\frac{n-2}{n}p_2p_1+\frac{(n-1)(n-2)}{3n^2}p_1^3$$

$$p_4-\frac{n-1}{n}p_3p_1+\frac{(n-2)(n-3)}{2n^2}p_2p_1^2-\frac{(n-1)(n-2)(n-3)}{1.2.4.n^3}p_1^4,$$

all which are still critical functions. But their present form is better adapted than their original one for the discussion upon which we are about to enter.

6. In the expression (see vol. iii. p. 177),

$$y^n+p_1y^{n-1}+p_2y^{n-2}+p_3y^{n-3}+\ldots+p_ry^{n-r}+\ldots+p_n \;\ldots\ldots(\text{A}),$$

suppose that

$$y=z-\frac{p_1}{n};$$

then, in seeking the coefficient of z^{n-r} in the development,

$$y^n \text{ gives}\ldots\ldots(-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2\ldots r}\left(\frac{p_1}{n}\right)^r,$$

$$p_1y^{n-1}\quad\ldots\ldots(-1)^{r-1}\frac{(n-1)(n-2)\ldots(n-r+1)}{1.2\ldots(r-1)}p_1\left(\frac{p_1}{n}\right)^{r-1},$$

$$p_2y^{n-2}\quad\ldots\ldots(-1)^{r-2}\frac{(n-2)(n-3)\ldots(n-r+1)}{1.2\ldots(r-2)}p_2\left(\frac{p_1}{n}\right)^{r-2},$$

etc. *etc.*

$$p_{r-1}y^{n-r-1}\ldots\ldots\ldots\ldots(-1)(n-r+1)p_{r-1}\cdot\frac{p_1}{n},$$

$$p_ry^{n-r}\ldots\ldots\ldots\ldots p_r.$$

Let

$$P_m = \frac{(n-m)(n-m-1)\ \ldots (n-r+1)}{1.2\ldots(r-m)\times n^{r-m}},$$

then the required coefficient of z^{n-r} may be represented by the formula

$$p_r - P_{r-1}p_{r-1}p_1 + P_{r-2}p^{r-2}p_1^2 - \ldots \pm P_m p_m p_1^{r-m} \pm \ldots$$

$$\ldots \pm P_3 p_3 p_1^{r-3} \pm P_2 p_2 p_1^{r-2} \pm \left(1 - \frac{1}{r}\right) P_1 p_1^r \ldots .(B).$$

7. By division we obtain

$$\frac{P_m}{P_{m+1}} = \frac{n-m}{n(r-m)} \ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots (C);$$

and, if, in this last equation, we change P_m into A_{r-m}, P_{m+1} into A_{r-m-1}, *etc.*, such change will be equivalent to the alteration of (B) into

$$p_r - A_1 p_{r-1} p_1 + A_2 p_{r-2} p_1^2 + \ldots \pm \frac{r-1}{r} A_{r-1} p_1^r \ \ \ldots .(C'),$$

and we shall have the relation

$$\frac{A_{r-m}}{A_{r-m-1}} = \frac{n-m}{n(r-m)}.$$

Now, by changing m into $r-m$ in the last condition, we find

$$\frac{A_m}{A_{m-1}} = \frac{n-r+m}{nm};$$

which coincides with the latter of the two equations (z) which I gave at page 191 of the 28th volume of the *Philosophical Magazine*, (ser. iii.). The former of those two equations (viz. $A_0 = 1$) needs no other remark than $P_r = 0$, the general expression failing—or rather changing its form—for that value of m. I may add that the equation (aa), of the page and work just cited, must have its last term corrected so as to be rendered conformable with the equation (B) above.

8. It will not be difficult to account *a priori* for these results. If z be the root of the general equation of the n^{th} degree, and

$$y = P + Qx^q + etc. \ldots\ldots\ldots\ldots\ldots\ldots (D),$$

the equation in y will be obtained by putting the above expression (A) equal to zero. And in general the coefficients $p_1, p_2, \ldots p_n$ contain P. Put the equation (D) under the form

$$y - P = Qx^q + ete. \ldots\ldots\ldots\ldots\ \ldots\ldots (E),$$

then the coefficients of the transformed equation in $y - P$ will be free from P. Now, by the structure of equations, we know that

$$-p_1 = nP + Q\Sigma(x^q) + etc.,$$

or

$$-P = \frac{p_1}{n} + \frac{1}{n} Q\Sigma(x^q) + etc.;$$

substituting this value of $-P$ in (E), and transposing terms, we find

$$y + \frac{p_1}{n} = Q\left\{(x^q - \frac{1}{n}\Sigma(x^q)\right\} + etc.,$$

and, since P does not enter into the right hand side of this last equation, the

transposed equation in $y + \frac{p_1}{n}$ has its coefficients free from P. Let

$$y + \frac{p_1}{n} = z, \text{ then } z = y - \frac{p_1}{n},$$

and this value of z coincides with that employed above; and we see that, since the equation in z will have its coefficients free from P, all functions of the form (B) are "*critical.*"

9. Let us denote the critical functions of the second, third, and fourth degrees, when put under the forms of paragraph 5, by C_2, C_3, and C_4 respectively. Then we have the following relations:—

$$C_3 = p_3 - \frac{1}{6n^2}(n-1)(n-2)\,p_1^3 - \frac{n-2}{n}C_2 p_1,$$

$$C_4 = p_4 - \frac{n-3}{n}C_3 p_1 + \frac{(n-)(n-3)}{2n^2}C_2 p_1^2 - \frac{(n-1)(n-2)(n-3)}{24n^3}p_1^4,$$

which may (see vol. iii. pp. 177 and 248,), be written as follows:—

$$C_3 = \beta - \frac{n-2}{n}p_1 C_2,$$

$$C_4 = \gamma + \frac{(n-2)(n-3)}{2n^2}p_1^2 C_2 - \frac{n-3}{n}p_1 C_3.$$

10. Hitherto we have been considering the subject of symmetric functions as a *theory*, it will now be seen that it may be regarded as a *calculus*. And I proceed to show its application to the transformation of equations, premising, that the general equation of the n^{th} degree in x is supposed to be transformed into another in y of the form (A), x and y being connected by the relation

$$y = \mathrm{P} + \mathrm{Q}x^q + \mathrm{R}x^r + \mathrm{S}x^s + etc.,$$

and also observing, that my principal object is to exemplify the use of the *critical* and other functions which we have obtained.

11. Let it, then, be required, that the coefficients of the equation in y should be subject to the conditions,

$$p_1 = a \ldots\ldots\ldots (1), \qquad C_2 = b \ldots\ldots\ldots\ldots (2),$$

then, both these conditions may be satisfied, *without elimination*, by the assumption

$$y = \mathrm{P} + \mathrm{Q}x + x^2 \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{F}).$$

For, (vol. iii. p. 177) P does not enter into (2). Hence (2) will enable us to determine Q by means of a simple* quadratic. And P may then be determined from (1). So, if we were to write C_3 or C_4 in place of C_2 in (2), we should arrive at similar results, the only difference being that we should, instead of a quadratic, have to solve a *cubic* or a *biquadratic* equation. We thus have *three* distinct transformations given us.

* I here use the word *simple* to denote an equation (in this case a quadratic) involving only one unknown quantity. On the subject of the nomenclature of equations I have already dwelt on various occasions in the *Mechanics' Magazine*, vol. xlv. pp. 364, 509, 510, and 582; vol. xlvi., pp. 104, 177; &c.; and in the *Philosophical Magazine*, s. iii. vol. xxxii., p. 354, &c. And see my second series of Notes on the Theory of Algebraic Equations, in vols. xlviii. and xlix. of the *Mechanics' Magazine*.

12. The system of equations

$$C_2 = b \ldots\ldots\ldots(2), \qquad \beta = c \ldots\ldots\ldots(3),$$

may be simultaneously satisfied by means of the above assumption (F), and that, too, by a quadratic and a linear equation only; for, P disappears from C_2, and (2) is satisfied by a quadratic in Q. And, the value of Q, thus obtained, being substituted in (3), we have only a *linear* equation in P to satisfy, since (vol. iii. pp. 177-8, and 246-7) β contains one, and only one, dimension of P.

13. The system

$$C_2 = b \ldots\ldots\ldots(2), \qquad \delta = d \ldots\ldots\ldots(4),$$

may also be solved by means of a quadratic and a linear equation only; the steps being the same as in the last paragraph; for, as we have already seen, δ contains (only) the first power of P. The conditions of this and the preceding paragraph may evidently still be satisfied when, in (2), we write C_3 or C_4 in place of C_2. The only effect of such a change being the introduction of a cubic or biquadratic in place of the quadratic mentioned above. Hence we have *six* additional transformations. In like manner the equations

$$C_2 = b \ldots\ldots\ldots(1), \qquad \gamma = e \ldots\ldots\ldots\ldots(5),$$

may be satisfied by the assumption (F), and by means of two quadratics. The only effect of changing C_2 into C_3 or C_4 is to introduce a cubic or biquadratic, respectively, in place of one of the quadratics; we thus have three new transformations. In all these cases the critical equation is to be first solved, and the value of Q, so deduced, to be substituted in the second equation. No other kind of elimination is necessary.

14. By reference to page 83 of the first volume of this Journal, it will be seen that, in the case of $n = 3$, we have

$$\pi(v) = -3C_2;$$

and that, in the case of $n = 4$, the relation

$$\pi(v) = -8C_3.$$

These last results are, perhaps, not altogether unworthy of notice. The reader will find some remarks upon the last of them in my first series of Notes on the Theory of Algebraic Equations, published in vol. xlvi. of the *Mechanics' Magazine*. (See page 517 of that volume.)

15. Before concluding this paper, I shall exhibit one or two other results relating to symmetric functions. Let us omit the suffix to f, and assume that

$$(a,b) = tfab - 2sfafb,$$

then, since, (Jerrard's *Mathematical Researches*, p. 57, note,),

$$fab = fafb - fa + b,$$

we have

$$(a,b) = (t - 2s)fafb - tfa + b,$$

and hence

$$(c,d) = (t - 2s)fcfd - tfc + d.$$

Let us make

$$a+b = \mathrm{A},\ c+d = \mathrm{B},\ fafbfcfd = \mathrm{E},\ \text{and}\ t - 2s = u,$$

then

$$(a,b) \times (c,d) = u^2\mathrm{E} - tu(fcfdf\mathrm{A} + fafbf\mathrm{B}) + t^2f\mathrm{A}f\mathrm{B}.$$

So, if we make

$$a+d=\mathrm{C}, \text{ and } b+c=\mathrm{D},$$

we shall find

$$(a,d)\times(b,c)=u^2\mathrm{E}-tu(\int b\int c\int \mathrm{C}+\int a\int d\int \mathrm{D})+t^2\int \mathrm{C}\int \mathrm{D};$$

hence, if we form the expression

$$\{(a,b)\times(c,d)-(a,d)\times(b,c)\}\div t \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{G}),$$

we shall have

$$u\{\int b\int c\int \mathrm{C}+\int a\int d\int \mathrm{D}-\int c\int d\int \mathrm{A}-\int a\int b\int \mathrm{B}\}$$
$$+t\{\int \mathrm{A}\int \mathrm{B}-\int \mathrm{C}\int \mathrm{D}\}.$$

Suppose that the quantities whose symmetric functions are considered are only two in number, then, the last expression will only contain two sets of terms, one set consisting of terms of the form $x_1^a\ x_2^{b+c+d}$, and the other set being composed of terms of the form $x_1^{a+b}\ x_2^{c+d}$. Now, all the terms of the first set have zero for coefficients, for

$$\text{coefficient of } x_1^a\ x_2^{b+c+d}=u\{1-1\}=0,$$

and so of all the terms of the set. But, for the coefficients of the terms of the second set, we shall find

$$\pm u\mp t=\pm 4s\mp t,$$

half the terms being positive and half negative. And, when $4s=t$, we see that *all* the terms of the second set vanish. Hence, for $n=2$ and $4s=t$, the expression (G) vanishes.

16. In what follows I shall, for convenience, denote by S_a, S_{ab},... the functions of x above denoted by $\int a, \int ab$,... the former form being better adapted for involution. This being premised, and n being still supposed equal to 2, let us examine the function

$$4p_2-p_1^2, \text{ or } 4y_1y_2-(y_1+y_2)^2\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{H}),$$

in the case in which

$$y_1=\mathrm{Q}x_1^q+\mathrm{R}x_1^r+h,$$
$$y_1=\mathrm{Q}x_2^q+\mathrm{R}x_2^r.$$

It will be found that (H) is equal to

$$\mathrm{Q}^2(4x_1^q\ x_2^q-\mathrm{S}_q^2)+\mathrm{Q}\{(4\mathrm{S}_{qr}-2\mathrm{S}_q\mathrm{S}_r)\mathrm{R}+(4x_2^q-2\mathrm{S}_q)h\}$$
$$+\mathrm{R}^2(4x_1^r\ x_2^r-\mathrm{S}_r^2)+\mathrm{R}(4x_2^r-2\mathrm{S}_r)h-h^2\ldots\ldots\ldots(\mathrm{I});$$

it will also be seen that

$$\{2x_2^q-\mathrm{S}_q\}^2=\{x_2^q-x_1^q\}^2=\mathrm{S}_{2q}-2x_1^q\ x_2^q,$$

and

$$4x_1^q\ x_2^q-\mathrm{S}_q^2=4x_1^q\ x_2^q-\mathrm{S}_{2q}-2x_1^q\ x_2^q=2x_1^q\ x_2^q-\mathrm{S}_{2q};$$

and so when r is written in place of q; hence, if we make

$$2x_2^q-\mathrm{S}_q={}'a, \text{ and } 2x_2^r-\mathrm{S}_r={}''a,$$

the function (I) becomes, writing for $2\mathrm{S}_{qr}-\mathrm{S}_q\,\mathrm{S}_r$ its equivalent $-{}'a\,{}''a$,

$${}'a^2\mathrm{Q}^2+2\{-{}'a\,{}''a\mathrm{R}+{}'ah\sqrt{-1}\}\mathrm{Q}+{}''a^2\mathrm{R}^2+2\,{}''ah\sqrt{-1}\,\mathrm{R}-h^2\ldots\ldots(\mathrm{J}).$$

Next, treat the last expression by the Method of Vanishing Groups, having previously multiplied it by $'a^2$. It will then take the form

$$\left\{'a^2Q - 'a''aR + 'ah\sqrt{-1}\right\}^2 + OR^2 + 4'a^{2''}ah\sqrt{-1}R,$$

and in order that this may be reduced to the form of a square we must have

$$4'a^{2'}ah\sqrt{-1}\,R = 0,$$

whence (neither h nor R being supposed to vanish) we obtain

$$'a^{2''}a = (x_2^q - x_1^q)(x_2^r - x_1^r) = 0,$$

which cannot be unless $x_1 = x_2$.

By way of rendering every step of the last paragraph perfectly clear, I may add, that

$$2S_{qr} - S_q\,S_r = S_{qr} + (S_{qr} - S_q\,S_r) = S_{qr} - S_{q+r}$$
$$= -(x_2^q - x_1^p)(x_2^r - x_1^r) = -'a''a.$$

The two last paragraphs of the above paper confirm the results which I gave at p. 133 of vol. xxviii. of the third series of the *Philosophical Magazine.*

2. *Church Yard Court, Temple,*
21st May, 1849.

ON SOME PROPOSED NEW SPIRALS.

[*S. M. Drach, Esq., F.R.A.S.*]

A spiral $r\theta = ab$ has been termed the reciprocal or hyperbolic spiral, from its equation resembling the orthogonal $xy = ab$ of a hyperbola referred to its asymptotes. Carrying out this resemblance, we might term the Archimedean spiral, $r = a\theta$, the rectilinear or uniform spiral. Thus again $a^2r^2 + b^2\theta^2 = a^2b^2$ would be the *elliptic* spiral, degenerating, when $a = b$, into the *circular* spiral. Generally $(ar)^{2n} + (b\theta)^{2n} = (ab)^{2n}$ represents a finite curve comprehended in the circle whose radius is a, and having a spiral form, with as many convolutions as there are circumferences (2π's) in a.

The general hyperbolic spiral $a^2r^2 = b^2\theta^2 - a^2b^2$, or more extendedly $(ar)^{2n} = (b\theta)^{2n} - (ab)^{2n}$, differs from the former by b changing into $b\sqrt{-1}$, although this passes through the origin when $\theta = \pm a$. It then makes an infinite number of revolutions, since whatever θ is, the equation can always be satisfied. If $n = 1$, and $a = b$; there results the equilateral hyperbolic spiral. In the orthogonal curves, the rectangle $4ab$ includes the ellipse and excludes the hyperbola. Here the circle whose diameter is $2b$ includes the elliptic, but does not exclude the hyperbolic spiral. In the latter, r and θ may be taken so great as to render ab inconsiderable, so that ultimately the latter spiral merges into the Archimedean $ar = b\theta$, which is therefore an *asymptotic spiral* to the hyperbolic one.

In the elliptic spiral, $n = 1$; hence if $a^4 + (1 - 2a^2)\theta^2 + \theta^4 = \Psi^2$, we find

$$\text{area} = \frac{b^2\theta(3a^2 - \theta^2)}{6a^2}; \ \text{arc} = \frac{b}{a}\int d\theta \left\{\frac{a^4+(1-2a^2)\theta^2+\theta^4}{a^2 - \theta^2}\right\}^{\frac{1}{2}} = \frac{b}{a}\int \frac{\Psi d\theta}{\sqrt{(a^2-\theta^2)}};$$

$$\text{tangent} = \frac{r\Psi}{\theta}; \ \text{subtangent} = \frac{b}{a\theta}\left(a^2-\theta^2\right)^{\frac{3}{2}}; \ \text{normal} = \frac{b^2\Psi}{a^2 r}; \ \text{subnormal} = -\frac{b^2\theta}{a^2 r}.$$

For the hyperbolic spiral, change b into $b\sqrt{-1}$. In the circular one, $a = b$; hence area $= \frac{\theta}{6}\left(3a^2 - \theta^2\right)$; arc $= \int \frac{\Psi d\theta}{\sqrt{(a^2-\theta^2)}}$; tangent $= \frac{r\Psi}{\theta}$; subtangent $= \frac{1}{\theta}\left(a^2 - \theta^2\right)^{\frac{3}{2}}$; normal $= \frac{\Psi}{r}$; and subnormal $= -\frac{\theta}{r}$.

If in the circular spiral, r, r', r'', successively correspond to $\theta, \theta + 2\pi, \theta + 4\pi$, or for three successive revolutions in the same direction, then

$$r''^2 - 2r'^2 + r^2 = 8\pi^2.$$

MODERN GEOMETRY.

[*Mr. Davies.*]

(Completed from vol. ii. p. 314.)

The termination of the existence of the Mathematician compels me to close this series of papers abruptly, and to abandon the discussion of many of the topics essential to conferring completeness on them. Upon those subjects, however, which have been introduced, it is necessary to say but little more; as with respect to them there were but two further objects contemplated. These were to establish the identity of the two modes of generating the hyperbolic paraboloïd* described at ii., 314, with some collateral particulars; and to give a collection of the principal known properties of the rule-surfaces of the second degree, which are so profusely scattered through the Continental Journals by M. Chasles and his disciples. For the latter purpose space cannot be afforded me, nor yet the time (as I am now situated) for properly checking the accuracy of my own notes. The former object, however, will be here accomplished; although even this, must be done more briefly than I originally intended, and with some details altogether omitted.

XLV.

If three straight lines be parallel to one plane, and no two of them in one plane: then all straight lines resting upon them will be divided proportionally.

For since the three lines are parallel to the same planes, a plane can be drawn through each of them parallel to that plane; and these planes will be

* For "hyperboloïd of one sheet" in the last paragraph of XLIII. (vol. ii. p. 314), read "hyperbolic paraboloïd."

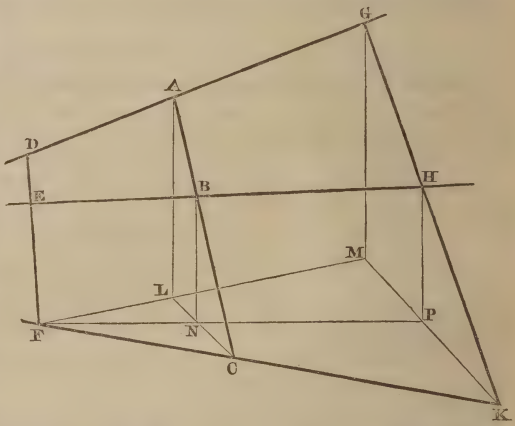
will be parallel to each other. Whence lines which rest upon those three lines are divided by those lines and by the parallel planes. But being divided by parallel planes they are divided proportionally, as stated in the proposition.

This proposition has, indeed been *virtually* proved, though not formally enunciated in XLIII.

XLVI.

If three straight lines be parallel to one plane, and no two of them in one plane: then all straight lines which rest upon them will be parallel to another plane.

Let DG, EH, FK be three straight lines parallel to the same plane (not drawn in the picture), no two of which either meet or are parallel; and let DF, AC, GK be any three straight lines resting on them: then will these lines be parallel to one plane.



For, through FK draw the plane KFM parallel to the given plane; and through A, B, G, H draw lines AL, BN, GM, HP parallel to DF, meeting the plane KFM in L, N, M, P respectively.

Then since LA, BN are parallel to the same line DF they are parallel to one another; and hence in one plane. But the line ABC is in that plane, and hence the points L,N,C are in the plane ACL and KFM; and consequently in their intersection. That is, the points L,N,C are in one straight line.

In the same manner it may be proved of M, P, K, or of any other line similarly related to the three given ones.

Again, the triangles ACL, BCN are similar; as are also the triangles GKM, HGP. Whence,

$$AC : CB :: LC : CN, \text{ and}$$

$$GK : KH :: MK : KP.$$

Moreover the three lines DG, EH, FK being parallel to the same plane *(hypoth.)* the lines AC, GK are divided proportionally in B and H (XLV.); that is

$$AC : CB :: GK : KH.$$

Wherefore

$$MK : KP :: LC : CN.$$

Now since DF, AL, GM are parallel, and the points D, A, G are in one straight line, these lines are in the same plane; and hence the points F,L,M

are in one line, namely the intersection of the plane DGMF with the plane KFM. And similarly, since the points E, B, H are in one line, the points F, N, P are in one line.

The two lines LC, MK are therefore divided proportionally by the diverging lines KF, PF, MF :—an indication of the parallelism of MK and LC; since only one straight line LC can be drawn through a given point N, so as to be divided by KF, FM in the same ratio as MK to KP. *(Leslie, Geom. Anal. b. i. prop.* 15.) Wherefore the lines LC, CA are parallel to MK, KH, each to each; and the planes LCA, MKG are also parallel.

Also DF is parallel to each of them, for it is parallel to a line in each, and a plane may be drawn through it parallel to each of those planes. The three lines are therefore situated in parallel planes; and are therefore parallel to any plane determined by means of two of them. Whence all lines situated as in the enunciation are parallel to one plane.

XLVII.

The same hypothesis being made, all lines which rest upon DG, EH, FK *divide them proportionally; and all lines which rest upon* DF, EH, FK *divide them proportionally.*

This may be deduced from the two preceding propositions, by a very obvious application of them; and might, but for unnecessarily encumbering the enunciation, have been made a part of the preceding proposition. It however flows readily from the following considerations:—

(*a.*) By the parallelograms ADLF, GDFM, EBNF, EHPF, we have AD = FL, GD = FM, EB = FN, EH = FP; and by the parallels CL, KM, we have

$$FC : FN : FL :: FK : FP : FM;$$

and hence,

$$PC : EB : DA :: FK : EH : DG.$$

The same may be shewn when any one of the other lines resting on DG, EH, FK is substituted for GK.

(*b.*) In a similar manner it may be shewn that DF = AL = GM, and EF = BN = HP; and that

$$DF : FE :: AL : BN :: AC : CB, \text{ and}$$

$$DF : FE :: GM : MP :: GK :: KH;$$

and so on for any number of such lines.

The double genesis of the hyperbolic paraboloïd according to the definition of Hachette, is now established by means of the mutuality of the two sets of lines, without deducing it as the connecting case of the hyperboloïd of one sheet. The same properties also enable us to identify the two geneses of the paraboloïd.

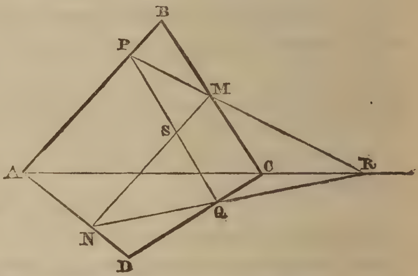
XLVIII.

If the opposite sides AB *and* CD *of a twisted quadrilateral* ABCD *be divided in* P *and* Q *in the same ratio, and likewise the opposite sides* AD *and* BC *be divided in* M *and* N *in the same ratio, viz.;—*

$$BM : MC :: AN : ND,$$
$$\text{and } AP : PB :: DQ : QC;$$

then the lines PQ, MN *being drawn, they will lie in the same plane, and intersect in a point* S, *such that*

$$PS : SQ :: BM : MC,$$
$$\text{and } MS : SN :: BP : PA.$$



For let the planes of the two component triangles meet in AC, and draw PM, NQ in those planes to meet AC. Suppose them, for the moment, to meet in different points, R and R′. Then, by transversals, we have,

$$AP.PM.CR = PB.MC.RA \text{ (trian. ABC and transv. PM)},$$
$$ND.QC.R'A = AN.DQ.R'C \text{ (...... ADC.............. NQ)},$$
$$BC.AN = BM.ND \text{ (first hypothesis)},$$
$$CD.AP = AB.CQ \text{ (second ....)}.$$

And compounding these we get

$$AR'.RC = AR.R'C;$$

and the points R, R′ are therefore easily shewn to coincide.

Wherefore the points P, M, Q, N are in one plane PRN; and the lines PQ, MN being in that plane, these lines intersect at some point S in that plane.

In the next place, after an analogous process,

$$PS.QN.RM = SQ.NR.MP \text{ (PQR and MN)},$$
$$CA.QD.RN = CD.QN.RA \text{ (CQR and AD)},$$
$$AR.BP.CM = AP.BM.CR \text{ (ABC and PR)},$$
$$PM.RC.AB = PB.RM.AC \text{ (PRA and BC)},$$
$$CD.AP = AQ.BD \text{ (second hypoth.)}$$

Whence compounding, PS.CM = QS.BM; or as a proportion,

$$PS : SQ :: BM : MC.$$

In a similar way, *mutatis mutandis*, is obtained

$$MS : SN :: BP : PA.$$

Scholium. This property might have been deduced from the preceding ones, perhaps with equal simplicity. This method of demonstration, however, has been chosen, mainly because it accords well with the application of the transversal method to this class of inquiries. A subordinate reason is, that it furnishes a substitute for prop. x, at p. 224, vol. ii. of my Edition of Hutton's Course. The hypothesis there made is not sufficiently restricted for the conclusion to be true in its general form; and that demon-

stration is only adapted to the restricted case. The restriction is that *also* AP : PB :: BM : MC. When this condition is fulfilled, what is there given is true : but it only becomes a particular case of the present one.

I am indebted to my friend Hugh Godfray, Esq., of St. John's College, Cambridge (and an able contributor to this work) for having pointed out this oversight and its correction, to me some years ago.

XLIX.

If a line move parallel to a plane and always rest upon two lines which meet that plane, it will generate the parabolic paraboloïd.

Let AC, GK (see fig. to XLVI) be two straight lines which meet the plane KFM in C, K, and let a line move parallel to KFM, always resting on AC, GK: it will generate the hyperbolic paraboloïd.

For, let EH, DG be two positions of the moving line, and KC the original one. Make

KC : CF :: HB : BE :: GA : AD,

this ratio being any whatever.

Then GK, AC, DF are in parallel planes, (XLVI.), and hence parallel to one plane. Wherefore, they are the directrices of a hyperbolic paraboloïd (XLIII.); of which FK, EH, DG are three generatrices. But these are also positions of a line resting on AC, GK and drawn parallel to the plane KFM. Whence the identity of the two geneses is established.

Schol. When the two lines AC, GK are in one plane, the surface becomes a plane ; which therefore may be considered as the limiting variety of the hyperbolic paraboloïd, and points out the origin of the name, *plan gauche*, or twisted plane.

It is very probable, however, that the system of lines will appear most simple to the mind which has been but little accustomed to this class of researches, when presented in reference to the twisted paraboloïd, as in XLVIII. The two sides AB and CD being viewed as the directrices of the line AD in its transition through BQ and BC, and always parallel to a plane determined by two of them, will correspond exactly with the second genesis. Under this aspect it has been often exhibited by the French writers of the present century. This view originated with M. Chasles, and it forms the subject of the *earliest* of his published Researches ; having been given in the *Correspondence sur l'Ecole Polytechnique*, tom. ii., 446, and iii., 6, when he was an *éléve* of that once-wonderful school. He gave more ample developments of it afterwards in Hachette's *Géométrie a trois Dimensions*, and elsewhere. Its interest, however, arises less from its difficulty than from its being the initial step in a career of brilliant discoveries in geometry such as have never yet been equalled, and probably never will be, by the genius of one man. Whatever Newton was in mathematico-physical research,—Chasles is yet the Newton of Geometry.

L.

A reference to the origin of the name *hyperbolic paraboloïd* was made in (XLIII.) It was then contemplated to give an investigation of the peculiarities of form of the Rule Surfaces of the second order, including the properties upon which the names are founded. To accomplish this, several

complicated figures would be required; and hence it must be confined to one or two general remarks, expressed in words only.

(*a.*) The general form of the hyperboloïd of one sheet may be conceived from the shape of a pulley, or of a backgammon-box. When the figure is one of revolution, the resemblance becomes close; and if it be conceived that a pair of opposite hyperbolas revolve about the conjugate axis, we have the hyperboloïd of revolution of one sheet, exactly placed before the mind. Wren's genesis of this figure was to suppose two lines not in the same plane to be invariably connected, and the one to be revolved about the other as an axis; in which case the moving line traced the surface of the hyperboloïd of revolution of one sheet. He devised a simple method of turning it upon the lathe; merely giving to the tool a position oblique to the axis of the lathe instead of parallel; and he proved with great elegance that the axial sections of this surface were hyperbolas. (See *Leyb. Rep.* vi. 12-15.) Moreover, the asymptotes of the hyperbola during the same revolution trace out an asymptotic cone, the conical form being *hollow*, or surrounding the the hyperboloïd of two sheets, which would be traced during the same revolution of the complete system of four hyperbolas, by the conjugate pair of hyperbolas.

(*b.*) If we suppose the circle described by the conjugate axis of the system to be gradually changed into a series of ellipses; and that the parallel ordinates of all sections parallel to any one of these ellipses be altered in the same ratio (as the ellipse is often conceived to be a transformation of the circle, or any hyperbola of the equilateral one), then the surface assumes its utmost generality of form.

(*c.*) It has been shewn in (XXXIII) that the section of this hyperboloïd and its asymptotic cone are similar figures; and as all three sections, the ellipse, parabola, and hyperbola can be cut from the cone, they can also be cut by the same planes from the hyperboloïd of one sheet.

(*d.*) A moment's reflection will shew that the hyperbolic paraboloïd is infinitely extended on both sides of each of the planes to which its generating lines are parallel, and that any plane, however drawn, will cut the surface in a curve of infinite extension. The ellipse is hence excluded from amongst the sections which it is possible to make in it; having the parabola and hyperbola as its only sections.

(*e.*) To frame some distinct conception of its form, the following considerations will be useful.

The ordinary paraboloïd of revolution is formed by the revolution of a parabola about its axis; and its figure is readily perceived by the mind. Let now the circular sections made by planes perpendicular to the axis be expanded or contracted into similar ellipses and similarly situated; and we have the most general form of the ordinary paraboloïd in considering it as the envelope or locus of these ellipses.

If now the paraboloïd be cut by planes parallel to the principal section, we get a series of parabolas. If then in the same planes, these parabolas be turned over their vertices being fixed, so that instead of having their axes within the paraboloïd they are in continuation of that position *backwards*

from the primitive parabola; the surface on which they will *then* be situated (or which envelopes them) will be the hyperbolic paraboloïd. For a small portion of it near the vertex, the back of a saddle may serve as a rude representation.

(*f.*) It will be obvious now why the name of hyperbolic paraboloïd was given to this surface. It is capable of being generated from the ordinary paraboloïd, and is in some senses the conjugate of it; whilst only hyperbolas and parabolas can be cut from it by a plane.

(*g.*) It can be proved that through each point on either of these surfaces as above described, two lines, and two only, can be drawn which shall be wholly on the surface; and that in the hyperbolic paraboloïd one of these lines will be parallel to one assignable plane and the other to another. It may, conversely, be proved (as has been proved for the hyperboloïd of one sheet) that every plane section is one of the sections of a cone:—in the hyperbolic paraboloïd the ellipse and its varieties being excluded.

Under these conditions the proof will be complete, and the character of the surfaces themselves be free from all obscurity. The reader to whom the subject is new will find it a useful process to establish these conclusions for himself; and to induce him to do so, I shall forbear giving references to books.

I cannot take leave of the many friends I have been accustomed to meet in these pages, and from whose writings I have derived much instruction, without a feeling of deep regret for the inevitable discontinuance of this work. It is a melancholy fact in the history of what is called "human progress,"—whether properly so called or not, cannot be discussed here,—that the "diffusion of useful knowledge" more extensively appears to be essentially interwoven with a deterioration of the qualities of abstract science, and an actual diminution of the number of its real and efficient cultivators. The science now acquired, or even sought for, is becoming more superficial:—the mere results of investigation, are alone objects of attraction. These results are only valued for their *uses* either in commerce or manufactures; or perchance occasionally for their aid in "getting up a popular and illustrated book for the market." Reasoning the most unsatisfactory, and assumption the most unwarrantable, replace the careful investigations and rigorous logic of the philosophers of other days. In Mathematics, the student is discouraged at every turn, and every obstacle is thrown in his way for bringing his researches before the world:—as effectually, indeed, as if it were a positive ordinance that abstract science should no longer be cultivated in England. As a people we are indeed become so *practical* and so *physical*, that there is now no journal open to the discussion of pure geometrical science:—except, indeed, it present itself occasionally under the guise of "practical utility" or "physical application;" or else assume a form so transcendental as to bewilder thoughtful readers, and excite the wondering admiration of the "oracle of the social party." Even journals that were formerly accustomed to insert a considerable number of mathematical papers, seldom give one now; and always when they do, it is with reluctance. Complaints constantly pour in upon the Editors of such journals, respecting "so much space being wasted upon such uninteresting topics." It is seldom, in truth, that an Editor will now venture upon such an insertion; except where the anthor has attained to a reputation

that shall in some degree conciliate the subscribers, by the notion that "no doubt there's something in it, for it is ————'s." I cannot, then, but regret that there is now one medium less, by which the Geometer can communicate with kindred minds. T. S. D.

Broomhall Cottage, Shooter's Hill,
August 5th, 1850.

NOTES ON "AN ANCIENT PROBLEM."

[*Mr. Davies.*]

I am desirous of correcting one or two oversights that have been made by me in the course of this series of notes; and likewise of adding a few remarks that appear to be necessary for completing the subjects.

1. *Mr. Noble's method of analysing porisms,* referred to at vol. iii., p. 76.

I have since that passage was printed been able to understand the *principle* upon which Mr. Noble proceeds; in making a single statement of the assumed porism fulfil the conditions of giving a construction, which Simson effects by two statements. Mr. Noble contracted his remarks into so small a compass, when they really required considerable expansion, that his views are expressed in a form approaching to enigmas. He was not comprehended by his able geometrical cotemporaries, Ivory, Lowry, Wallace, and Hampshire; and hence I may well be pardoned for being in the same predicament. His views required a long memoir for their development, instead of half a duodecimo page in the "Companion."

In the notes on Mr. Pott's translation of Simson's Porisms, I shall give a sufficiently full account of Mr. Noble's views and method, and in doing this, I have the advantage of that venerable gentleman's own suggestions, and such assistance as I may need to seek from him.

2. *The sphérical inscribed polygon,* referred to at vol. iii. p. 316.

I had misapprehended this problem. The polygon was not required to be composed of arcs of *great* circles, but arcs of less circles, each of these circles being required to pass through two given points on the sphere. Under this aspect, the number of given conditions is correct: but the problem takes a hopelessly difficult form.

Owing to a defect in the copy of the "Companion" which I had before me, and which I had not observed, I was led to say that no solution had been offered. In this I was wrong: two so-called solutions were given in the following year's number. My readers who are curious can consult them: but I need make no further remark here. I am indebted to Mr. Wilkinson for this correction.

3. *Sir William Hamilton's extension of the problem.*

Since the appearance of the preceding number of the *Mathematician,* Sir William Hamilton, Astronomer Royal of Ireland, has enunciated a theorem of a very remarkable character. It must suffice here to give merely the enunciation; as I have not had time at my disposal to attempt either an independent investigation or to form any analysis of the method by which its

discoverer obtained it. Besides, the method of "Quaternions," from which he first deduced it, has been but little studied by *geometers*; and it would be out of place here to dwell upon it under any other aspect. This is the theorem *(Phil. Mag.*, July, 1849).

"I. An *ellipsoid* (E) being given, and also a system of any *even* number of points of space, A_1, A_2, ...A_{2m}, of which points it is here supposed that none are situated on the surface of the ellipsoid; it is, in general, possible to *inscribe* in this ellipsoid *two*, and *only two*, distinct and *real polygons of* $2m$ *sides*, $BB_1...B_{2m-1}$ and $B'B'_1...B'_{2m-1}$, such that the sides of each of these two polygons (B)(B′) shall pass, respectively and successively, through the $2m$ given points; or in other words, so that BA_1B_1, $B_1A_2B_2$...... $B_{2m-1}A_{2m}B'$, and also $B'A_1B'_1$, $B'_1A_2B'_2...B'_{2m-1}\ A_{2m}B'$, shall be straight lines, while $B, B_1...B_{2m-1}$, and also B', $B'_1...B_{2m-1}$, shall be points upon the surface of the ellipsoid.

[It should be noted that there are also, in general, what may, by the use of a known phraseology, be called *two other*, but *geometrically imaginary*, modes of inscribing a polygon, under the same conditions, in an *ellipsoid*: which modes *may become real, by imaginary deformation*, in passing to *another* surface of the second order.]

"II. If we now take any *other* and *variable point* P on the ellipsoid (E) *instead* of B, or B′, and make *it* the *first corner* of an inscribed polygon of $2m+1$ sides, of which the *first* $2m$ *sides* shall pass, respectively and successively, through the $2m$ given points (A), in such a manner that $PA_1P_1, P_1\ A_2P_2\ ..P_{2m-1}A_{2m}P_{2m}$, shall be straight lines, while P, P_1, $P_2...P_{2m}$ shall all be points on the surface of the ellipsoid: then the *last side*, or *closing chord*, $P_{2m}P$, of this new and *variable polygon* (P) thus inscribed in the ellipsoid (E), shall *touch*, in all its positions, a certain *other ellipsoid* (E′)."

These theorems respecting the ellipsoid are precisely analogous to those of M. Poncelet and Mr. Gaskin, respecting lines of the second order; and there can, I think, be little doubt of their applying in all surfaces of the second degree. In the absence of direct proof, however, this can only be deemed conjectural, highly probable though it be.

It appears from the report of the recent Meeting of the British Association, (given in the *Athenæum)* that this eminent analyst gave to the mathematical section an outline of a method of investigating these properties independently of the quarternion theory. I sincerely hope he will publish the investigation *in extenso:* for extended as above hinted, it seems likely to be the crowning point of this system of inquiries. I take this means of suggesting to him the inquiry into the relation of the two ellipsoïds (E) and (E′); especially whether they touch, and if so (which it appears they must) whether in isolated points or in a continuous curve; and again, what will take place when the number of points is $2m+1$? It would, moreover, be interesting toknow whether any construction can be devised for the elements of the ellipsoid (E′); and conversely.

Broomhall Cottage, Shooter's Hill,
August 22.

SOLUTIONS OF MATHEMATICAL EXERCISES.

CXXXVII. *James Cockle, Esq., M.A., Barrister-at-Law.*

Let $U = 0$ be the equation of a surface of the second degree referred to the axes X, Y, Z, of which axes let P represent one; let $u = 0$ be the same surface referred to any other system of axes x, y, z, of which let p be one; also let $V = 0$, $W = 0$, be the equations of planes referred to the former axes, and $v = 0$, $w = 0$, those of planes referred to the latter. Then, $F(P) = 0$, and $f(p) = 0$ denoting quadratics (in P and p respectively) with real and unequal roots, it is required to show that if $U = 0$ can be put under the form

$$VW + F(P) = 0,$$

then $u = 0$ can also be put under the form

$$v\,w + f(p) = 0.$$

[SOLUTION. *Mr. Cockle, the Proposer.*]

My object in proposing the above exercise was to obtain a purely analytical demonstration of a property of the *hyperboloïd of one sheet.* In order to restrain the equation $U = 0$ to the representation of that surface the planes $V = 0$ and $W = 0$ must be neither parallel nor coincident.

There is also another restriction on the generality of the question which the reader will be kind enough to impart into it. It is as follows;—

Let the equation

$$px' + qy' + rz' + s = 0$$

represent successively the planes of xy, yz, and zx, referred to the principal planes of the hyperboloïd as coordinate planes. Then, if for all of those planes, (of xy, yz, zx), there subsists the relation

$$a^2p^2 + b^2q^2 - c^2r^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

(where a, b, and c, are the principal axes of the hyperboloïd) the property in question cannot be predicated of the equation $u = 0$.

I have not now time to attempt such a discussion as I have alluded to above. But the following remarks, founded partly on geometric considerations, may not be improper here.

For convenience let P denote Z. Then for two (unequal) values of Z, $U = 0$ takes the form $VW = 0$. In other words the given surface can be rectilineally by two (and only two) distinct planes parallel to the plane of XY. Consequently it is the *hyperboloïd of one sheet* that $U = 0$ represents. (See the third of my "Chapters on Analytical Geometry," pp. 271-2 of vol. xlvi. of the *Mechanics' Magazine).*

And in general (and excluding the cases in which (a) subsists) the equation to the hyperboloid of one sheet may be put under the form

$$uv + f(p) = 0.$$

(Vide *Ibid.*)

In connection with this subject, I may remark that Professor Hearn has (*supra*, pp. 195-200) given a process for determining the nature of surfaces

of the second degree, to which I have already adverted (*supra*, p. 249.) That learned analyst's method closely resembles the one which I have pursued in my "Chapters on Analytical Geometry," commenced in vol. xlv. and continued through vols. xlvi. and xlvii. of the *Mechanics' Magazine*, where they concluded. My process is founded on the reduction of the general quadratic between three unknowns, which I gave at p. 114 of vol. i. of the *Mathematician*, and of which I have given a praxis at pp. 504-5 of of vol. xlvii. of the *Mechanics' Magazine*. Professor Hearn has adopted the convenient plan of starting from the primitive forms of the equations to quadratic surfaces. He has shown (*sup*. p. 199) how to distinguish surfaces of revolution, and has indicated a method of finding cyclic planes, etc. (*Ibid*). When some or all of the squares are wanting, Mr. Hearn and I have followed different courses. Professor Hearn's will be seen (*supra*. p. 198) and mine will be found at p. 547 of vol. xlvii. of the *Mechanics' Magazine*, and also in my "Chapters on Analytical Geometry."

CLXXIV. *James Cockle, Esq., M.A., Barrister-at-Law.*

Prove that the equation

$$x^{12}+ax^3+bx^2+\frac{b^2}{a}x+d=0$$

admits of finite algebraic solution.

[SOLUTION. *Mr. Cockle, the Proposer.*]

The finite solution of the above equation of the 12th degree depends on the discussion of the more general form given below, and alluded to at page 116 of Vol. I. of the *Mathematician*. Of course when n is greater than 4, all that we can, in general, do is to reduce the solution of the equation of the $3n$th degree, of that form, to the solution of the general equation of the nth degree.

Let us, then, assume

$$x^{3n}+ax^3+bx^2+cx+d=0,$$

$$\text{or } -x^{3n}=ax^3+bx^2+cx+d.$$

If $x=y+z$; then

$$-(y+z)^{3n}=\left.\begin{array}{l} ay^3+3azy^2+3az^2y+az^3 \\ \qquad\quad +\ by^2\ +2bzy+bz^2 \\ \qquad\qquad\qquad +\ cy\ \ +cz \\ \qquad\qquad\qquad\qquad\quad +d \end{array}\right\}$$

$$=ay^3+(3az+b)y^2+(3az^2+2bz+c)y+az^3+bz^2+cz+d$$

$$=ay^3+Ay^2+By+C \text{ (suppose.)}$$

Multiplying by 3AB, gives

$$-3AB\,(y+z)^{3n}=3aABy^3+3BA^2y^2+3B^2Ay+3ABC,$$

and the right hand side will be a complete cube, if,

$$(1) \qquad 3aAB = A^3 \text{ or } 3AB = A^2,$$

$$(2) \qquad 3ABC = B^3 \text{ or } 3AC = B^2;$$

therefore we must have $3aB = A^2$, that is

$$3a(3az^2 + 2bz + c) = (3az + b)^2,$$

$$\text{or,} \quad 9a^2z^2 + 6abz + 3ac = 9a^2z^2 + 6abz + b^2,$$

$$\text{or,} \qquad 3ac = b^2.$$

Also we must have $3AC = B^2$; that is

$$3(3az + b)(az^3 + bz^2 + cz + d) = (3az^2 + 2bz + c)^2.$$

By actual multiplication, and simplifying,

$$(3ac - b^2)z^2 + (9ad - bc)z + 3bd - c^2 = 0,$$

$$\text{or since} \quad 3ac - b^2 = 0, \quad (9ad - bc)z + 3bd - e^2 = 0,$$

as at p. 116 of Vol. I. of the *Mathematician*.

Hence if z be determined from this quadratic, the equation

$$x^{3n} + ax^3 + bx + \frac{b^2}{3a} x + d = 0, \text{ may be made to take the form}$$

$$-3AB (y + z)^{3n} = (Ay + B)^3,$$

and may consequently be depressed to the n^{th} degree. In fact, extracting the cube root of both sides, gives

$$\sqrt[3]{-3AB}\,(y + 2)^n = \rho(Ay + B),$$

where ρ is a cube root of unity; and when n is less than 5, as is the case with the question which forms the basis of this discussion, (where $3n = 12$ and $n = 4$), the equation may of course be subjected to finite algebraic solution.

2, *Pump Court, Temple, 2nd Sept.*, 1850.

CLXXXII. *Dr. Burns, Rochester.*

Given $\sin\theta = \sin\theta^2 = -\sin\theta^3 = \sin\theta^4 = -\sin\theta^5 = \sin\theta^6 = -$ *etc.*; find θ a whole number of degrees greater than 1° and less than 90°.

[First Solution. *Dr. Burns, the proposer.*]

Since by hypothesis $\sin\theta^2 = \sin\{(2n+1)\pi - \theta\}$; therefore $\theta^2 = (2n+1)\pi - \theta$, and consequently $\theta^2 + \theta = (2n + 1)\pi$. Now substituting 0, 1, 2, 3, *etc.* for n, it is found that there are only two whole numbers between 1° and 90° which satisfy the first equation, viz., 35° and 44°, and of these only 44° satisfies the rest; consequently $\theta = 44°$.

[Second Solution. *By A. D. B., Hammersmith.*]

By trigonometry we have

$$\sin\theta = \sin(m\pi - \theta), \text{ when } m \text{ is odd}$$

$$\sin\theta = -\sin(m_1\pi + \theta), \ldots\ldots m_1 \ldots\ldots$$

$$\sin\theta = \sin(m_2\pi - \theta), \ldots\ldots m_2 \ldots\ldots$$

etc. *etc.*

By the question, $m\pi - \theta = \theta^2$; therefore $\theta^2 + \theta = 180m$

$$m_1\pi + \theta = \theta^3; \;\ldots\ldots\; \theta^3 - \theta = 180m_1$$
$$m_2\pi - \theta = \theta^4; \;\ldots\ldots\; \theta^4 + \theta = 180m_2, \text{ and so on};$$

hence $\dfrac{\theta^2+\theta}{m} = \dfrac{\theta^3-\theta}{m_1} = \dfrac{\theta^4+\theta}{m_2} = etc.$; consequently we get from the first of these equalities, $\dfrac{1}{m} = \dfrac{\theta-1}{m_1}$ or $m_1 = m(\theta - 1)$. Similarly we find

$$m_2 = m(\theta^2 - \theta + 1);\; m_3 = m(\theta^3 + 1)(\theta - 1), etc.$$

Now m, m_1, m_2, m_3, *etc.* being odd integers, we must have θ even, otherwise the values of m_1, m_3, m_5, *etc.* would become even if θ were supposed odd. We have therefore to seek the first odd value of m which will give an even integral value to the equation $\theta^2 + \theta = 180m$. In this manner we find $m = 11$ and $\theta = 44°$. The value $m = 7$ would give $\theta = 35°$ which satisfies only the equalities $\sin\theta = \sin\theta^2 = \sin\theta^4 = \sin\theta^6 = etc.$

CLXXXV. *Mr. Robert Harley.*

Required the value of the expectation of a person who is to receive a number of crowns, of which the digits only are known.

[SOLUTION. *Mr. Nicholas Smyth, Galway.*]

If $a, b, c, d \ldots l$ be the n given digits; then will

$$1.2.3.\ldots(n-1) \times \{10^{n-1} + 10^{n-2} + \ldots + 1\} \times (a+b+c+\ldots l)$$

be the sum of the $1.2.3..n$ different numbers which may be formed of these digits. And since each of these numbers is equally likely to be the expectancy, it follows that the mean $\dfrac{10^n - 1}{9n}(a + b + c + \ldots\ldots l)$ $= \dfrac{111\ldots\text{to } n \text{ places}}{n}(a+b+c+\ldots l)$, is the value of the expectation in crowns.

Dr. Burns observes "that if N be the number of permutations that can be formed of the given digits, and S the sum of these permutations; then $\dfrac{S}{N}$ is the mean, and consequently the value of the expectation.

Solutions were received from Messrs. Roy, Harley, Ashton, and Johnson.

*** In consequence of the announcement of the discontinuance of the publication of the *Mathematician* in the last number of that work, the Proposer of this Exercise, thinking that no solution of it would appear in the *Mathematician*, sent it to the Editor of the *Western Miscellany*, for insertion in that periodical. It forms question 42 of that work, and the Proposer's solution of it is given at pp. 411, 412.

CLXXXVII. *Dr. Rutherford.*

Find the values of the unreal roots of the equation

$$x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$$

[FIRST SOLUTION. *Dr. Rutherford, the proposer.*]

By the usual methods for determining the nature of the roots of equations, it is readily found that the proposed equation has only one *real* root, and

consequently it has four *unreal* or *impossible* roots. The real root is 1.05910900346 (see *Lady's Diary* for 1839, p. 47), and to find the unreal roots, we must first divest the equation of its second term by increasing each of its roots by ·6. Thus

1	3	2	—3	—2	—2	(—·6
	—·6	—1·44	—·336	+2·0016	—·00096	
	2·4	·56	—3·336	·0016	—2·00096	
	—·6	—1·08	·312	1·8144		
	1·8	—·52	—3·024	1·816		
	—·6	—·72	·744			
	1·2	—1·24	—2·28			
	—·6	—·36				
	·6	—1·6				
	—·6					
	0					

The transformed equation whose roots are each greater by ·6 than those of the given equation, is therefore

$$y^5 - 1{\cdot}6y^3 - 2{\cdot}28y^2 + 1{\cdot}816y - 2{\cdot}00096 = 0 \ldots\ldots\ldots (A).$$

In this equation let $a + \sqrt{-\beta}$ be substituted for y; then equating the rational part of the resulting equation to zero, and likewise the irrational part, there will arise the two equations

$$5a\beta^2 - (10a^3 - 4{\cdot}8a - 2{\cdot}28)\beta + a^5 - 1{\cdot}6a^3 - 2{\cdot}28a^2 + 1{\cdot}816a - 2{\cdot}00096 = 0 \ldots (B)$$

$$\beta^2 - (10a^2 - 1{\cdot}6)\beta + 5a^4 - 4{\cdot}8a^2 - 4{\cdot}56a + 1{\cdot}816 = 0 \ldots\ldots\ldots\ldots (C).$$

By means of these two equations, eliminate β, and the resulting equation in a is

$$\left.\begin{array}{l} a^{10} - 1{\cdot}2a^8 - {\cdot}285a^7 + {\cdot}1395a^6 + {\cdot}91583a^5 - {\cdot}054425a^4 - {\cdot}016258a_2 \\ \quad + {\cdot}12386835a^2 + {\cdot}004765305a - {\cdot}0202574584 = 0 \end{array}\right\} \ldots (D).$$

Employing Budan's criterion for the detection of imaginary roots, we get the following results:

In $(x - a)$ ..... + + + + + + + + + + + ...(5 variations lost),

... $\left(\dfrac{1}{x} - 1\right)$ ... + + + + + — — — — — — ...(1 variation left),

... $(-x)$........ + + — + + — — + + — — ...(5 variations),

... $(-x - 1{\cdot}2)$ ... + + + + + + + + + + + ...(5 variations lost),

...$\left(-\dfrac{1}{x} - \dfrac{1}{1{\cdot}2}\right)$... + + + + + + + + + + — ...(1 variation left).

Hence equation (D) has *eight* unreal, and only two real roots. Reducing the roots of (D) successively by ·1 it is found that the positive real root is situated between ·3 and ·4, as in the following calculation.

1+0	−1·2	−·285	+·1395	+·91583	·054425	−·0162580	+·12386835	+·004765305	−·0202574584	(·35204853
3	09	−333	−1854	−1377	270618	648579	1457997	41534496	138899403	
3	−1·11	−618	−459	90206	216193	485999	13844832	46299801	−63675181	
3	18	−279	−2691	−9450	242268	1375383	5584146	58286934	60785548	
6	−93	−897	−3150	80756	458461	1861382	19428978	10458673 5	−2889633	
3	27	−198	−3285	−19305	184353	1928442	11369472	1698436 1	2820779	
9	−66	−1095	−6435	61451	642814	3789824	307984 50	12157109 6	−68854	
3	36	−90	−3555	−29970	94443	2211771	31702 71	1864935	56750	
12	−30	−1185	−9990	31481	737257	6001 595	339687 21	1402204 5	−12104	
3	45	45	−3420	40230	−26247	338 947	33300	8185 3	11351	
15	15	−1140	−13410	−8749	71 1010	6340 542	372987	1410389 8	−753	
3	54	207	−2799	−48627	−3 3115	320	34825	8214 4	709	
18	69	−933	−16209	−57376	67 7895	6660	4078 12	14186 0 4	−44	
3	63	396	−1611	−8854	−3	305	14 53	1 6	43	
21	132	−537	17820	−166230	64	6965	4092 65	14187 6	−1	
3	72	612	112		−3	290	14 55	1 6		
24	201	75	−17708		61	7 255	4107 2	14,1,8,9		
3	84	150			−3	11	14 6			
27	285	225			58	7 266	41 22			
3	15				−3	11				
30	300				..55	7 277				
						11				
						7 288				

Now the real root of equation (A) is $r = 1\cdot65919000346$, and if $a_1 \pm \sqrt{-\beta_1}$ and $a_2 \pm \sqrt{-\beta_2}$ be the four imaginary roots, we must have $r + 2a_1 + 2a_2 = 0$; hence $a_2 = -\frac{r}{2} - a_1$; where $a_1 = \cdot35204853$; *

$$\therefore a_2 = -\cdot829554502 - \cdot35204853 = -1\cdot18160303.$$

Dividing (B) by (C) we get the remainder $(40a^3 - 3\cdot2a + 2\cdot28)\beta - 24a^5 + 22\cdot4a^3 + 20\cdot52a^2 - 7\cdot264a - 2\cdot00096$, which is the greatest common measure of these equations (B) and (C); consequently

$$\beta = \frac{24a^5 - 22\cdot4a^3 - 20\cdot52a^2 + 7\cdot264a + 2.00096}{40a^3 - 3\cdot2a + 2\cdot28}$$

$$= \frac{3a^2}{5} - \cdot512 + \frac{-2\cdot736a^2 + \cdot7032a + \cdot39604}{5a^3 - \cdot4a + \cdot285}.$$

Now if $a_1 = \cdot35204853$, then $\beta_1 = \cdot402745321$; but if $a_2 = 1\cdot18160303$, then $\beta_2 = 893699913$; consequently the four unreal roots of equation (A) are $\cdot35204853 \pm \sqrt{-\cdot402745321}$, and $-1\cdot18160303 \pm \sqrt{-\cdot893699913}$; and therefore the four unreal roots of the original equation, being less by $\cdot6$ than those of (A), are

$$-\cdot24795147 \pm \sqrt{-\cdot402745321} \text{ and } -1\cdot78160303 \pm \sqrt{-\cdot893699913}.$$

* The real positive root of eq. (D) is $\cdot352048527667318098 3$, which is correct as far as 17 or 18 decimals.

[SECOND SOLUTION. *Mr. W. H. Levy, Shalbourne.*]

The process for determining all the roots of a numerical equation is given in an elegant little work lately written by the Proposer of this exercise*. I shall follow the process there given. Now it is shewn (Lady's Diary for 1839, pp. 46-50), that this equation has only one real root, viz. $1\cdot05910900346$, and putting $z - \cdot6$ for x in the proposed equation, in order to divest it of its second term, we get

$$z^5 - 1\cdot6z^3 - 2\cdot28z^2 + 1\cdot816z - 2\cdot00096 = 0,$$

the root (r) of which is consequently $= 1\cdot65910900346$; hence (*Rutherford's Complete Solution of Numerical Equations*, page 20, eq. 5) we find the equation $a^{10} - 1\cdot2a^8 - \cdot285a^7 + \cdot1395a^6 + \cdot91583a^5 - \cdot054425a^4 - \cdot016258a^3 + \cdot12386835a^2 + \cdot004765305a - \cdot0202574584 = 0$, one real root of which is found by Horner's method $= \cdot352048527258 = a_1$.

Also $a_2 = -\frac{r}{2} - a_1 = -1\cdot181603028989$. Now taking these values of

α_1 and α_2 we find the two values of $\beta =$ ·402745312370 and ·893699913966 (*ibid*, p. 21); consequently the five roots are

$$1\cdot05910900346;\ -\cdot247951472742 \pm \sqrt{-\cdot402745312370} \text{ and}$$
$$-1\cdot781603028989 \pm \sqrt{-\cdot893699913966}.$$

* "The Complete Solution of Numerical Equations." London, G. Bell, 186, Fleet Street. There is an error in the printing of equation (6) p. 21. In the second expression for the value of β read "$(4b^2-25d)$" for "$(b^2-25d$."

CLXXXVIII. *Mr. Thomas Weddle.*

If two triangles be inscribed in, or circumscribed about a conic section, there exists another conic section to which the angles of each of the triangles are the poles of the opposite sides.

[Solution. *By the Proposer.*]

Let
$$X=0,\quad Y=0,\quad Z=0 \ldots\ldots\ldots\ldots(1)$$
be the equations to the sides of one of the triangles, and

$$\left.\begin{array}{l} X = b_1Y + c_1Z \\ X = b_2Y + c_2Z \\ X = b_3Y + c_3Z \end{array}\right\} \ldots\ldots\ldots\ldots(2),$$

be the equations to the sides of the other triangles. In the 'Mathematician,' Vol. I., p. 327, I have shown that the condition (at once necessary and sufficient) which is required in order that both the triangles may be either inscribed in or circumscribed about, the same conic section, will be found by eliminating M and N from the following equations,

$$\left.\begin{array}{l} b_1c_1 + Mc_1 + Nb_1 = 0 \\ b_2c_2 + Mc_2 + Nb_2 = 0 \\ b_3c_3 + Mc_3 + Nb_3 = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(3).$$

Now (Salmon's Conic Sections, p. 205), in order that the conic mentioned in the enunciation, may be such that the angular points of the second triangle shall be the poles of the opposite sides, its equation must be of the form

$$(-X+b_1Y+c_1Z)^2+h(-X+b_2Y+c_2Z)^2+k(-X+b_3Y+c_3Z)^2=0\ldots(4),$$

and this, when expanded, must reduce to the form

$$eX^2+fY^2+gZ^2=0,$$

in order that the angles of the first triangle may be the poles of the opposite sides. Hence the coefficients of XY, XZ, and YZ in (4), must each $= 0$,

$$\left.\begin{aligned} \therefore\ b_1 + hb_2 + kb_3 &= 0 \\ c_1 + hc_2 + kc_3 &= 0 \\ b_1c_1 + hb_2c_2 + kb_3c_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Multiply the first two equations by n and m respectively, then add the three equations, and equate the coefficients of h and k to zero;

$$\left.\begin{aligned} \therefore\ b_1c_1 + mc_1 + nb_1 &= 0 \\ b_2c_2 + mc_2 + nb_2 &= 0 \\ b_3c_3 + mc_3 + nb_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

and if m and n be eliminated from these equations, we shall get the condition, necessary and sufficient, in order that the two triangles may be such that the angular points of each shall, with respect to a certain conic, be the poles of the opposite sides. But (6) differs from (3) only in having m and n instead of M and N, and hence the condition will be the same whether the triangles be inscribed in a conic, circumscribed about a conic, or be such that the angular points of each are, with respect to a certain conic, the poles of the opposite sides.

The equation of the last mentioned conic will be obtained by finding the values of h and k from (5), and substituting them in (4). It is to be observed that when h and k are *both positive*, this conic (as is evident from (4),) will be imaginary.

The property constituting this exercise, was inferred from the analogous theorem in three dimensions, which will shortly appear in the "Cambridge and Dublin Mathematical Journal," in a second paper on the "Theorems in Space analogous to those of Pascal and Brianchon in a Plane."

Good solutions were received from Messrs. John Fraser, Percy Street Academy, Newcastle, and Nicholas Smyth.

END OF VOL. III.

E. JONES, PRINTER, THOMAS STREET, WOOLWICH.